

Morley's theorem on Omitting types

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Abstract

We think about Morley's omitting types theorem for countable first-order theory. Then I introduce the result of having been related to Morley's theorem shown by [4].

1 Introduction

Definition.($\beth$ -number) $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, $\beth_\delta = \sup_{\alpha < \delta} \beth_\alpha$.

Fact.(Erdős-Rado) Let α be infinite cardinal and $n < \omega$. Then $\beth_n^+ \rightarrow (\omega^+)^{n+1}_{\omega^+}$, $\beth_{\alpha+n}^+ \rightarrow (\beth_\alpha^+)^{n+1}_{\beth_\alpha^+}$.

Note. $\alpha \rightarrow (\beta)_\gamma^n$ means whenever $|X| = \alpha$ and given any function f from $[X]^n$ into γ , there exists a subset Y of X with $|Y| = \beta$ and an $i < \gamma$ such that for all $\bar{y} \in [Y]^n$, $f(\bar{y}) = i$.

Theorem.(Stretching) Let $\mathcal{L}$ be countable language, M be a model of theory T of $\mathcal{L}$, $\langle A, < \rangle$ be an infinite set of indiscernibles in M , and $\langle B, < \rangle$ be an arbitrary infinite lineary ordered set. Then there exist a model N of T such that $\langle B, < \rangle$ is a set of indiscernibles in N , and for any $a_1 < \dots < a_n \in A$ and $b_1 < \dots < b_n \in B$, $tp(a_1, \dots, a_n) = tp(b_1, \dots, b_n)$.

Proof. Put $\Sigma := \{t(x_1, \dots, x_n) : t \text{ is term in } \mathcal{L}\}$. We define an equivalence relation $\sim$ on Σ as follows. If $t(x_1, \dots, x_n), t'(x_1, \dots, x_n) \in \Sigma$, define $t \sim t'$ iff for any $a_1 < \dots < a_n \in A$, $M \models t(a_1, \dots, a_n) = t'(a_1, \dots, a_n)$. Put $\bar{N} := \{t(b_1, \dots, b_n) : t(x_1, \dots, x_n) \in \Sigma, b_1 < \dots < b_n \in B\}$. We define an equivalence relation $\approx$ on $\bar{N}$ as follows. If $t(b_1, \dots, b_n), t'(b'_1, \dots, b'_m) \in \bar{N}$, define $t \approx t'$ iff $t_0(z_1, \dots, z_s) \sim t'_0(z_1, \dots, z_s)$, where $\{z_1, \dots, z_s\} := \{x_1, \dots, x_n\} \cup \{x'_1, \dots, x'_m\}$ and $t_0(z_1, \dots, z_s) := t(x_1, \dots, x_n)$, $t'_0(z_1, \dots, z_s) := t'(x'_1, \dots, x'_m)$. Put $N := \{t(\bar{b})^\approx : t(\bar{b}) \in \bar{N}\}$. Note that for any $t_1(\bar{b}_1)^\approx, \dots, t_n(\bar{b}_n)^\approx \in N$ there exists, for some $\bar{b}' \in B$ and $t'_i \in \Sigma$ such that $t_i(\bar{b}_i)^\approx = t'_i(\bar{b}')^\approx$. We treat B as a subset of N by identifying each $b \in B$ with $b^\approx$.

N can be made into a $\mathcal{L}$ -structure by defining constants, functions and relations as follows:

(Constants) $N \models c_N = c^\approx$.

(Functions) $N \models F(t_1(\bar{b})^\approx, \dots, t_n(\bar{b})^\approx) = (F(t_1(\bar{b}), \dots, t_n(\bar{b})))^\approx$.

(Relations) $N \models R(t_1(\bar{b})^\approx, \dots, t_n(\bar{b})^\approx)$

$\stackrel{\text{def}}{=} \text{for all } a_1 < \dots < a_m \in A, M \models R(t_1(\bar{a}), \dots, t_n(\bar{a}))$.

This definition dose not depend on the choice of representatives of the equivalence classes under $\approx$.

By induction on the complexity of formulas and use Skolem function it can be shown that for any $b_1 < \dots < b_n \in B$ and $\phi(x_1, \dots, x_n) \in \mathcal{L}$,

$$\begin{aligned} N \models \phi(b_1, \dots, b_n) \quad \text{iff} \quad & \text{for all } a_1 < \dots < a_n \in A, \\ M \models \phi(a_1, \dots, a_n). \end{aligned}$$

By indiscernibility of $\langle A, < \rangle$, $\langle B, < \rangle$ is a set of indiscernibles in N and for any $a_1 < \dots < a_n \in A$ and $b_1 < \dots < b_n \in B$, $tp(a_1, \dots, a_n) = tp(b_1, \dots, b_n)$. In particular, $N \equiv M$, hence N is a model of T .

