

Self-similar solutions of two-point free boundary problem for heat equation

Jong-Shenq Guo

Department of Mathematics, National Taiwan Normal University

and

Yoshihito Kohsaka

(高坂 良史)

Mathematical Institute, Tohoku University

(東北大学大学院理学研究科数学専攻)

1 Introduction and main result

We study the following two-point free boundary problem :

$$\begin{cases} u_t = u_{xx}, & -\xi_1(t) < x < \xi_2(t), \quad t > 0 \\ u_x(-\xi_1(t), t) = \tan(\theta_1 - \beta_1), & u(-\xi_1(t), t) = \xi_1(t) \tan \beta_1, \\ u_x(\xi_2(t), t) = \tan(\beta_2 - \theta_2), & u(\xi_2(t), t) = \xi_2(t) \tan \beta_2, \\ u(x, 0) = u_0(x), & \xi_1(0) = \xi_{01}, \quad \xi_2(0) = \xi_{02}, \end{cases} \quad (1.1)$$

where β_i and θ_i are given constants satisfying $\beta_i \in [0, \pi/2)$ and $\theta_i \in (0, \beta_i + \pi/2)$, $i = 1, 2$, ξ_{01} and ξ_{02} are positive constants, $u_0 \in C^2[-\xi_{01}, \xi_{02}]$ satisfying the compatibility conditions, and $u_0 > 0$ in $(-\xi_{01}, \xi_{02})$. In this problem (u, ξ_1, ξ_2) are unknown functions to be found.

This type of free boundary problem arises in the combustion theory to describe flame propagation. It is motivated by mathematical modeling of combustion in [1, 5]. Note that the prescribed angle condition at each free boundary makes the problem (1.1) different from the Stefan problem. For a detailed overview of more general or different models we refer the reader to the work of Vazquez [5].

The purpose in this talk is to prove the existence of self-similar solutions for the problem (1.1), which is classified by angle conditions, and also to analyze the stability of them.

The problem (1.1) has fundamental properties as follows. Set

$$\begin{aligned} \Gamma(t) &:= \{(x, u(x, t)) \mid -\xi_1(t) < x < \xi_2(t)\} \subset \mathbf{R}^2, \\ \partial\Omega_1 &:= \{(x, z) \mid z = -(\tan \beta_1)x, \quad x \leq 0\}, \quad \partial\Omega_2 := \{(x, z) \mid z = (\tan \beta_2)x, \quad x \geq 0\}. \end{aligned}$$

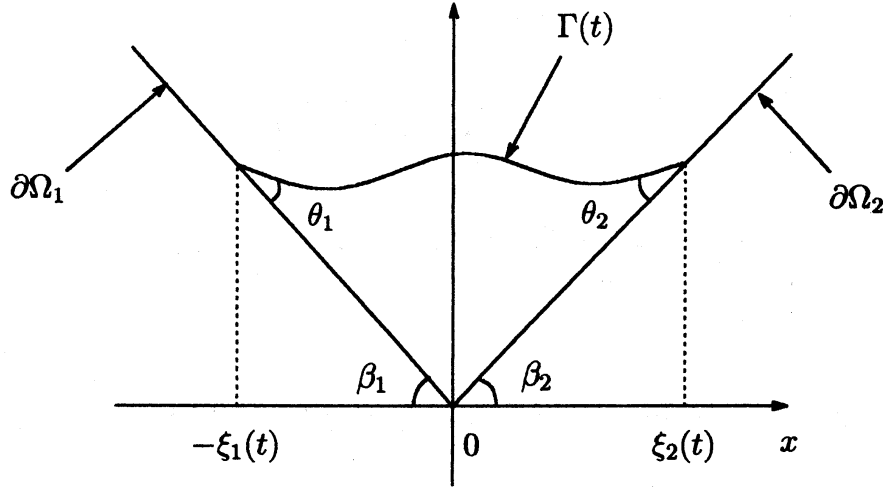


Figure 1: The situation of (1.1)

