

GENERALIZATION OF YOUNG DIAGRAMS AND HOOK FORMULA

KENTO NAKADA
OKAYAMA UNIVERSITY

1. PRELIMINARIES

First, we give several notations for root systes. We always fix a root datum $(A; \mathfrak{h}, \mathfrak{h}^*, \Pi, \Pi^\vee)$:

$A = (a_{i,j})_{i,j \in I}$: a generalized Cartan matrix.

$\mathfrak{h}$: $\mathbb{R}$ -vector space,

$\mathfrak{h}^*$: the dual space of $\mathfrak{h}$,

$\langle, \rangle : \mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{R}$: the canonical bilinear form.

$\Pi := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$: linearly independent subset

$\Pi^\vee := \{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$: linearly independent subset

such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$.

For each $i \in I$, we define the *simple reflection* $s_i \in \text{GL}(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

$$\text{equivalently, } s_i : h \mapsto h - \langle \alpha_i, h \rangle \alpha_i^\vee, \quad h \in \mathfrak{h}.$$

$W := \langle s_i \mid i \in I \rangle$: the Weyl group

We define a (real) root system and a (real) coroot system:

$$\Phi := W\Pi \left(\subseteq \bigoplus_{i \in I} \mathbb{Z}\alpha_i \right) : \text{(real) root system}$$

$$\Phi_+ := \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i : \text{(real) positive root system}$$

$$\Phi_- := \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i : \text{(real) negative root system}$$

$$\Phi = \Phi_+ \amalg \Phi_- \quad (\text{disjoint union})$$

$$\Phi^\vee := W\Pi^\vee \left(\subseteq \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \right) : \text{(real) coroot system}$$

$$\Phi_+^\vee := \Phi^\vee \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee : \text{(real) positive coroot system}$$

$$\Phi_-^\vee := \Phi^\vee \cap \bigoplus_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i^\vee : \text{(real) negative coroot system}$$

$$\Phi^\vee = \Phi_+^\vee \amalg \Phi_-^\vee \quad (\text{disjoint union})$$

For a real root $\beta = w(\alpha_i) \in \Phi$, we define the dual coroot $\beta^\vee \in \Phi^\vee$ of β by:

$$\beta^\vee = w(\alpha_i^\vee).$$

Remark 1. This is independent from the choice of $w \in W$ and $\alpha_i \in \Pi$.

The map $\Phi \ni \beta \mapsto \beta^\vee \in \Phi^\vee$ is a bijection.

For each $\beta \in \Phi$, we define the reflection $s_\beta \in W$ by:

$$\begin{aligned} s_\beta(\lambda) &= \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*, \\ s_\beta(h) &= h - \langle \beta, h \rangle \beta^\vee, \quad h \in \mathfrak{h}. \end{aligned}$$

Definition 1. Let $w \in W$. We define the inversion set $\Phi(w)$ of w by:

$$\Phi(w) := \{ \gamma \in \Phi_+ \mid w^{-1}(\gamma) < 0 \}.$$

Definition 2. Let $w \in W$. We denote by $\text{Red}(w)$ the set of reduced decompositions of w :

$$\text{Red}(w) := \{ s_{i_1} s_{i_2} \cdots s_{i_d} \mid \text{reduced decompositions of } w \}.$$

Definition 3. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if:

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

The set of integral weights is denoted by P .

Definition 4. An integral weight $\lambda \in P$ is said to be *dominant* if:

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} = \mathbb{N}, \quad i \in I.$$

The set of dominant integral weights is denoted by $P_{\geq 0}$.

2. MINUSCULE ELEMENTS AND PETERSON-PROCTOR HOOK FORMULA

Definition 5 (Peterson (see [1])). Let $\Lambda \in P_{\geq 0}$. An element $w \in W$ is said to be Λ -minuscule if there exists a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_d} \in \text{Red}(w)$ of w such that

$$\langle s_{i_{k+1}} \cdots s_{i_d}(\Lambda), \alpha_{i_k}^\vee \rangle = 1, \quad k = 1, 2, \dots, d.$$

Remark 2. This definition is independent from the choice of reduced decompositions of w .

Example 1. A Grassmannian permutation is a Λ -minuscule element in the Weyl group of type A (symmetric group).

Theorem 2.1 (Proctor (see e.g. [7])). Suppose that the underlying generalized Cartan matrix is simply-laced. Then there exists a one-to-one correspondence between $\{(\Lambda, w)\}$ and d -complete posets.

