

On the First Cohomology Group of
a Minimal Set

by

Ippei ISHII

(Keio Univ.)

Notations and Definitions

Let (Y, ρ_t) be a flow on a compact metric space Y .

(i) The flow (Y, ρ_t) is said to be a minimal flow on Y , if every orbit is dense in Y .

(ii) A subset Σ of Y is said to be a local section if it satisfies: (a) $h: \overline{\Sigma} \times (-\mu, \mu) \rightarrow \{\rho_t(y) \mid y \in \overline{\Sigma}, -\mu < t < \mu\}$ defined by $h(y, t) = \rho_t(y)$ for some $\mu > 0$ (such μ is called a collar-size for Σ), and (b) $\{\rho_t(y) \mid y \in \Sigma, t \in J\}$ is open for any open $J \subset \mathbb{R}$. Moreover if Σ is compact, then we call it a global section.

(iii) $\bar{H}^*(Y)$ denotes the Alexander cohomology of Y with the real coefficients. For a presheaf Γ of modules on Y , $\check{H}^*(Y; \Gamma)$ denotes the Čech cohomology with the coefficient Γ .

1. Preliminaries

At the meeting last year, I have reported the following results. (For the precise proof, see [1].)

PROPOSITION 1. For a minimal flow (M, ξ_t) and a local section Σ , we can construct a minimal flow $(\tilde{M}, \zeta_t)$ with the following properties: (a) $\tilde{M}$ is a compact metric space, (b) there is a continuous map $p: \tilde{M} \rightarrow M$ such that $p \circ \zeta_t = \xi_t \circ p$, (c) $\overline{p^{-1}(\Sigma)}$ is a global section of $(\tilde{M}, \zeta_t)$, and (d) $p^{-1}(\Sigma)$ is totally disconnected; i.e., $\dim(\overline{p^{-1}(\Sigma)}) = 0$.

Let (M, ξ_t) be a minimal flow and Σ be a local section with a collar-size μ . And let $(\tilde{M}, \zeta_t)$ be a minimal flow which is constructed in the previous proposition. Define X to be $X = M \cup \{\xi_t(X) \mid x \in \Sigma, -\mu < t < 0\}$, and $\tilde{X}$ to be $\tilde{X} = \{\zeta_t(\tilde{x}) \mid \tilde{x} \in p^{-1}(\Sigma), -\mu < t < 0\}$. Let Γ_j ($j = 1, 2, 3$) be presheaves defined by $\Gamma_1(U) = \bar{H}^0(U)$, $\Gamma_2(U) = \bar{H}^0(p^{-1}(U))$ and $\Gamma_3(U) = \text{Coker}(P^*)$ where p^* is the homomorphism $\bar{H}^0(U) \rightarrow \bar{H}^0(p^{-1}(U))$ induced by $p: p^{-1}(U) \rightarrow U$. Then we have

PROPOSITION 2. There is an exact sequence

$$\check{H}^0(X; \Gamma_2) \rightarrow \check{H}^0(X; \Gamma_3) \rightarrow \bar{H}^1(X) \rightarrow 0.$$

2. Results

Using the exact sequence in Proposition 2, we can give a method for calculating the first cohomology of a 3-dimensional minimal set.

In what follows, (M, ξ_t) will be a minimal flow on a 3-dimensional compact manifold which is generated by a C^1 -vector field.

Notations

(a) For a real valued function F defined on a subset D of M , $\hat{F}$ denotes a map $\hat{F}: D \rightarrow M$ defined by $\hat{F}(x) = \xi_{F(x)}(x)$.

(b) Let Σ be a local section, then we use the following notations.

$T_\Sigma: M \rightarrow \mathbb{R}$ defined by $T_\Sigma(x) = \inf \{t > 0 \mid \xi_t(x) \in \overline{\Sigma}\}$,

$B_\Sigma^1 \subset \partial\Sigma: B_\Sigma^1 = \{x \in \partial\Sigma \mid \hat{T}_\Sigma(x) \in \partial\Sigma\}$,

$B_\Sigma^j \subset \partial\Sigma: B_\Sigma^j = \{x \in \partial\Sigma \mid \hat{T}_\Sigma(x) \in B_\Sigma^{j-1}\} \quad (j = 2, 3, \dots)$

$A_\Sigma^j \subset \Sigma: A_\Sigma^j = \{x \in \Sigma \mid \hat{T}_\Sigma(x) \in B_\Sigma^j\} \quad (j = 1, 2, 3, \dots)$

$C_\Sigma \subset \Sigma: C_\Sigma = \{x \in \Sigma \mid \hat{T}_\Sigma(x) \in \partial\Sigma\}$.

Let Σ be a local section of (M, ξ_t) which is homeomorphic to a 2-disk. Here we make an assumption.