Let $D(t)$ be the domain enclosed by $\Gamma(t)$, $\partial\Omega_1$, and $\partial\Omega_2$. By a simple calculation, we have

$$\frac{d}{dt}\mu(D(t)) = u_x(\xi_2(t), t) - u_x(-\xi_1(t), t) = \tan(\beta_2 - \theta_2) - \tan(\theta_1 - \beta_1)$$

where $\mu(D)$ is the area of D . This implies that

$$\frac{d}{dt}\mu(D(t)) \begin{cases} > 0 & \text{if } \theta_1 + \theta_2 < \beta_1 + \beta_2, \\ = 0 & \text{if } \theta_1 + \theta_2 = \beta_1 + \beta_2, \\ < 0 & \text{if } \theta_1 + \theta_2 > \beta_1 + \beta_2. \end{cases} \quad (1.2)$$

It is natural to expect that if $\theta_1 + \theta_2 < \beta_1 + \beta_2$, then $\Gamma(t)$ expands with time t ; if $\theta_1 + \theta_2 = \beta_1 + \beta_2$, then $\Gamma(t)$, whose area is preserved in time t , tends to a fixed line as $t \rightarrow \infty$; if $\theta_1 + \theta_2 > \beta_1 + \beta_2$, then $\Gamma(t)$ shrinks with time t and vanishes in a finite time $T = T(u_0, \xi_{01}, \xi_{02})$.

To analyze the asymptotic behavior of $\Gamma(t)$, we define the following.

Definition 1.1 (Self-similar) Let $\rho > 0$ and set

$$u^\rho(x, t) := \rho^{-1}u(\rho(x - x_0) + x_0, \rho^2(t - t_0) + t_0).$$

We say that u is self-similar with the center (x_0, t_0) if $u^\rho(x, t) = u(x, t)$ for any $\rho > 0$.

Note that the problem (1.1) is invariant for the rescaling $u \mapsto u^\rho$. If u is self-similar and is also a solution of (1.1) for some u_0 , ξ_{01} , and ξ_{02} , then we call such u a self-similar solution of (1.1).

There are several references studying self-similar solutions for this type of free boundary problem. For the case $\beta_1 = 0$ and $\beta_2 = \pi/2$, which is one-point free boundary

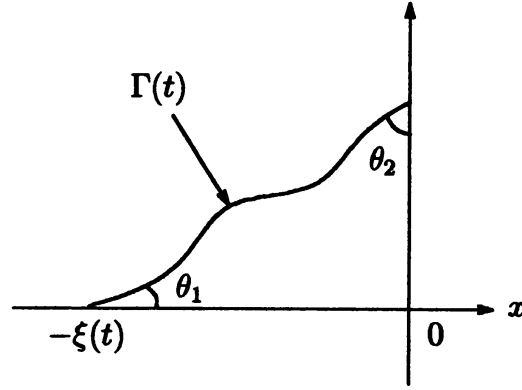


Figure 2: $\beta_1 = 0$ and $\beta_2 = \pi/2$ (see [3, 4])

problem, we refer to [3, 4]. In [4] the author proved the existence and uniqueness of a self-similar solution for a quasilinear parabolic equation $u_t = (a(u_x))_x$ in the case $\theta_1 + \theta_2 < \pi/2$, where $a \in C^2(\mathbb{R})$ and the first derivative of a is positive. The asymptotic stability of this self-similar solution was also obtained. In [3] they considered a focusing problem with $\theta_1 = \pi/4$ and $\theta_2 = \pi/2$. It was proved that a self-similar solution, which vanishes in finite time, exists uniquely and that all solutions are asymptotically equal to this self-similar solution. For the case $\beta_1 = \beta_2 = 0$, we refer to [2]. In [2] they studied in several space dimension and established a theory of existence, uniqueness and regularity for radial symmetric solutions having bounded support. They also investigated the focusing behavior, which is shown to be self-similar, for solutions whose support expands in finite time to fill a hole. We remark that the one-dimensional problem in their model is a special case of our problem.

In order to investigate the existence of self-similar solutions of (1.1), we consider the following problems. Now we set $\alpha_1 := \theta_1 - \beta_1$ and $\alpha_2 := \beta_2 - \theta_2$.

Case $\alpha_1 < \alpha_2$: Analyze *forward self-similar solutions*. That is, for

$$u(x, t) = \sqrt{2t}v(x/\sqrt{2t}), \quad \xi_1(t) = \sqrt{2t}p, \quad \xi_2(t) = \sqrt{2t}q,$$

we study

$$\begin{cases} v'' + \eta v' - v = 0, & -p < \eta < q, \\ v'(-p) = \tan \alpha_1, & v(-p) = p \tan \beta_1, \\ v'(q) = \tan \alpha_2, & v(q) = q \tan \beta_2. \end{cases} \quad (1.3)$$

In the problem (1.3), v, p, q are unknown function and constants to be found.