2 Morley's Theorem

Theorem. (Morley's omitting types theorem) Let T be a theory of countable language $\mathcal{L}$, Γ a set of partial types in finitely many variables over $\emptyset$, $\mu = (2^\omega)^+$. Suppose $\{M_\alpha : \alpha < \mu\}$ is a sequence of models of T such that

1. $|M_\alpha| > \beth_\alpha$,
2. M_α omits each member of Γ .

Then for every $\lambda \geq \omega$, there is a model N with $|N| = \lambda$ of T such that N omits each member of Γ .

Proof. Assume to simplify an argument T has built-in Skolem functions and the set of formulas Γ in the unary. Let $C = \langle c_i : i < \omega \rangle$ be a sequence of new constant symbols, $\mathcal{L}^* = \mathcal{L} \cup C$.

Now we construct the consistent $\mathcal{L}^*$ -theory Φ as following properties:

1. $T \cup \{c_i \neq c_j : i < j < \omega\} \subset \Phi$;
2. for each term $t(x_1, \dots, x_n)$ and $p \in \Gamma$, there is a $\phi_p \in p$ such that for all $i_1 < \dots < i_n < \omega$,

$$\neg \phi_p(t(c_{i_1}, \dots, c_{i_n})) \in \Phi$$
;
3. for any $\psi(x_1, \dots, x_n) \in \mathcal{L}$ if $i_1 < \dots < i_n < \omega$ and $j_1 < \dots < j_n < \omega$,

$$\psi(c_{i_1}, \dots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \dots, c_{j_n}) \in \Phi.$$

Notation. $F := \{(M_\alpha, A_\alpha) : \alpha < \mu\}$ is a sequence such that M_α is satisfied the hypotheses of the theorem and A_α is subset of M_α with $|A_\alpha| > \beth_\alpha$.

We say that $F' = \{(M'_\alpha, B_\alpha) : \alpha < \mu\}$ is subsequence of F if for each M'_α there is $\beta \geq \alpha$ such that $M'_\alpha = M_\beta$ and $B_\alpha \subset A_\beta$ with $|B_\alpha| > \beth_\alpha$.

Fix a linear ordering of each M_α in an arbitrary fashion denoting them all by $<$.

Claim 1. Fix a term $t(x_1, \dots, x_n)$. There is subsequence F' of F as following property: for each $p \in \Gamma$ there is a $\phi_p \in p$ such that for any $(M'_\alpha, B'_\alpha) \in F'$, if $i_1 < \dots < i_n < \omega$ and $b_{i_j} \in B'_\alpha$ then $M'_\alpha \models \neg \phi_p(t(b_{i_1}, \dots, b_{i_n}))$.

Proof of claim 1. Note $|\Gamma| \leq 2^\omega$. Let $N_\alpha = M_{\alpha+n}$. Define, for all $\alpha < \mu$, $f_\alpha : [A_{\alpha+n}]^n \rightarrow \mathcal{L}^\Gamma$ ($\bar{a} \mapsto f_\alpha(\bar{a})$) where $f_\alpha(\bar{a}) : \Gamma \rightarrow \mathcal{L}$ ($p \mapsto (f_\alpha(\bar{a}))(p) := \phi_{\bar{a},p} \in p$) such that $N_\alpha \models \neg \phi_{\bar{a},p}(t(\bar{a}))$ such a $\phi_{\bar{a},p}$ exists since N_α omits p .

Now $|A_{\alpha+n}| > \beth_{\alpha+n}$ and for $\alpha \geq 3$, $\beth_\alpha \geq |\mathcal{L}^\Gamma|$.

By Erdős-Rado Theorem, $(\beth_{\alpha+n})^+ \rightarrow (\beth_\alpha^+)^n_{|\mathcal{L}^\Gamma|}$. Thus we obtain $B_\alpha \subset A_{\alpha+n}$ and $\phi_{\alpha,p} \in \mathcal{L}$ such that

1. $|B_\alpha| > \beth_\alpha$,
2. for all $\bar{b} \in [B_\alpha]^n$, $N_\alpha \models \neg \phi_{\alpha,p}(t(\bar{b}))$.

Namely, for all $\bar{b} \in [B_\alpha]^n$, $f_\alpha(\bar{b}) = \text{constant}$.

As $\mu = (2^\omega)^+$, by Erdős-Rado, there is subsequence $\{M'_\alpha : \alpha < \mu\}$ of $\{N_\alpha : \alpha < \mu\}$ such that for all $\bar{b} \in [B'_\alpha]^n$ and $p \in \Gamma$, $(f_\alpha(\bar{b}))(p) = \text{constant}$. Thus $\{(M'_\alpha, B'_\alpha) : \alpha < \mu\}$ and $\phi_p := (f_\alpha(\bar{b}))(p)$ are required.

