

射影空間内の点の配置とtheta関数

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In this note, we announce a structure theorem on the graded ring of modular forms on the bounded symmetric domain of type $I_{2,2}$ with respect to the principal congruence subgroup of level $(1+i)$. Relations between the period map of certain K3 surfaces, the hypergeometric functions and the present theorem have been studied in [MSY2].

We first explain the classical model, which prepares the understanding of our case. Theta functions $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau)$ on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$ are defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau) := \sum_{n \in \mathbb{Z}} e[(n + \frac{1}{2}a)^2 \tau + b n],$$

where $\tau \in \mathbb{H}$, $a, b \in \mathbb{Z}$ and $e[x] = \exp(\pi i x)$. These functions have the following properties:

$$(i) \quad \vartheta \begin{bmatrix} a+r \\ b+s \end{bmatrix}(\tau) = \exp(\frac{1}{2}\pi i b \cdot r) \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau), \text{ where } r, s \in 2\mathbb{Z},$$

$$(ii) \quad \text{if } a \cdot b \notin 2\mathbb{Z}, \text{ then } \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau) \text{ vanishes.}$$

By the properties (i) and (ii), it is enough to consider only three theta functions $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau)$, where $a, b \in \{0, 1\}$ and $a \cdot b = 0$. The group $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ by

$$g \cdot \tau = (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathbb{H}, \quad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in SL(2, \mathbb{R}).$$

Let $\Gamma(2)$ be the congruence subgroup of level 2:

$$\Gamma(2) = \{g \in SL(2, \mathbb{Z}) \mid g \equiv I_2 \pmod{2}\}.$$

Definition. A holomorphic function f on $\mathbb{H}$ is called a modular form of weight k relative to $\Gamma(2)$, if the following condition is satisfied:

$$f(g \cdot \tau) = (\det(C\tau + D))^k f(\tau), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(2).$$

Let $\text{Mod}_k(2)$ denote the vector space spanned by modular forms of weight k relative to $\Gamma(2)$ and let

$$\text{Mod}(2) := \bigoplus_{k \geq 0} \text{Mod}_k(2)$$

be the graded ring of modular forms relative to $\Gamma(2)$. It is well known that

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\tau)^4 \in \text{Mod}_2(2),$$

where $a, b \in (0, 1)$ and $a \cdot b = 0$, and that these functions satisfy the following relation:

$$\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau)^4 - \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)^4 + \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau)^4 = 0.$$

Let $\overline{\Gamma(2) \backslash \mathbb{H}}$ be the compactification of the quotient space $\Gamma(2) \backslash \mathbb{H}$. We define a holomorphic map ϕ of $\overline{\Gamma(2) \backslash \mathbb{H}}$ into the complex projective plane $\mathbb{P}^2$ by

$$\phi: \tau \mapsto [\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau)^4, \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)^4, \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(\tau)^4].$$

By the above relation, the image of ϕ is included in a line in $\mathbb{P}^2$, which will be denoted by X . Moreover, the map ϕ gives an isomorphism between $\overline{\Gamma(2) \backslash \mathbb{H}}$ and $X (\simeq \mathbb{P}^1)$. This fact implies that the graded ring $\text{Mod}(2)$ is generated by $\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)^4$ and $\vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(\tau)^4$.

Next we explain our case. A classical bounded symmetric domain of type $I_{2,2}$ is defined by

$$D := \{ W = (w_{jk}) \mid 1 \leq j, k \leq 2 \mid \frac{W - W^*}{2i} > 0 \}, \text{ where } W^* = {}^t \overline{W}.$$

The group $\text{Aut}(D)$ of automorphisms of D is generated by $U(2,2)$ and T , where

$$T(W) = {}^t W,$$

$$U(2,2) = \{g \in GL(4, \mathbb{C}) \mid g^* J g = J, J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}\},$$

which acts on D by

$$(g \cdot T^j) \cdot W = \{A(T^j \cdot W) + B\} \{C(T^j \cdot W) + D\}^{-1},$$

where $W \in D$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2,2)$, $j \in \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$; hence we have

$$\text{Aut}(D) \simeq U(2,2) \rtimes \langle T \rangle, \quad \langle T \rangle = \{\text{id}, T\}.$$

We define theta functions on D as follows:

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W) := \sum_{n \in \mathbb{Z}[i]^2} e^{[(n + \frac{1}{1+i}a)^* W (n + \frac{1}{1+i}a) + 2\text{Re}(\langle \frac{1}{1+i}b \rangle^* n)]},$$

where $W \in D$ and $a, b \in \mathbb{Z}[i]^2$. These functions have the following properties:

- (i) $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W) = \Theta \begin{bmatrix} a \\ b \end{bmatrix} ({}^t W),$
- (ii) $\Theta \begin{bmatrix} \delta a \\ \varepsilon b \end{bmatrix} (W) = \Theta \begin{bmatrix} a \\ b \end{bmatrix} (W),$ where δ and ε are units of $\mathbb{Z}[i]$,
- (iii) $\Theta \begin{bmatrix} a+r \\ b+s \end{bmatrix} (W) = \exp(\pi i \text{Re}({}^t b \cdot r)) \Theta \begin{bmatrix} a \\ b \end{bmatrix} (W),$ where $r, s \in (1+i)\mathbb{Z}[i]^2$,
- (iv) if ${}^t a \cdot b \notin (1+i)\mathbb{Z}[i]$, then $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)$ vanishes.

By the properties (ii), (iii) and (iv), it is enough to consider only ten theta functions $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)$, where $a, b \in \{0,1\}^2$ and ${}^t a \cdot b = 0$. Let $\Gamma(1+i)$ be the congruence subgroup

$$\Gamma(1+i) := \{g \in GL(4, \mathbb{Z}[i]) \mid g^* J g = J, g \equiv I_4 \pmod{(1+i)}\},$$

of level $(1+i)$, and let

$$\Gamma_T(1+i) := \Gamma(1+i) \rtimes \langle T \rangle \subset \text{Aut}(D).$$

Definition. A holomorphic function f on D is called a hermitian

modular form of weight $2k$ relative to $\Gamma_T(1+i)$ (with the character $\det: \Gamma(1+i) \rightarrow$ the group of units of $\mathbb{Z}[i]$), if the following conditions are satisfied:

$$(i) \quad f(T \cdot W) = f(W),$$

$$(ii) \quad f(g \cdot W) = \{\det(g)\}^k \{\det(CW+D)\}^{2k} f(W), \quad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma(1+i).$$

Let $\text{Mod}_{2k}(1+i)$ denote the vector space spanned by hermitian modular forms of weight $2k$ relative to $\Gamma_T(1+i)$ and let

$$\text{Mod}(1+i) := \bigoplus_{k \geq 0} \text{Mod}_{2k}(1+i)$$

be the graded ring of hermitian modular forms relative to $\Gamma_T(1+i)$.

Proposition 1. The ten functions $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)^2$ ($a, b \in \{0,1\}^2$, $t_{a \cdot b} = 0$) are hermitian modular forms of weight 2 relative to $\Gamma_T(1+i)$.

Proposition 1 is essentially proved in [Fr].

Proposition 2. Theta functions $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)$ ($a, b \in \{0,1\}^2$, $t_{a \cdot b} = 0$) satisfy the following relations:

$$t_{a \cdot b} \sum_{c, d \in \mathbb{Z}} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)^2 e^{[t_{ca} + t_{db}]} = 0, \quad c, d \in \{0,1\}^2, \quad t_{c \cdot d} = 1.$$

Remark 3. The proposition gives six linear relations between the theta functions $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)^2$; five relations among the six are linearly independent.

Let $\overline{\Gamma_T(1+i) \backslash \mathbb{D}}$ be the Satake compactification of the quotient space $\Gamma_T(1+i) \backslash \mathbb{D}$. We define a holomorphic map $\Phi: \overline{\Gamma_T(1+i) \backslash \mathbb{D}} \rightarrow \mathbb{P}^9$ by

$$\Phi: W \mapsto [\dots, \Theta \begin{bmatrix} a \\ b \end{bmatrix} (W)^2, \dots].$$

By Remark 3, it is clear that the image of Φ is included in a 4-dimensional linear subspace of $\mathbb{P}^9$, which will be denoted by Y . Now we state the main result of the present paper:

Theorem 4. The map Φ gives an isomorphism between $\overline{\Gamma_T(1+i)} \setminus D$ and $Y (\simeq \mathbb{P}^4)$.

Corollary 5. Any five linearly independent theta functions $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (w)^2$ are free generators of the graded ring $\text{Mod}(1+i)$.

Remark 6. As the isomorphism $\phi: \overline{\Gamma(2)} \setminus \mathbb{H} \rightarrow X (\simeq \mathbb{P}^1)$ connects the analytic moduli and the algebraic moduli of the family of elliptic curves, the isomorphism $\Phi: \overline{\Gamma_T(1+i)} \setminus D \rightarrow Y (\simeq \mathbb{P}^4)$ connects the analytic moduli and the algebraic moduli of a 4-dimensional family of K3 surfaces, which are double cover of $\mathbb{P}^2$ branching along 6 lines. For more details, see [MSY2] and [Ma2].

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