Case $\alpha_1 = \alpha_2$: Analyze *stationary self-similar solutions*. In this case, a family of the straight lines, namely $u(x) = (\tan \alpha)x + d$ where $\alpha := \alpha_1 = \alpha_2$ and d is any positive constant, is stationary solutions of (1.1).

Case $\alpha_1 > \alpha_2$: Analyze *backward self-similar solutions*. That is, for

$$u(x, t) = \sqrt{-2t}v(x/\sqrt{-2t}), \quad \xi_1(t) = \sqrt{-2t}p, \quad \xi_2(t) = \sqrt{-2t}q,$$

$$\begin{cases} v'' - \eta v' + v = 0, & -p < \eta < q, \\ v'(-p) = \tan \alpha_1, & v(-p) = p \tan \beta_1, \\ v'(q) = \tan \alpha_2, & v(q) = q \tan \beta_2. \end{cases} \quad (1.4)$$

In the problem (1.4), v, p, q are unknown function and constants to be found.

We are ready to state our main results.

Theorem 1.1 *The following hold:*

- (i) *Assume that $\alpha_1 < \alpha_2$. Then there exists a unique (up to the translation of time t) forward self-similar solution for (1.1). Moreover, it is asymptotically stable.*
- (ii) *Assume that $\alpha_1 = \alpha_2$. Then there exists a unique stationary self-similar solution for (1.1) with a given D_0 , which is the domain enclosed by $\Gamma_0 := \{(x, u_0(x)) | -\xi_{01} \leq x \leq \xi_{02}\}$, $\partial\Omega_1$, and $\partial\Omega_2$.*
- (iii) *Assume that $\alpha_1 > \alpha_2$. Then there is a constant $G_c (< -\tan \beta_1)$ depending only on α_1 and β_1 such that the following hold.*
 - (iii-a) *There exists at least one backward self-similar solution for (1.1) if $-\beta_1 \leq \alpha_2 < \alpha_1 \leq \beta_2$.*
 - (iii-b) *There exist at least two backward self-similar solutions for (1.1) if $\tan^{-1} G_c < \alpha_2 < -\beta_1 < \alpha_1 \leq \beta_2$.*
 - (iii-c) *There exists at least one backward self-similar solution for (1.1) if $\tan^{-1} G_c < \alpha_2 < -\beta_1$ and $\beta_2 < \alpha_1$.*
 - (iii-d) *There exists at least one backward self-similar solution for (1.1) if $\bar{\alpha} \leq \alpha_2 \leq \tan^{-1} G_c$ for some $\bar{\alpha} \in (-\pi/2, \alpha_1)$ depending only on α_1, β_1 , and β_2 .*

Remark 1.1 The exact existence for the case (iii) and the stability for the cases (ii), (iii) are still open.

2 Case: $\alpha_1 < \alpha_2$

Give α_1, β_1, p, q , with $\beta_1 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, $p > 0$, $q > 0$. Let us consider the initial value problem:

$$\begin{cases} v'' + \eta v' - v = 0, & \eta > -p, \\ v'(-p) = \tan \alpha_1, & v(-p) = p \tan \beta_1. \end{cases} \quad (2.1)$$

Let $F(\eta) = \eta v'(\eta) - v(\eta)$. Then by (2.1) we have $F'(\eta) = -\eta F(\eta)$. It follows that

$$v''(\eta) = -F(\eta) = p A_1 e^{(p^2 - \eta^2)/2}, \quad \eta > -p, \quad (2.2)$$

where $A_1 := \tan \alpha_1 + \tan \beta_1$. Note that $A_1 > 0$, since $\alpha_1 > -\beta_1$. This implies that $v'' > 0$. By an integration of (2.2) from $-p$ to η ($> -p$), we obtain

$$v'(\eta) = \tan \alpha_1 + pA_1 e^{p^2/2} [I^-(p) + I^-(\eta)] \quad (2.3)$$

where

$$I^-(\eta) = \int_0^\eta e^{-s^2/2} ds.$$

Set

$$G(p, q) := v'(q) = \tan \alpha_1 + pA_1 e^{p^2/2} [I^-(p) + I^-(q)].$$

Moreover, by integrating (2.3) from $-p$ to η ($> -p$), we obtain that

$$v(\eta) = \eta \tan \alpha_1 + \eta p A_1 e^{p^2/2} [I^-(p) + I^-(\eta)] + p A_1 e^{(p^2 - \eta^2)/2}.$$