Claim 2. Fix a $\mathcal{L}$ -formula $\psi(x_1, \dots, x_n)$. There is subsequence F' of F as following property: for any $(M'_\alpha, B_\alpha) \in F'$ if $i_1 < \dots < i_n < \mu$, $j_1 < \dots < j_n < \mu$ and $b_{i_k}, b_{j_k} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \dots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \dots, b_{j_n}).$$

Proof of Claim 2. Define, for all $\alpha < \mu$, $h_\alpha : [A_\alpha]^n \rightarrow 2$ as follows:

$$h_\alpha(\bar{a}) = \begin{cases} 0 & \text{if } M_\alpha \models \psi(\bar{a}), \\ 1 & \text{otherwise.} \end{cases}$$

By Erdős-Rado theorem, there is $B_\alpha \subset A_\alpha$ such that $|B_\alpha| > \beth_\alpha$ and for all $\bar{b} \in [B_\alpha]^n$, $h_\alpha(\bar{b}) = \text{constant}$. Thus, $\{(M_\alpha, B_\alpha) : \alpha < \mu\}$ is required.

Let $\{t_i : i < \omega, t_i \text{ is a term of } \mathcal{L}\}$ and $\{\psi_i : i < \omega, \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$ be enumerations of all the terms of $\mathcal{L}$ and all the $\mathcal{L}$ -formula, respectively. Now we construct Φ by induction on $i < \omega$. Suppose $F_0 := \{(M_\alpha, M_\alpha) : \alpha < \mu\}$ and $\Phi_0 := T \cup \{c_i \neq c_j : i < j < \omega\}$. Clearly, for any $(M_\alpha, M_\alpha) \in F_0$, $M_\alpha \models \Phi_0$ and $|M_\alpha| > \beth_\alpha$.

Case 1 ($i < \omega$ is even). Assume we have found F_i and Φ_i . We take new term $t(x_1, \dots, x_n) \in \{t_i : i < \omega, t_i \text{ is a term of } \mathcal{L}\}$, by claim 1, there is subsequence F_{i+1} of F_i as following property: for each $p \in \Gamma$, there is a $\phi_p \in p$ such that for any $(M'_\alpha, B_\alpha) \in F_{i+1}$, if $i_1 < \dots < i_n < \omega$ and $b_{i_j} \in B_\alpha$,

$$M'_\alpha \models \neg \phi_p(t(b_{i_1}, \dots, b_{i_n})).$$

We put $\Phi_{i+1} = \Phi_i \cup \{\neg \phi_p(t(c_{i_1}, \dots, c_{i_n})) : p \in \Gamma, i_1 < \dots < i_n < \omega\}$.

Case 2 ($i < \kappa$ is odd). Assume we have found F_i and Φ_i . We take new formula $\psi(x_1, \dots, x_n) \in \{\psi_i : i < \omega, \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$, by claim 2, there is subsequence F_{i+1} of F_i as following property: for any $(M'_\alpha, B_\alpha) \in F_{i+1}$ if $i_1 < \dots < i_n < \mu$, $j_1 < \dots < j_n < \mu$ and $b_{i_k}, b_{j_k} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \dots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \dots, b_{j_n}).$$

We put $\Phi_{i+1} := \Phi_i \cup \{\psi(c_{i_1}, \dots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \dots, c_{j_n}) : i_1 < \dots < i_n < \omega, j_1 < \dots < j_n < \omega\}$.

If put $\Phi := \bigcup_{i < \omega} \Phi_i$ then it is required $\mathcal{L}^*$ -theory. We take any $(M_\alpha, A_\alpha) \in F := \bigcap_{i < \omega} F_i$. By construction $M_\alpha \models \Phi$.

Let A be the set of all interpretation $C = \{c_i : i < \omega\}$ in M_α , N be Skolem closure of A in M_α . Thus N is model of T , omitting Γ , indiscernibles in M_α , and $|N| = \omega$.

Take any $\lambda \geq \omega$. By stretching theorem, there is a model of T which the cardinality of λ such that omitting Γ . Note that if $|\Gamma| \leq \omega$ then it is sufficient $\mu = \omega_1$, see [2].

It is known that Morley's theorem is proved in infinitary logic, and it is effective means to show existence of models in infinitary logic that the compactness theorem is false generally, see [1], [3].

3 Related Result

The following result is related to Morley's omitting type theorem. This theorem says the thing that is stronger than Morley's theorem under a certain condition.

Theorem.(Tsuboi) Let T be a countable complete $\mathcal{L}$ -theory and Γ a set of complete types with $|\Gamma| < 2^\omega$. Suppose that for each $\alpha < \omega_1$, there is a model $M_\alpha \models T$ with the following properties:

1. $|M_\alpha| > \beth_\alpha$,
2. M_α omits each member of Γ .