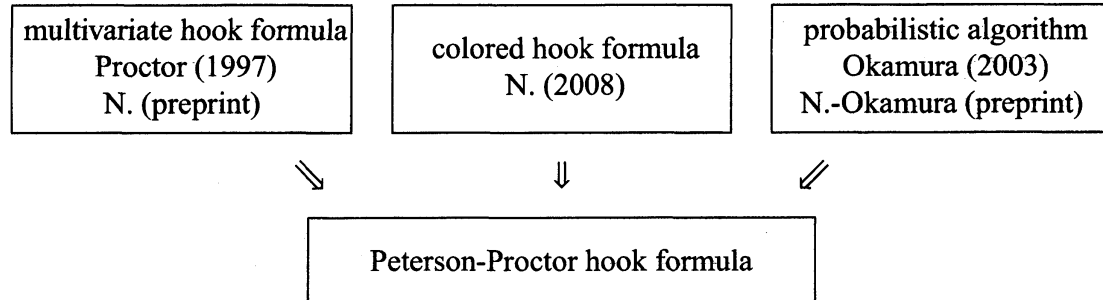
Theorem 2.2 (Peterson-Proctor (see [1])). Let $\Lambda \in P_{\geq 0}$ and $w \in W$ a Λ -minuscule element. Then we have:

$$\#\text{Red}(w) = \frac{\ell(w)!}{\prod_{\beta \in \Phi(w)} \text{ht}(\beta)}.$$

This hook formula is, of course, a generalization of hook length formula for a Young diagram due to Frame-Robinson-Thrall [2], and a shifted Young diagram due to Thrall [9].

In terms of d -complete posets, this counts the number of linear extensions of the d -complete posets.

Now, we have three approaches to prove Peterson-Proctor hook formula.



3. FINITE PREDOMINANT INTEGRAL WEIGHTS

Definition 6. An integral weight $\lambda \in P$ is said to be *pre-dominant* if:

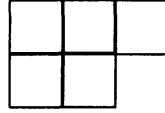
$$\langle \lambda, \beta^\vee \rangle \geq -1, \quad \beta \in \Phi_+.$$

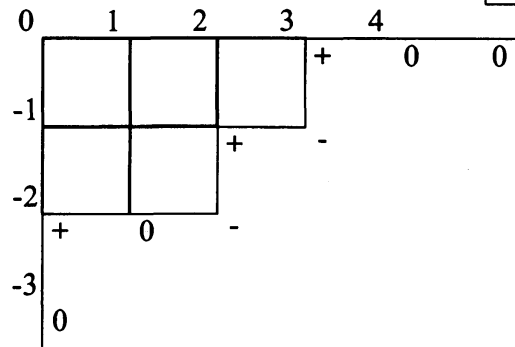
The set of pre-dominant integral weights is denoted by $P_{\geq -1}$.

Definition 7. Let $\lambda \in P_{\geq -1}$. We define a set $D(\lambda)$ by:

$$D(\lambda) := \{ \beta \in \Phi_+ \mid \langle \lambda, \beta^\vee \rangle = -1 \}.$$

The set $D(\lambda)$ is called a *diagram* of λ . A pre-dominant integral weight λ is said to be *finite* if $\#D(\lambda) < \infty$. The set of finite pre-dominant integral weights is denoted by $P_{\geq -1}^{\text{fin}}$.

Example 2. As an example, we consider how Young diagram  is realized as $D(\lambda)$.



According to the above picture, we put $\lambda := 1\Lambda_{-2} + (-1)\Lambda_0 + 1\Lambda_1 + (-1)\Lambda_2 + 1\Lambda_3$, in the root system of type A_6 with index $I = \{-2, -1, 0, 1, 2, 3\}$, where Λ_i denotes i -th fundamental weight. Then we have $\lambda \in P_{\geq -1}^{\text{fin}}$ such that $(D(\lambda); <)$ is order-isomorphic to the original Young diagram.

$$D(\lambda) = \begin{array}{|c|c|c|} \hline \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 & \alpha_0 + \alpha_1 + \alpha_2 & \alpha_2 \\ \hline \alpha_{-1} + \alpha_0 & \alpha_0 & \\ \hline \end{array}$$

Thus, we recover the original Young diagram.