Set

$$H(p, q) := \frac{v(q)}{q} = \tan \alpha_1 + pA_1 e^{p^2/2} [I^-(p) + I^-(q)] + \frac{p}{q} A_1 e^{(p^2 - q^2)/2}.$$

It is easy to compute that

$$\begin{cases} \frac{\partial G}{\partial p}(p, q) = pA_1 + (p^2 + 1)A_1 e^{p^2/2} [I^-(p) + I^-(q)] (> 0), \\ \frac{\partial G}{\partial q}(p, q) = pA_1 e^{(p^2 - q^2)/2} (> 0), \\ \frac{\partial H}{\partial p}(p, q) = pA_1 + (p^2 + 1)A_1 e^{p^2/2} \left\{ [I^-(p) + I^-(q)] + \frac{1}{q} e^{-q^2/2} \right\} (> 0), \\ \frac{\partial H}{\partial q}(p, q) = -\frac{p}{q^2} A_1 e^{(p^2 - q^2)/2} (< 0). \end{cases}$$

For given $\alpha_2 (> \alpha_1)$ and $\beta_2 \in [0, \pi/2)$, we want to solve the equations

$$G(p, q) = \tan \alpha_2 \quad \text{and} \quad H(p, q) = \tan \beta_2. \quad (2.4)$$

for some $p > 0$ and $q > 0$. If we can find the pair of (p, q) satisfying (2.4), (v, p, q) is the solution of (1.3).

Remark 2.1 Clearly $G(p, q) > \tan \alpha_1$. This claims that if $\alpha_1 \geq \alpha_2$, there are no (p, q) satisfying (2.4). That is, there are no forward self-similar solutions of (1.1) for $\alpha_1 \geq \alpha_2$.

Let consider the equation $G(p, q) = \tan \alpha_2$ for a given $\alpha_2 (> \alpha_1)$. We first observe that $G(p, q)$ is monotone increasing in p and q . Note that the limit of $I^-(q)$ as $q \uparrow +\infty$ exists and is also finite. Since $G(0, +\infty) = \tan \alpha_1$ and $G(+\infty, +\infty) = +\infty$, there is a unique $p_\infty > 0$ such that

$$G(p_\infty, +\infty) = \tan \alpha_2.$$

In addition, since $G(0, 0) = \tan \alpha_1$ and $G(+\infty, 0) = +\infty$, there is a unique $p_0 > 0$ such that

$$G(p_0, 0) = \tan \alpha_2.$$

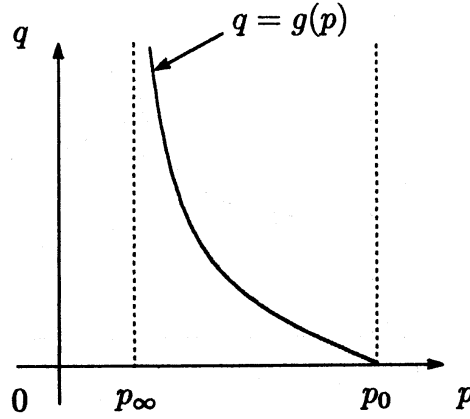


Figure 3: (p, q) -line satisfying $G(p, q) = \tan \alpha_2$

From the monotonisity in p of G , it follows that $p_\infty < p_0$. Then we have for $p \in (p_\infty, p_0)$

$$G(p, 0) < G(p_0, 0) = \tan \alpha_2 = G(p_\infty, +\infty) < G(p, +\infty)$$

The monotonisity in q of G implies that for each $p \in (p_\infty, p_0)$ there is a unique $q = g(p) > 0$ such that $G(p, g(p)) = \tan \alpha_2$. Note that $g(+\infty) = p_\infty$ and $g(0) = p_0$. Differentiating $G(p, g(p)) = \tan \alpha_2$ with respect to p , we obtain

$$\frac{\partial G}{\partial p}(p, g(p)) + \frac{\partial G}{\partial q}(p, g(p)) \cdot g'(p) = 0.$$

Thus we are led to $g'(p) < 0$, since $\partial G / \partial p > 0$ and $\partial G / \partial q > 0$ (see Figure 3).

