

SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS OF SURFACES AND SECONDARY INVARIANTS

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1. INTRODUCTION

Let Σ_g be a closed oriented surface of genus $g \geq 1$ and $\mathcal{M}_g$ its mapping class group consisting of the isotopy classes of orientation preserving diffeomorphisms of Σ_g . We denote the 2-sphere with 3-holes by P . For any $a, b \in \mathcal{M}_g$, let $N_{a,b}$ be the Σ_g -bundle over P with monodromies a^{-1} and b^{-1} .

Meyer's signature 2-cocycle

$$\text{sign}_g: \mathcal{M}_g \times \mathcal{M}_g \rightarrow \mathbb{Z}$$

is defined by $\text{sign}_g(a, b) = \text{sign}(N_{a,b})$, where $\text{sign}(N_{a,b})$ is the signature of 4-manifold $N_{a,b}$ (see [10, 1]). Novikov additivity for the signature of manifolds shows that sign_g satisfies the cocycle condition. Meyer also defined a 2-cocycle τ_g on $Sp(2g, \mathbb{Z})$ over $\mathbb{Z}$, which is also called signature 2-cocycle. It is well-known that the equality $\text{sign}_g = \zeta_g^* \tau_g$ holds, where ζ_g is the standard representation of $\mathcal{M}_g$ to $Sp(2g, \mathbb{Z})$ induced from the obvious action of $\mathcal{M}_g$ on the first cohomology group of Σ_g .

Let ι be the hyperelliptic involution on Σ_g depicted in Figure 1.

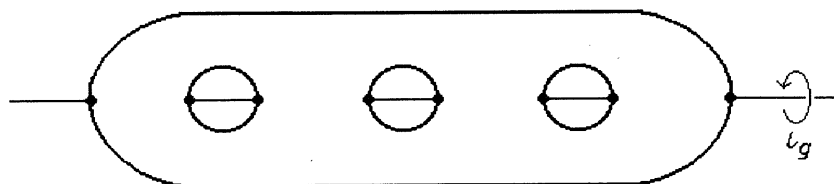


FIGURE 1. The hyperelliptic involution ι on Σ_g .

The hyperelliptic mapping class group $\mathcal{H}_g$ of Σ_g is the subgroup of $\mathcal{M}_g$ consisting of elements which commute with the class of ι . It is known that $\mathcal{M}_1 = \mathcal{H}_1 = SL(2, \mathbb{Z})$, $\mathcal{M}_2 = \mathcal{H}_2$ and that $\mathcal{H}_g (g \geq 3)$ is a subgroup of $\mathcal{M}_g$ of infinite index.

Meyer's signature cocycle $sign_g$ defines a nontrivial class of the second cohomology group of $\mathcal{M}_g$ with coefficients in $\mathbb{Z}$ and its restriction to $\mathcal{H}_g$ is also nontrivial. But it is trivial in the cohomology group of $\mathcal{H}_g$ with coefficients in $\mathbb{Q}$. Thus there exists a function or a 1-cochain

$$\phi_g: \mathcal{H}_g \rightarrow \mathbb{Q}$$

such that $sign_g = \delta\phi_g$, where δ denotes the coboundary operator defined by $\delta\phi_g(a, b) = \phi_g(b) - \phi_g(ab) + \phi_g(a)$ for $a, b \in \mathcal{H}_g$. It follows that ϕ_g is unique from the fact that the first cohomology group of $\mathcal{H}_g$ vanishes. This function ϕ_g is called Meyer function. It is known that it is conjugacy invariant. Its values are contained in $\frac{1}{2g+1}\mathbb{Z}$ and concrete values on Lickorish generators and BSCC maps are calculated by Endo [4], Matsumoto [9] and Morifuji [11].

In the case of $g = 1$, under the identification $\mathcal{M}_1 \cong \mathcal{H}_1 \cong SL(2, \mathbb{Z})$, Meyer [10] and Atiyah [1] gave the explicit expression of the Meyer function using the Dedekind sums (see also [7]). Thus we can compute the values of it. Moreover Atiyah [1] put various geometric interpretations on the values of ϕ_1 on hyperbolic elements. Hereafter we regard $SL(2, \mathbb{Z}) (= Sp(2, \mathbb{Z}))$ as the domain of ϕ_1 . Hence we have $\delta\phi_1 = \tau_1$.

In this paper we study some representations induced from the actions of subgroups of the mapping class groups of a surface on the first cohomology group of $\pi_1(\Sigma_g)$ with coefficients in the module obtained from the nontrivial representation of $\pi_1(\Sigma_g)$ to $\mathbb{Z}_2 = Aut(\mathbb{Z})$. As an application of them, in the case of $g = 1, 2$ (see also [5, 6]) and 3, we define some functions on subgroups of $\mathcal{H}_g$ using Atiyah-Patodi-Singer ρ -invariants and state that the difference of our function from the Meyer function is a nontrivial homomorphism on the subgroup. Moreover we state that the Meyer function coincides with the average of our functions on a certain subgroup.