Theorem 3.1. Let $\Lambda \in P_{\geq 0}$ and $w \in W$ a Λ -minuscule element. Then we have $w(\Lambda) \in P_{\geq -1}^{\text{fin}}$. Furthermore, this correspondence is bijective between $P_{\geq -1}^{\text{fin}}$ and the set of such pairs (Λ, w) .

$$\begin{array}{ccc} \{(\Lambda, w)\} & \rightarrow & P_{\geq -1}^{\text{fin}} \\ \downarrow \Psi & & \downarrow \Psi \\ (\Lambda, w) & \mapsto & w(\Lambda) \end{array}$$

Put $\lambda := w(\Lambda)$. Then we have

$$\Phi(w) = D(\lambda).$$

Definition 8. Let $\lambda \in P_{\geq -1}^{\text{fin}}$ and $\beta \in D(\lambda)$. We define a set $H_\lambda(\beta)$ by:

$$H_\lambda(\beta) := \{ \gamma \in D(\lambda) \mid s_\beta(\gamma) < 0 \} = D(\lambda) \cap \Phi(s_\beta).$$

We call the set $H_\lambda(\beta)$ the *hook at β* .

Proposition 3.2. Let $\lambda \in P_{\geq -1}^{\text{fin}}$ and $\beta \in D(\lambda)$. Then we have:

- (1) $\#H_\lambda(\beta) = \text{ht}(\beta)$.
- (2) $s_\beta(\lambda) \in P_{\geq -1}^{\text{fin}}$.
- (3) $D(s_\beta(\lambda)) = s_\beta(D(\lambda) \setminus H_\lambda(\beta))$.

Definition 9. Let $\lambda \in P_{\geq -1}^{\text{fin}}$. A sequence $(\beta_1, \beta_2, \dots, \beta_l)$ ($l \geq 0$) of positive real roots is said to be a λ -path if:

$$\beta_k \in D(s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda)), \quad (k = 1, 2, \dots, l).$$

The set of λ -paths is denoted by $\text{Path}(\lambda)$.

Definition 10. Let $\lambda \in P_{\geq -1}^{\text{fin}}$. A λ -path of maximal length is called a *maximal λ -path*. The set of maximal λ -paths is denoted by $\text{MPath}(\lambda)$.

Note that if $\#D(\lambda) = d$ then length of maximal λ -path is d , and hence that maximal λ -path is of a form $(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_d})$.

Example 3. Back to Example 2, put $\lambda := \Lambda_{-1} - \Lambda_0 + \Lambda_1 - \Lambda_2 + \Lambda_3$. Then we have 5 maximal λ -paths below:

$$\begin{array}{l}
 (\alpha_0, \alpha_{-1}, \alpha_2, \alpha_1, \alpha_0) \cdots \begin{array}{|c|c|c|} \hline 5 & 4 & 3 \\ \hline 2 & 1 & \\ \hline \end{array} \\
 (\alpha_0, \alpha_2, \alpha_{-1}, \alpha_1, \alpha_0) \cdots \begin{array}{|c|c|c|} \hline 5 & 4 & 2 \\ \hline 3 & 1 & \\ \hline \end{array} \\
 (\alpha_2, \alpha_0, \alpha_{-1}, \alpha_1, \alpha_0) \cdots \begin{array}{|c|c|c|} \hline 5 & 4 & 1 \\ \hline 3 & 2 & \\ \hline \end{array} \\
 (\alpha_0, \alpha_2, \alpha_1, \alpha_{-1}, \alpha_0) \cdots \begin{array}{|c|c|c|} \hline 5 & 3 & 2 \\ \hline 4 & 1 & \\ \hline \end{array} \\
 (\alpha_2, \alpha_0, \alpha_1, \alpha_{-1}, \alpha_0) \cdots \begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 4 & 2 & \\ \hline \end{array}
 \end{array}$$

Now we restate the Peterson-Proctor hook formula:

Theorem 3.3. Let $\lambda \in P_{\geq -1}^{\text{fin}}$. Put $d := \#D(\lambda)$. Then we have:

$$\#\text{MPath}(\lambda) = \frac{d!}{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}.$$

We give two of three approaches to prove the above theorem in section 4 and 5.