Let consider the equation $H(p, q) = \tan \beta_2$ for a given $\beta_2 \in [0, \pi/2)$. We observe that $H(p, q)$ is monotone inscreasing in p for all $q > 0$ and monotone decreasing in q for all $p > 0$. Note that $H(p, +\infty) = G(p, +\infty)$ and $\alpha_2 < \beta_2$. Since $H(p_\infty, 0^+) = +\infty$ and $H(p_\infty, +\infty) = \tan \alpha_2$, there is a unique $\bar{q} > 0$ such that

$$H(p_\infty, \bar{q}) = \tan \beta_2.$$

In addition, since $H(p_\infty, +\infty) = \tan \alpha_2$ and $H(+\infty, +\infty) = +\infty$, there is a unique $p_*(> p_\infty)$ such that

$$H(p_*, +\infty) = \tan \beta_2.$$

Then we have for $p \in (p_\infty, p_*)$

$$H(p, +\infty) < H(p_*, +\infty) = \tan \beta_2 = H(p_\infty, \bar{q}) < H(p, \bar{q}).$$

The monotonisity in q of H implies that for each $p \in (p_\infty, p_*)$ there is a unique $q = h(p) > \bar{q}$ such that $H(p, h(p)) = \tan \beta_2$. Note that $h(p_\infty) = \bar{q}$ and $h(p_*) = +\infty$. Differentiating $H(p, h(p)) = \tan \beta_2$ with respect to p , we derive

$$\frac{\partial H}{\partial p}(p, h(p)) + \frac{\partial H}{\partial q}(p, h(p)) \cdot h'(p) = 0.$$

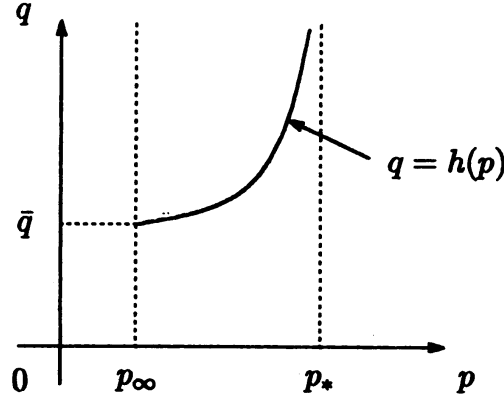


Figure 4: (p, q) -line satisfying $H(p, q) = \tan \beta_2$

Therefore we find $h'(p) > 0$, since $\partial H / \partial p > 0$ and $\partial H / \partial q < 0$ (see Figure 4).

We are ready to state and prove the following theorem.

Theorem 2.1 *Assume that $\alpha_1 < \alpha_2$. Then for given $\beta_1, \beta_2 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, and $\alpha_2 \in (\alpha_1, \beta_2)$ there is a unique solution (v, p, q) to the problem (1.3).*

Proof. Combine Figure 3 and Figure 4. Then we see that there is a unique (p, q) with $p \in (p_\infty, \bar{p})$, where $\bar{p} := \min\{p_0, p_*\}$, and $q > \bar{q}$ such that $g(p) = h(p) = q$. This proves the theorem. $\square$

We state the following theorem without the proof.

Theorem 2.2 *(Stability of a forward self-similar solution) Assume that $\alpha_1 < \alpha_2$. Also assume that $u_0 \in C^2[-\xi_{01}, \xi_{02}]$ satisfies the compatibility conditions and $u_0 > 0$ in $(-\xi_{01}, \xi_{02})$. Let $\Gamma(t)$ be a smooth solution of (1.1) with the initial data $\Gamma_0 = \{(x, u_0(x)) \mid -\xi_{01} \leq x \leq \xi_{02}\}$ and $S(t)$ be a forward self-similar solution denoted as*

$$S(t) := \{(\hat{x}, \sqrt{2t}v(\hat{x}/\sqrt{2t})) \mid -\sqrt{2t}p \leq \hat{x} \leq \sqrt{2t}q\}$$

where (v, p, q) is a solution of (1.3). Then $S(t)$ is asymptotically stable in the sense :

$$d_H(\Gamma(t), S(t)) \leq Ct^{-\delta}, \quad t > 1,$$

for some $\delta \in (0, 1/2)$ and a constant $C(> 0)$, which depends on the initial data Γ_0 . Here d_H denotes the Hausdorff distance.