2. SOME REPRESENTATIONS OF SUBGROUPS OF THE MAPPING CLASS GROUPS

Let Σ_g be a closed oriented surface of genus $g \geq 1$ and $*$ $\in \Sigma_g$ a base point. Let $\omega: \pi_1(\Sigma_g, *) \rightarrow \mathbb{Z}_2$ be a nontrivial homomorphism which is also regarded as an element of $H^1(\Sigma_g; \mathbb{Z}_2)$. If we regard $\mathbb{Z}_2$ as $Aut(\mathbb{Z})$, then using ω , we can obtain

$\pi_1(\Sigma_g, *)$ -module $\mathbb{Z}$, which is denoted by $\mathbb{Z}_\omega$. We consider the first cohomology group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)$ which is isomorphic to $\mathbb{Z}^{2(g-1)} \oplus \mathbb{Z}_2$. Moreover it has a natural pairing defined by the cup product, the pairing $\mathbb{Z}_\omega \otimes \mathbb{Z}_\omega \cong \mathbb{Z}$ and the evaluation on the fundamental class of Σ_g . It is found that this pairing induces a symplectic form on the quotient group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/\text{torsion}$ and that it is isomorphic to the standard one on $\mathbb{Z}^{2(g-1)}$.

Let $\mathcal{M}_{g*}$ be the mapping class group of Σ_g with a base point and $\mathcal{M}_{g*}^\omega$ the subgroup of it consisting of elements which preserve ω . This subgroup acts on the group $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/\text{torsion}$ by pullback. Since this action preserves the symplectic form, if we take a symplectic basis for it, we have the representation

$$\zeta_{g*}^\omega: \mathcal{M}_{g*}^\omega \rightarrow Sp(2(g-1), \mathbb{Z}).$$

These representations are related to prym representations of Looijenga [8]. Some properties of ζ_{g*}^ω were investigated in [5, 6].

In this section we study the restrictions of them to subgroups of the hyperelliptic mapping class group of genus $g \geq 3$.

The hyperelliptic mapping class group $\mathcal{H}_g$ of Σ_g is naturally isomorphic to the group of isotopy classes of orientation preserving diffeomorphisms which commute with ι under isotopy which also commutes with ι [3]. This description of $\mathcal{H}_g$ shows that it acts the set of the fixed points of ι . Thus we have the representation $\sigma: \mathcal{H}_g \rightarrow \mathfrak{S}_{2g+2}$, where $\mathfrak{S}_{2g+2}$ denotes the symmetric group of degree $2g+2$ which is the number of the fixed points of ι . Let $\mathcal{H}_g^\sigma$ be the kernel of the representation of σ . Let $j: \mathcal{M}_{g*} \rightarrow \mathcal{M}_g$ be the natural homomorphism, then we have the short exact sequence $1 \rightarrow \pi_1(\Sigma_g, *) \rightarrow \mathcal{M}_{g*} \xrightarrow{j} \mathcal{M}_g \rightarrow 1$. Put $\mathcal{H}_{g*} = j^{-1}(\mathcal{H}_g)$ and $\mathcal{H}_{g*}^\sigma = j^{-1}(\mathcal{H}_g^\sigma)$. The following lemma is known.

Lemma 1. *For any $a \in \mathcal{H}_g^\sigma$, the induced homomorphism a^* on $H^1(\Sigma_g; \mathbb{Z}_2)$ is the identity.*

By this lemma, we have $\mathcal{H}_g^\sigma \subset \mathcal{M}_g^\omega$ and $\mathcal{H}_{g*}^\sigma \subset \mathcal{M}_{g*}^\omega$ for any $\omega \neq 0 \in H^1(\Sigma_g; \mathbb{Z}_2)$.

We denote the class of ι in $\mathcal{H}_g^\sigma$ by the same letter ι .

Lemma 2. For any lift $\tilde{\iota}$ of $\iota \in \mathcal{H}_g^\sigma$ to $\mathcal{H}_{g*}^\sigma$, the image of $\tilde{\iota}$ by ζ_{g*}^ω commutes with those of all elements of $\mathcal{H}_{g*}^\sigma$.

The fundamental group $\pi_1(\Sigma_g, *)$ of Σ_g is presented by $\langle \alpha_i, \beta_i \mid 1 \leq i \leq g \mid \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle$, where the generators are depicted in Figure 2.

