

作用素の q -正定値性と q -正規拡大について (On q -positive definiteness and q -normal extensions of an operator)

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調和振動子に現れる生成作用素 $S = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right)$ は Segal-Bargmann 空間上の独立変数を掛ける積作用素にユニタリー同値になる (V. Bargmann [1961])。よって、 S はより広いヒルベルト空間上の正規作用素に拡大される。近年、量子群の理論に関連して、多様な q -調和振動子が調べられている。その中のひとつに、 $S^*S - qSS^* = 1$ がある。一方、作用素に関しても 1 次元量子平面の生成元や Wess グループにより導入された q -ハイゼンベルグ代数の生成元に現れる要素や関係式等から、通常の子規作用素等に対応して q -正規作用素等の変形作用素が導入され調べられている。この講演では上記 q -調和振動子に現れる q -生成作用素の q -正規拡大に関連して、作用素に「 q -正定値性」の概念を導入し、作用素の q -正規拡大に関して得られたいくつかの結果を報告する。

1 q -formally normal operators

Let q be a real positive number such that $q \neq 1$. For operators S and T in $\mathcal{H}$, the relation $S \subseteq T$ means that $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $S\eta = T\eta$ for all $\eta \in \mathcal{D}(S)$.

Let T be a closed densely defined operator in $\mathcal{H}$. If T satisfies

$$TT^* = qT^*T,$$

then T is called a *deformed normal operator with deformation parameter q* and we will simply say T is *q -normal*. The q -normality of a closed densely defined operator T is equivalent to the conditions:

$$\mathcal{D}(T) = \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| = \sqrt{q}\|T\eta\| \quad (\eta \in \mathcal{D}(T)).$$

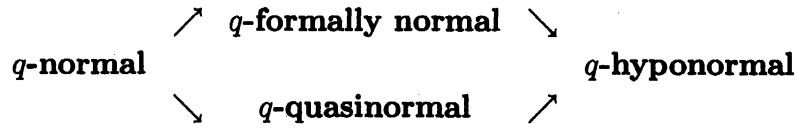
A densely defined operator T is called a *q -hyponormal* if it satisfies

$$\mathcal{D}(T) \subseteq \mathcal{D}(T^*) \quad \text{and} \quad \|T^*\eta\| \leq \sqrt{q}\|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$. If a q -hyponormal operator satisfies

$$\|T^*\eta\| = \sqrt{q} \|T\eta\|$$

for all $\eta \in \mathcal{D}(T)$, then T is called **q -formally normal**. Let T be a closed densely defined operator in $\mathcal{H}$ with polar decomposition $T = U|T|$. If T satisfies the equality $U|T| = \sqrt{q}|T|U$, then T is called a **q -quasinormal operator**.



Let T be a q -hyponormal operator in $\mathcal{H}$. Then there exists uniquely a contraction K_T such that

$$T^* \supseteq \sqrt{q}K_T T \quad \text{and} \quad \ker K_T \supseteq \ker T^*.$$

If T is q -formally normal, then K_T is a partial isometry with the initial space $\overline{\mathcal{R}}(T)$ and the final space $\overline{\mathcal{R}}(T^*|_{\mathcal{D}(T)})$.

命题 1 *A non-trivial q -formally normal operator is unbounded.*

命题 2 *If a closed q -formally normal operator T is q -quasinormal, then T is q -normal.*

命题 3 *Let T be a q -formally normal operator in a Hilbert space. If the domain $\mathcal{D}(T)$ is a core for T^* , then the closure $\tilde{T}$ is q -normal.*

定理 4 *Let T be a q -formally normal operator in a Hilbert space $\mathcal{H}$. If K_T is unitary, T is injective and the inverse T^{-1} is also q -formally normal.*

系 5 *The spectrum of a q -formally normal operator in $\mathcal{H}$ must contain zero if $\mathcal{R}(T^*|_{\mathcal{D}(T)})$ is dense in $\mathcal{H}$.*

Proof. Assume T has a bounded inverse T^{-1} . Then, T^{-1} is also q -formally normal. This is a contradiction since T^{-1} must be unbounded. $\square$

Every q -quasinormal operator has a q -normal extension in a possibly larger Hilbert space.

命題 6 *If a densely defined operator T has a q -formally normal extension in a possibly larger Hilbert space, then T is q -hyponormal.*

2 q -positive definiteness

Suppose a densely defined operator S in $\mathcal{H}$ has invariant domain, namely; $S\mathcal{D}(S) \subseteq \mathcal{D}(S)$. Then, q -positive definite condition means that

$$\sum_{i,j=0}^n q^{ij} \langle S^i f_j, S^j f_i \rangle \geq 0 \quad (\text{q-PD})$$

for $f_0, \dots, f_n \in \mathcal{D}(S)$, $n \in \mathbb{N}_0 = \{0, 1, \dots\}$.