To prove this theorem, we construct a sub-solution and a super-solution, which converge to $S(t)$ asymptotically as $t \rightarrow \infty$, and apply the strong maximum principle.

3 Case: $\alpha_1 = \alpha_2$

In this case, there is a family of stationary self-similar solutions of (1.1), that is,

$$u_d(x, t) = u_d(x) = (\tan \alpha)x + d,$$

where $\alpha := \alpha_1 = \alpha_2$ and d is any positive constant. The corresponding fixed end points to u_d are given by

$$p = \frac{d}{\tan \alpha_1 + \tan \beta_1}, \quad q = \frac{d}{\tan \beta_2 - \tan \alpha_2}.$$

According to (1.2), the condition $\alpha_1 = \alpha_2$ implies the area-preserving property. Let D_0 be the domain enclosed by $\Gamma_0 := \{(x, u_0(x)) | -\xi_{01} \leq x \leq \xi_{02}\}$, $\partial\Omega_1$, and $\partial\Omega_2$. Set $A_1 := \tan \alpha_1 + \tan \beta_1$ and $A_2 := \tan \beta_2 - \tan \alpha_2$. Let

$$d_* = \sqrt{\frac{2A_1A_2}{A_1 + A_2} \mu(D_0)}.$$

Then a stationary self-similar solution of (1.1) is uniquely determined as $u_d(x) = ax + d_*$ for a given D_0 .

4 Case: $\alpha_1 > \alpha_2$

Give α_1, β_1, p, q , with $\beta_1 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, $p > 0$, $q > 0$. Let us consider the initial value problem:

$$\begin{cases} v'' - \eta v' + v = 0, & \eta > -p, \\ v'(-p) = \tan \alpha_1, & v(-p) = p \tan \beta_1. \end{cases} \quad (4.1)$$

Then as before we have

$$v''(\eta) = -pA_1e^{-(p^2-\eta^2)/2}, \quad \eta > -p, \quad (4.2)$$

where $A_1 = \tan \alpha_1 + \tan \beta_1 > 0$. Note that $v'' < 0$. By an integration of (4.2) from $-p$ to η ($> -p$), we obtain

$$v'(\eta) = \tan \alpha_1 - pA_1e^{-p^2/2}[I^+(p) + I^+(\eta)], \quad \eta > -p, \quad (4.3)$$

where

$$I^+(\eta) = \int_0^\eta e^{s^2/2} ds.$$

Set

$$\hat{G}(p, q) := v'(q) = \tan \alpha_1 - pA_1e^{-p^2/2}[I^+(p) + I^+(q)].$$

In addition, by integrating (4.3) again, we derive that

$$v(\eta) = \eta \tan \alpha_1 - \eta p A_1 e^{-p^2/2}[I^+(p) + I^+(\eta)] + p A_1 e^{-(p^2-\eta^2)/2}.$$

Also set

$$\hat{H}(p, q) := \frac{v(q)}{q} = \tan \alpha_1 - p A_1 e^{-p^2/2}[I^+(p) + I^+(q)] + \frac{p}{q} A_1 e^{-(p^2-q^2)/2}.$$

It is easy to compute that

$$\begin{cases} \frac{\partial \hat{G}}{\partial p}(p, q) = -pA_1 + (p^2 - 1)A_1e^{-p^2/2}[I^+(p) + I^+(q)], \\ \frac{\partial \hat{G}}{\partial q}(p, q) = -pA_1e^{-(p^2-q^2)/2} (< 0), \\ \frac{\partial \hat{H}}{\partial p}(p, q) = -pA_1 + (p^2 - 1)A_1e^{-p^2/2} \left\{ [I^+(p) + I^+(q)] - \frac{1}{q}e^{q^2/2} \right\}, \\ \frac{\partial \hat{H}}{\partial q}(p, q) = -\frac{p}{q^2}A_1e^{-(p^2-q^2)/2} (< 0). \end{cases}$$

For given $\alpha_2 (< \alpha_1)$ and $\beta_2 \in [0, \pi/2)$, we want to solve the equations

$$\hat{G}(p, q) = \tan \alpha_2 \quad \text{and} \quad \hat{H}(p, q) = \tan \beta_2. \quad (4.4)$$

for some $p > 0$ and $q > 0$. If we can find the pair of (p, q) satisfying (4.4), (v, p, q) is the solution of (1.4).