4. COLORED HOOK FORMULA

Let $\lambda \in P_{\geq -1}^{\text{fn}}$, and put $d = D(\lambda)$. Then we have:

Theorem 4.1 ([4]).

$$\sum_{(\beta_1, \beta_2, \dots, \beta_l) \in \text{Path}(\lambda), l \geq 0} \frac{1}{\beta_1} \frac{1}{\beta_1 + \beta_2} \cdots \frac{1}{\beta_1 + \cdots + \beta_l} = \prod_{\beta \in D(\lambda)} \left(1 + \frac{1}{\beta}\right).$$

Taking the lowest degree, we get:

Corollary 4.2.

$$\sum_{(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_d}) \in \text{MPath}(\lambda)} \frac{1}{\alpha_{i_1}} \frac{1}{\alpha_{i_1} + \alpha_{i_2}} \cdots \frac{1}{\alpha_{i_1} + \cdots + \alpha_{i_d}} = \prod_{\beta \in D(\lambda)} \frac{1}{\beta}.$$

Taking the specialization $\alpha_i \mapsto 1$, we get:

Corollary 4.3 (Peterson-Proctor hook formula).

$$\#\text{MPath}(\lambda) = \frac{d!}{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}.$$

5. PROBABILISTIC ALGORITHM

For simplicity of description, we assume that the underlying root datum is simply-laced. We call the following algorithm the *algorithm A* for Γ :

GNW1.: Set $k := 0$ and set $\lambda_0 := \lambda$.

GNW2.: (Now $D(\lambda_k)$ has $d - k$ roots.) Pick a root $\beta \in D(\lambda_k)$ with the probability $1/(d - k)$.

GNW3.: If $\#H_{\lambda_k}(\beta) - \{\beta\} \neq 0$, then pick a $\gamma \in H_{\lambda_k}(\beta) - \{\beta\}$ with the probability $1/\#(H_{\lambda_k}(\beta) - \{\beta\})$, put $\beta := \gamma$ and repeat GNW3.

GNW4.: (Now $\#(H_{\lambda_k}(\beta) - \{\beta\}) = 0$.) ($\beta = \alpha_{i_k}$.) Set $\alpha_{i_{k+1}} := \alpha_i$ and set $\lambda_{k+1} := s_i(\lambda_k)$.

GNW5.: Set $k := k + 1$. If $k < d$, return to GNW2; if $k = d$, terminate.

Then, by the definition of the algorithm A for λ , the sequence $(\mathcal{B} =)(\alpha_{i_1}, \dots, \alpha_{i_d})$ generated above is a maximal λ -path. We denote by $\text{Prob}_\lambda(\mathcal{B})$ the probability we get $\mathcal{B} \in \text{MPath}(\lambda)$ by the algorithm A. The algorithm A for λ gives a probability measure $\text{Prob}_\lambda()$ over (a finite set) $\text{MPath}(\lambda)$.

Theorem 5.1 (S. Okamura [6], N-S. Okamura [5]). *Let $\mathcal{B} \in \text{MPath}(\lambda)$. Then we have:*

$$(5.1) \quad \text{Prob}_\lambda(\mathcal{B}) = \frac{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}{d!}.$$

Since the right-hand side of (5.1) is independent from the choice of $\mathcal{B} \in \text{MPath}(\lambda)$, the probability measure is uniform. Hence, taking the inverse, we get:

Corollary 5.2 (Peterson-Proctor hook formula).

$$\#\text{MPath}(\lambda) = \frac{d!}{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}.$$

See [3] for Young diagram case due to Greene-Nijenhuis-Wilf, and [8] for shifted Young diagram case due to Sagan.

REFERENCES

- [1] J. B. Carrell, *Vector fields, flag varieties, and Schubert calculus*, Proc. Hyderabad Conference on Algebraic Groups (ed. S. Ramanan), Manoj Prakashan, Madras, 1991.
- [2] J. S. Frame, G. de B. Robinson, and R. M. Thrall, *The hook graphs of symmetric group*, Canad. J. Math. **6** (1954), 316-325.
- [3] C. Greene, A. Nijenhuis, and H. S. Wilf, *A probabilistic proof of a formula for the number of Young tableaux of a given shape*, Adv. in Math. **31** (1979), 104-109.
- [4] K. Nakada, *Colored hook formula for a generalized Young diagram*, Osaka J. of Math. **Vol. 54 No. 4** (2008), 1085-1120.
- [5] K. Nakada, and S. Okamura, *Uniform generation of standard tableaux of a generalized Young diagram*, preprint.
- [6] S. Okamura, *An algorithm which generates a random standard tableau on a generalized Young diagram* (in Japanese), master's thesis, Osaka university, 2003.
- [7] R. A. Proctor, *Dynkin diagram classification of λ -minuscule Bruhat lattices and of d -complete posets*, J. Algebraic Combin. **9** (1999), 61-94.
- [8] B. E. Sagan, *On selecting a random shifted Young tableaux*, J. Algorithm **1** (1980), 213-234.
- [9] R. M. Thrall, *A combinatorial problem*, Mich.Math.J. **1** (1952), 81-88.