Then for each $\lambda \geq \omega$ there is a model N omitting Γ and with $|N| = \lambda$.

Proof. Let $X = \omega_1$ and $\{I_i : i \in X\}$ be a set of infinite indiscernible sequences and $\{t_n; n < \omega\}$ be an enumeration of all the $\mathcal{L}$ -terms. We may assume that t_n has n -variables. We will say that the set $\{I_i : i \in X\}$ is t_n -uniform if the following condition holds: If $i, j \in X$, then $tp(t_n(I_i)) = tp(t_n(I_j))$ where $tp(t_n(I_i)) := tp(t_n(a_0, \dots, a_{n-1}))$ ($a_0 < \dots < a_{n-1} \in I_i$). We will say that $\{I_i : i \in X\}$ is essentially t_n -uniform if there is an uncountable subset Y of X such that $\{I_i : i \in Y\}$ is t_n -uniform. For a formula $\phi(x)$, define $X^{\phi, t_n} := \{i \in X : \phi(x) \in tp(t_n(I_i))\}$. Put $X_\emptyset = \omega_1$, and for each $i \in X_\emptyset$ we fix a sequence $I_\emptyset(i)$ enumerating the universe M_i .

Using the argument in the paper([4]), for $\eta \in 2^{<\omega}$ and $k < \omega$, we can inductively choose $X_\eta \subset \omega_1$, $\{I_\eta(i) : i \in X_\eta\}$ and formulas $\phi_{\eta, k}$ with the following properties:

1. If $\eta < \nu$, then
 - (a) X_ν is an uncountable subset of X_η ;
 - (b) $I_\nu(i)$ is a subsequence of $I_\eta(i)$ for each $i \in X_\nu$.
2. $i < j \Rightarrow |I_\eta(i)| < |I_\eta(j)|$, and $\sup\{|I_\eta(i)| : i \in X_\eta\} \geq \beth_{\omega_1}$.

3. If $\eta \in 2^n$ then

- (a) each $I_\eta(i)$ is an infinite indiscernible sequence;
- (b) $\{I_\eta(i) : i \in X_\eta\}$: essentially t_n -uniform $\Rightarrow$ it is t_n -uniform.

4. If $\eta \in 2^n$ and $k \leq n$ then

$$\{I_\eta(i) : i \in X_\eta\} : \text{not } t_n\text{-uniform} \\ \Rightarrow X_{\eta \smallfrown 0} \subset (X_\eta)^{(\phi_{\eta,k}), t_k} \text{ and } X_{\eta \smallfrown 1} \subset (X_\eta)^{(\neg \phi_{\eta,k}), t_k}.$$

For all $\nu \in 2^\omega$, we define the following:

- 1. K_ν is the set of all $n < \omega$ such that $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is not t_n -uniform;
- 2. for $n \in K_\nu$, $\Delta_\nu^n(x) := \bigcup_{n \leq m < \omega} \{\phi_{\nu|m,n}(x) : \nu(m) = 0\} \cup \bigcup_{n \leq m < \omega} \{\neg \phi_{\nu|m,n}(x) : \nu(m) = 1\}$;
- 3. $\Phi_\nu := \{\{x_i\}_{i < \omega} \text{ is indiscernible}\} \cup \bigcup_{n \in K_\nu} \Delta_\nu^n(t_n(\bar{x}_n)) \cup \bigcup_{n \notin K_\nu} p_{\nu|n}(t_n(\bar{x}_n))$;
- 4. $F_\nu := \{(M_i^\nu, I_\nu(i)) : i \in X_\nu\} \quad (I_\nu(i) \subset M_i^\nu)$.

$$2^\omega \left\{ \begin{array}{l} \overbrace{F_{\langle 0 \dots \rangle} : (M_0^{(0 \dots)}, I_{\langle 0 \dots \rangle}(0)), \dots, (M_i^{(0 \dots)}, I_{\langle 0 \dots \rangle}(i)), \dots}^{\omega_1} \models \Phi_{\langle 0 \dots \rangle} \\ \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ F_\nu \quad : (M_0^\nu, I_\nu(0)), \dots, (M_i^\nu, I_\nu(i)), \dots \models \Phi_\nu \\ \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ F_{\langle 1 \dots \rangle} : (M_0^{(1 \dots)}, I_{\langle 1 \dots \rangle}(0)), \dots, (M_i^{(1 \dots)}, I_{\langle 1 \dots \rangle}(i)), \dots \models \Phi_{\langle 1 \dots \rangle} \end{array} \right.$$

We can take $\nu \in 2^\omega$ well, see [4], such that if $\{c_i : i < \omega\}$ realizing Φ_ν in M_0^ν , and N is Skolem closure of $\{c_i : i < \omega\}$ in M_0^ν then N omits Γ . The rest of the statement is clear from Stretching Theorem.

References

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