Remark 4.1 Clearly $\hat{G}(p, q) < \tan \alpha_1$. This claims that if $\alpha_1 \leq \alpha_2$, there are no (p, q) satisfying (4.4). That is, there are no backward self-similar solutions of (1.1) for $\alpha_1 \leq \alpha_2$.

In order to solve (4.4), let study the fuctions $\hat{G}(p, q)$ and $\hat{H}(p, q)$. Now set

$$J(p) := \frac{p}{p^2 - 1}e^{p^2/2} - I^+(p) \quad \text{for } p \neq 1,$$

$$K(q) := I^+(q) - \frac{1}{q}e^{q^2/2} \quad \text{for } q > 0.$$

We compute that

$$J'(p) = -\frac{1}{(p^2 - 1)^2}e^{p^2/2} < 0 \quad \text{for } p \neq 1, \quad (4.5)$$

$$K'(q) = \frac{1}{q^2}e^{q^2/2} > 0 \quad \text{for } q > 0, \quad (4.6)$$

and observe that

$$J(0) = 0, \quad J(1^-) = -\infty, \quad J(1^+) = +\infty, \quad J(+\infty) = -\infty, \quad (4.7)$$

$$K(0^+) = -\infty, \quad K(+\infty) = +\infty. \quad (4.8)$$

It follows from (4.6) and (4.8) that there is a unique $r_0 > 0$ such that

$$K(q) \begin{cases} < 0 & \text{if } 0 < q < r_0, \\ = 0 & \text{if } q = r_0, \\ > 0 & \text{if } q > r_0. \end{cases}$$

First we study the function $\hat{G}(p, q)$. Note that

$$\begin{cases} \frac{\partial \hat{G}}{\partial p}(p, q) < 0 & \text{for } p \in (0, 1], \\ \frac{\partial \hat{G}}{\partial p}(p, q) = A_1(p^2 - 1)e^{-p^2/2}[I^+(q) - J(p)] & \text{for } p > 1. \end{cases}$$

Since $(I^+)'(q) > 0$, $I^+(0) = 0$, and $I^+(\infty) = +\infty$, there is a unique $p_c(q) > 1$ such that $J(p_c(q)) = I^+(q)$ for each $q > 0$. We have $p_c(0^+) \in (1, +\infty)$, $p_c(+\infty) = 1$, and $p'_c(q) < 0$. These imply that for all $q > 0$

$$\frac{\partial \hat{G}}{\partial p}(p, q) \begin{cases} < 0 & \text{if } p < p_c(q); \\ = 0 & \text{if } p = p_c(q); \\ > 0 & \text{if } p > p_c(q). \end{cases} \quad (4.9)$$

Then we find

$$G_c := \hat{G}(p_c(0), 0) = -\tan \beta_1 - \frac{1}{p_c^2(0) - 1}(\tan \alpha_1 + \tan \beta_1) (< -\tan \beta_1). \quad (4.10)$$

Next we study the function $\hat{H}(p, q)$. Note that

$$\begin{cases} \frac{\partial \hat{H}}{\partial p}(p, q) A_1(p^2 - 1)e^{-p^2/2}[K(q) - J(p)] & \text{for } p \neq 1, \\ \frac{\partial \hat{H}}{\partial p}(1, q) = -A_1 < 0. \end{cases}$$

Consider the case $0 < p < 1$. For $q \geq r_0$

$$\frac{\partial \hat{H}}{\partial p}(p, q) = -pA_1 + A_1(p^2 - 1)e^{-p^2/2}[K(q) + I^+(p)] < 0. \quad (4.11)$$

On the other hand, by virtue of (4.5) and (4.7), we see that $J(p) < 0$ for $p \in (0, 1)$. This implies that for each $p \in (0, 1)$ there exists a unique $q_s(p) \in (0, r_0)$ such that

$$K(q_s(p)) = J(p).$$

Note that $q_s(0^+) = r_0$, $q_s(1^-) = 0$, and $q'_s(p) < 0$. Thus we derive for $0 < q < r_0$

$$\frac{\partial \hat{H}}{\partial p}(p, q) \begin{cases} > 0 & \text{if } 0 < q < q_s(p); \\ = 0 & \text{if } q = q_s(p); \\ < 0 & \text{if } q > q_s(p) \end{cases} \quad (4.12)$$