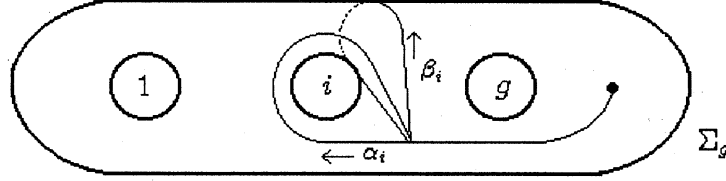


FIGURE 2. The generators of $\pi_1(\Sigma_g, *)$.

Let α_i^*, β_i^* ($1 \leq i \leq g$) be the dual basis for $H^1(\Sigma_g; \mathbb{Z}_2)$ to the one for $H_1(\Sigma_g; \mathbb{Z}_2)$ which is given by the homology classes of α_i, β_i .

Lemma 3. For any nonzero class $\omega \in H^1(\Sigma_g; \mathbb{Z}_2)$, there exists $a \in \mathcal{H}_g$ such that $a^*\omega = \alpha_k^*$ for some k .

Direct computations show that the representation matrix of $\zeta_{g*}^{\alpha_k^*}(\tilde{\iota})$ with respect to a symplectic basis for $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/\text{torsion}$ is given by $\pm(I_{2(k-1)} \oplus (-I_{2(g-k)}))$, where $I_{2(k-1)}$ and $I_{2(g-k)}$ are the identity matrices of rank $2(k-1)$ and $2(g-k)$ respectively. And $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/\text{torsion}$ decomposes to the direct sum of two symplectic submodules over $\mathbb{Z}$. This result and Lemma 2 imply the following lemma.

Lemma 4. For any nonzero $\omega \in H^1(\Sigma_g; \mathbb{Z}_2)$, the representation matrix of $\zeta_{g*}^\omega(\tilde{\iota})$ with respect to some symplectic basis is $\pm(I_{2(k-1)} \oplus (-I_{2(g-k)}))$ for some k . Moreover $H^1(\pi_1(\Sigma_g, *), \mathbb{Z}_\omega)/\text{torsion}$ is decomposed to the direct sum of two symplectic submodules over $\mathbb{Z}$ on which $\zeta_{g*}^\omega(\tilde{\iota})$ is $\pm$ the identity.

If we take a fixed point e of ι as a base point $*$ of Σ_g , we can consider the group $\mathcal{H}_g^\sigma$ as a subgroup of $\mathcal{H}_{g*}^\sigma$.

Corollary 5. The representation $\zeta_{g_e}^\omega$ induces two representations of $\mathcal{H}_g^\sigma$ to $Sp(2(k-1), \mathbb{Z})$ and $Sp(2(g-k), \mathbb{Z})$, where k is the integer in Lemma 3.

3. SOME FUNCTIONS ON SUBGROUPS OF $\mathcal{H}_{g*}$ OF LOW GENUS.

In this section we consider the case of $g = 1, 2$ and 3 .

Let H' be the set $H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}$ for $g = 1, 2$ and the set $\{\omega \in H^1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\} \mid k = 2 \text{ in Lemma 4}\}$ for $g = 3$.

Lemma 6. *The number $\sharp H'$ of the elements of H' is 3, 15 and 35 for $g = 1, 2$ and 3 respectively.*

For each $\omega \in H'$, let Δ_{g*}^ω denote $\mathcal{H}_{g*} \cap \mathcal{M}_{g*}^\omega$ for $g = 1, 2$ and $\mathcal{H}_{g*}^\sigma$ for $g = 3$. For any $\omega \in H'$, the image of Δ_{g*}^ω by ζ_{g*}^ω is contained in $\{id\}$, $SL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ for $g = 1, 2$ and 3 respectively under an appropriate choice of a symplectic basis for the representation space. In the case of $g = 3$, let $\zeta_{g*}^{\omega+}$ and $\zeta_{g*}^{\omega-}$ be the composition of ζ_{g*}^ω with the projection from $SL(2, \mathbb{Z})$ to the first and second factor $SL(2, \mathbb{Z})$ respectively.

For each $\omega \in H'$, the function

$$\Phi_{g*}^\omega: \Delta_{g*}^\omega \rightarrow \frac{1}{3}\mathbb{Z}$$

is defined by $0, (\zeta_{2*}^\omega)^*\phi_1$ and $(\zeta_{3*}^{\omega+})^*\phi_1 + (\zeta_{3*}^{\omega-})^*\phi_1$ for $g = 1, 2$ and 3 respectively. It is easy to see that these functions are well defined.

Lemma 7. *The equality $\delta\Phi_{g*}^\omega = (\zeta_{g*}^\omega)^*\tau_{g-1}$ holds on Δ_{g*}^ω for each $\omega \in H'$.*

4. THE MAIN THEOREM

In this section we define some functions on subgroups of the mapping class groups and state the main theorem.