Assumption I. $A_\Sigma^j = \emptyset$ for $j \geq 2$, and A_Σ^1 is a finite set.

Let $A_\Sigma^1 = \{a_1, a_2, \dots, a_N\}$ consist of N -points. We denote by $C_1, C_2, \dots, C_{2N}$ the components of $C_\Sigma \setminus A_\Sigma^1$. (It is easy to see that if A_Σ^1 consists of N -points, then $C_\Sigma \setminus A_\Sigma^1$ has $2N$ connected components.) Then, for each point a_k of A_Σ^1 , we can take a neighborhood $S_k \subset \Sigma$ of a_k with the properties: (a) there are continuous functions $\sigma_{k,j}$ ($j = 1, 2, 3$) such that $\delta_{k,j}(S_k) \subset \Sigma'$ ($j = 1, 2$), $\delta_{k,3}(S_k) \subset \Sigma$, and $\delta_{k,j}(a_k) = \hat{T}_\Sigma^j(a_k)$ ($j = 1, 2, 3$), where Σ' is a local section which includes the closure of Σ . We make another assumption on Σ .

Assumption II. $S_k \cap (C_\Sigma \setminus A_\Sigma^1)$ has exactly three components $\gamma_{k,j}$ ($j = 1, 2, 3$) such that $\delta_{k,2}(\gamma_{k,1}) \subset \Sigma$, $\delta_{k,2}(\gamma_{k,2}) \cap \overline{\Sigma} = \emptyset$, and $\delta_{k,2}(\gamma_{k,3}) \subset \partial \Sigma$.

REMARK. We can show that there is a local section which satisfies the Assumptions I and II.

Fixing a numbering of the components of A_Σ^1 and $C_\Sigma \setminus A_\Sigma^1$, for each k ($1 \leq k \leq N$), we define integers $k(j)$ ($j = 1, 2, 3, 4$ and $1 \leq k(j) \leq 2N$) so that $C_{k(j)} \cap \gamma_{k,j} \neq \emptyset$ ($j = 1, 2, 3$) and $\hat{T}_\Sigma(a_k) \in \overline{C}_{k(j)}$. And a $2N \times 2N$ matrix $\Lambda_\Sigma = [\lambda_1, \lambda_2, \dots, \lambda_{2N}]$ (λ_j is a $2N$ -vector) is defined by

$$\begin{aligned} (u_1, u_2, \dots, u_{2N}) \lambda_{2k-1} &= u_{k(1)} - u_{k(2)} \\ (u_1, u_2, \dots, u_{2N}) \lambda_{2k} &= u_{k(2)} - u_{k(3)} + u_{k(4)} \end{aligned} \quad (k = 1, \dots, N).$$

Numbering the components of A_Σ^1 and $C_\Sigma \setminus A_\Sigma^1$ as in the figure, we have

$$\begin{aligned}
 C_1(1) &= C_1, & C_1(2) &= C_2, & C_1(3) &= C_4, & C_1(4) &= C_{10}, \\
 C_2(1) &= C_3, & C_2(2) &= C_2, & C_2(3) &= C_5, & C_2(4) &= C_{11}, \\
 C_3(1) &= C_7, & C_3(2) &= C_4, & C_3(3) &= C_6, & C_3(4) &= C_{12}, \\
 C_4(1) &= C_8, & C_4(2) &= C_5, & C_4(3) &= C_6, & C_4(4) &= C_9, \\
 C_5(1) &= C_9, & C_5(2) &= C_{10}, & C_5(3) &= C_7, & C_5(4) &= C_3, \\
 C_6(1) &= C_{12}, & C_6(2) &= C_{11}, & C_6(3) &= C_8, & C_6(4) &= C_1.
 \end{aligned}$$

Hence the equation $u\Lambda_\Sigma = 0$ becomes as follows:

$$\begin{aligned}
 u_1 - u_2 &= 0, & u_2 - u_4 + u_{10} &= 0, \\
 u_3 - u_2 &= 0, & u_2 - u_5 + u_{11} &= 0, \\
 u_7 - u_4 &= 0, & u_4 - u_6 + u_{12} &= 0, \\
 u_8 - u_5 &= 0, & u_5 - u_6 + u_9 &= 0, \\
 u_9 - u_{10} &= 0, & u_{10} - u_7 + u_3 &= 0, \\
 u_{12} - u_{11} &= 0, & u_{11} - u_8 + u_1 &= 0.
 \end{aligned}$$

One can easily see that this equation has three independent solutions. Therefore, using the Theorem, we get $\bar{H}^1(T^3) \simeq \mathbb{R}^3$.

REFERENCE

[1] Ishii, I., On the first cohomology group of a minimal set, to appear in Tokyo Journal of Mathematics Vol.1 No.1.