例 7 (q -oscillator) *Suppose S is closable in $\mathcal{H}$. If $\mathcal{D}(S)$ is invariant for S and S^* and such that*

$$S^*S - qSS^* = 1 \quad \text{on } \mathcal{D}(S),$$

then S satisfies $(q^{-1}\text{-PD})$.

命題 8 *Suppose S has invariant domain. If S has a q -formally normal extension N in a possibly larger Hilbert space such that*

$$N\mathcal{D}(N) \subseteq \mathcal{D}(N) \quad \text{and} \quad N^*\mathcal{D}(N) \subseteq \mathcal{D}(N), \quad (\star)$$

then it satisfies the q -positive definite condition (q -PD).

Proof. For $f_0, \dots, f_n \in \mathcal{D}(S)$, $n \in \mathbb{N}_0$,

$$\begin{aligned} \sum_{i,j=0}^n q^{ij} \langle S^i f_j, S^j f_i \rangle &= \sum_{i,j=0}^n q^{ij} \langle N^i f_j, N^j f_i \rangle \\ &= \sum_{i,j=0}^n \langle N^{*j} f_j, N^{*i} f_i \rangle \\ &= \left\| \sum_{i=0}^n N^{*i} f_i \right\|^2 \end{aligned}$$

□

例 9 Let $\mathcal{H}$ be a separable Hilbert space and $\{e_n\}_{n \in \mathbb{N}_0}$ be an orthonormal basis of $\mathcal{H}$. Define an operator S in $\mathcal{H}$ by

$$\mathcal{D}(S) = \text{linear span of } \{e_n : n \in \mathbb{N}_0\}$$

and

$$S e_n = \left(\frac{1}{\sqrt{q}} \right)^n e_{n+1}$$

for all $n \in \mathbb{N}_0$.

定理 10 Suppose S in $\mathcal{H}$ has invariant domain. If S satisfies the q -positive definite condition (q -PD), then S has a q -formally normal extension N in a possibly larger Hilbert space such that the condition $(\star)$ holds.

Proof. Define $K : (\mathbb{N}_0 \times \mathcal{D}(S)) \times (\mathbb{N}_0 \times \mathcal{D}(S)) \rightarrow \mathbb{C}$ by

$$K((m, x), (n, y)) = q^{mn} \langle S^n x, S^m y \rangle$$

for $x, y \in \mathcal{D}(S)$ and $m, n \in \mathbb{N}_0$.

- K is a positive definite kernel on $\mathbb{N}_0 \times \mathcal{D}(S)$. The corresponding R.K.H.S is denoted by $\mathcal{K}$.
- Define

$$K_{(m,x)}(n, y) = K((m, x), (n, y)) \quad \text{and put}$$

$$\mathcal{D} \equiv \text{linear span of } \{K_{(m,x)} : m \in \mathbb{N}_0, x \in \mathcal{D}(S)\}$$

Then, $\mathcal{D}$ is dense in $\mathcal{K}$ and $\langle K_{(m,x)}, K_{(n,y)} \rangle_{\mathcal{K}} = K_{(m,x)}(n, y)$.

- Define an operator N on $\mathcal{D} \equiv \mathcal{D}(N)$ by

$$N K_{(m,x)} = q^m K_{(m,Sx)} \quad m \in \mathbb{N}_0, x \in \mathcal{D}(S).$$

Then,

1. $\mathcal{D} \subseteq \mathcal{D}(N^*)$ and $N^* K_{(n,y)} = K_{(n+1,y)}$
2. $\langle N K_{(m,x)}, N K_{(n,y)} \rangle_{\mathcal{K}} = q^{mn+m+n} \langle S^{n+1}x, S^{m+1}y \rangle_{\mathcal{H}}.$
3. $\langle N^* K_{(m,x)}, N^* K_{(n,y)} \rangle_{\mathcal{K}} = q^{(m+1)(n+1)} \langle S^{n+1}x, S^{m+1}y \rangle_{\mathcal{H}}.$

Finally, Define an isometry $V : \mathcal{D}(S) \ni x \rightarrow K_{(0,x)} \in \mathcal{K}$ (and it is extended to $\mathcal{H}$).
By $VS \subseteq NV$, N extend S . $\square$

注意 11

$$\mathcal{D}(N) = \text{linear span of } \left\{ N^{*n} f : f \in \mathcal{D}(S) \right\}.$$

定理 12 Suppose S has invariant domain. If S satisfies the q -positive definite condition (q -PD), then S satisfies the $(q'$ -PD) for every $q' > q$.

Proof. This follows from the fact (by Man-Duen Choi): “If $q > 1$, then the matrix $(q^{ij})_{i,j=0}^n$ is positive semi-definite for any $n \in \mathbb{N}_0$ ”.

系 13 Suppose S has invariant domain. If $q > 1$, then for S to satisfy the positive definite condition (1-PD), implies S to satisfy the $(q$ -PD).

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