Let ω be a nonzero class in $H^1(\Sigma_g; \mathbb{Z}_2)$. For any $a \in \mathcal{M}_{g*}^\omega$, put $M_a := \Sigma_g \times [0, 1]/(x, 0) \sim (a(x), 1)$. Then M_a is a Σ_g -bundle over $S^1 = [0, 1]/0 \sim 1$ with the identification i of Σ_g with the fiber at $0 \in S^1$ and with the section $s: S^1 \rightarrow M_a$ defined by the base point $*$ of Σ_g . It is easily checked that there is a unique homomorphism $\omega_a: \pi_1(M_a, s(0)) \rightarrow \mathbb{Z}_2 = \{\pm 1\} \subset U(1)$ satisfying the equalities $i^*\omega_a = \omega$ and $s^*\omega_a = 1$. We define the function $\rho_\omega: \mathcal{M}_{g*}^\omega \rightarrow \mathbb{Q}$ by $\rho_\omega(a) := \rho_{\omega_a}(M_a)$ for each $a \in \mathcal{M}_{g*}^\omega$. Here $\rho_{\omega_a}(M_a)$ is the Atiyah-Patodi-Singer ρ -invariant for (M_a, ω_a) .

In general, the Atiyah-Patodi-Singer ρ -invariant is a diffeomorphism invariant for a pair of a closed oriented manifold of odd dimension and a unitary representation of the fundamental group of it to $U(n)$. If a metric on the manifold is given, then the invariant is defined by the difference of the η -invariant of the signature operator on the manifold and n times that of signature operator with coefficients in the flat bundle obtained from the unitary representation. Thus ρ -invariants take their values in $\mathbb{R}$. If a unitary representation factors through a finite group, then the value of the ρ -invariant belongs to $\mathbb{Q}$.

For each $\omega \in H'$, we define a rational valued function μ_{g*}^ω on Δ_{g*}^ω by

$$\mu_{g*}^\omega = \rho_\omega + \Phi_{g*}^\omega.$$

These functions have the following properties.

Lemma 8. *For any $a \in \Delta_{g*}^\omega$ and $f \in \mathcal{H}_{g*}$, the following hold.*

1. $\mu_{g*}^\omega(1) = 0$,
2. $\mu_{g*}^\omega(a^{-1}) = -\mu_{g*}^\omega(a)$,
3. $\mu_{g*}^{(f^{-1})^*\omega}(faf^{-1}) = \mu_{g*}^\omega(a)$,
4. $\text{sign}_g = \delta\mu_{g*}^\omega$ on Δ_{g*}^ω .

The main property in this lemma is 4. In order to prove it, we need the following theorem proved by Atiyah, Patodi and Singer.

Theorem 9 (Atiyah-Patodi-Singer [2]). *Let M be a closed oriented manifold of odd dimension and $\alpha: \pi_1(M) \rightarrow U(n)$ a unitary representation. If M is the boundary of a compact oriented manifold N with α extending to a unitary representation of $\pi_1(N)$ then $\rho_\alpha(M) = n \text{sign}(N) - \text{sign}_\alpha(N)$.*

We consider the Σ_g -bundle $N_{a,b}$ over P , where $a, b \in \mathcal{M}_{g*}^\omega$. There is a unique homomorphism $\omega_{a,b}: \pi_1(N_{a,b}) \rightarrow \mathbb{Z}_2 \subset U(1)$ satisfying the same condition as ω_a . We apply Atiyah-Patodi-Singer's theorem to the pair $(N_{a,b}, \omega_{a,b})$ and use the Leray-Serre spectral sequence of the fibration $N_{a,b} \rightarrow P$. Then we have the property 4 in Lemma 8. Using Lemma 8, it is easy to see that the function μ_{g*}^ω descends to a function μ_g^ω on $\Delta_g^\omega := j(\Delta_{g*}^\omega)$ for any $\omega \in H'$.

Theorem 10. *The difference $\phi_g - \mu_g^\omega$ is a nontrivial homomorphism from Δ_g^ω to $\mathbb{Q}$ for any $\omega \in H'$ and the equality $\phi_g = \frac{1}{\#H'} \sum_{\omega \in H'} \mu_g^\omega$ holds on $\mathcal{H}_g^\sigma$ for $g = 1, 2$ and 3 .*

Since the Meyer function ϕ_g has the same properties as those in Lemma 8, the former part of this theorem follows from Lemma 8 and nontrivial examples which can be given explicitly. The latter follows from Lemma 8 and $H^1(\mathcal{H}_g^\sigma, \mathbb{Q})^{\mathfrak{S}_{2g+2}} = \{0\}$ which is obtained from the fact of $H^1(\mathcal{H}_g, \mathbb{Q}) = \{0\}$ using the Hochschild-Leray-Serre spectral sequence of the short exact sequence $1 \rightarrow \mathcal{H}_g^\sigma \rightarrow \mathcal{H}_g \rightarrow \mathfrak{S}_{2g+2} \rightarrow 1$.

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