Consider the case $p > 1$. It follows from (4.5)-(4.8) that there exists a unique $q_u(p) > 0$ such that

$$K(q_u(p)) = J(p).$$

Note that $q_u(1^+) = +\infty$, $q_u(+\infty) = 0$, and $q'_u(p) < 0$. Therefore we are led to

$$\frac{\partial \hat{H}}{\partial p}(p, q) \begin{cases} < 0 & \text{if } 0 < q < q_u(p); \\ = 0 & \text{if } q = q_u(p); \\ > 0 & \text{if } q > q_u(p). \end{cases} \quad (4.13)$$

Let consider the equation $\hat{G}(p, q) = \tan \alpha_2$ for a given $\alpha_2 (< \alpha_1)$. We separate into three cases; (a) $-\beta_1 \leq \alpha_2$, (b) $\tan^{-1} G_c < \alpha_2 < -\beta_1$, (c) $\alpha_2 \leq \tan^{-1} G_c$. For the sake of convenience, we analyze them in order of (c) $\rightarrow$ (b) $\rightarrow$ (a).

Case $\alpha_2 \leq \tan^{-1} G_c$: If $\alpha_2 = \tan^{-1} G_c (= \tan^{-1}[\hat{G}(p_c(0), 0)])$, $(p_c(0), 0)$ is a solution of $\hat{G}(p, q) = \tan \alpha_2$. Thus we study the case $\alpha_2 < \tan^{-1} G_c$. Recalling (4.9), we have $\tan \alpha_2 < G_c = \hat{G}(p_c(0), 0) \leq \hat{G}(p, 0)$ for all $p > 0$. We also find $\hat{G}(p, +\infty) = -\infty$. It follows from $\partial \hat{G} / \partial q < 0$ that for each $p > 0$ there is a unique $q = \hat{g}(p) > 0$ such that $\hat{G}(p, \hat{g}(p)) = \tan \alpha_2$. Note that for $(\tilde{p}, \tilde{q})$ on the line $p = p_c(q)$ we also have $\tilde{q} = \hat{g}(\tilde{p}) \in (0, +\infty)$ satisfying $\hat{G}(\tilde{p}, \hat{g}(\tilde{p})) = \tan \alpha_2$. Differentiating $\hat{G}(p, \hat{g}(p)) = \tan \alpha_2$ with respect to p , we obtain

$$\frac{\partial \hat{G}}{\partial p}(p, \hat{g}(p)) + \frac{\partial \hat{G}}{\partial q}(p, \hat{g}(p)) \cdot \hat{g}'(p) = 0.$$

Recalling (4.9) again, this implies that

$$\begin{cases} \hat{g}'(p) < 0 & \text{for } 0 < p < \tilde{p}, \\ \hat{g}'(p) > 0 & \text{for } p > \tilde{p}. \end{cases}$$

In addition, using the reduction to absurdity, we see $g(0^+) = +\infty$ and $g(+\infty) = +\infty$ (see Figure 5(c))

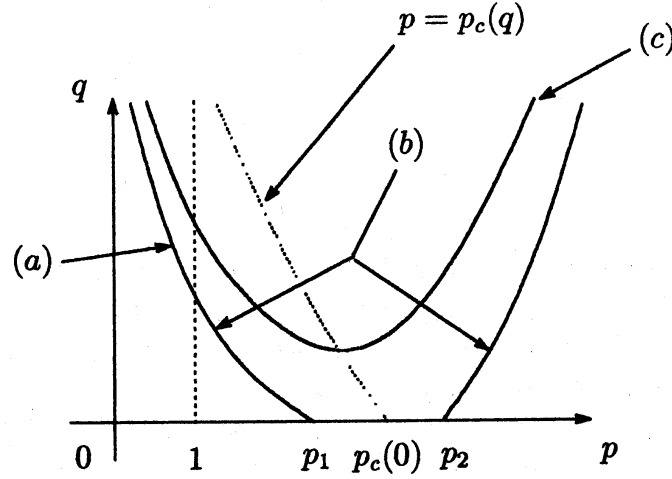
Case $\tan^{-1} G_c < \alpha_2 < -\beta_1$: We observe that $\hat{G}(0, 0) = \tan \alpha_1 (> \tan \alpha_2)$ and

$$\begin{aligned} \hat{G}(p, 0) &= \tan \alpha_1 - p A_1 e^{-p^2/2} I^+(p) \\ &\rightarrow \tan \alpha_1 - A_1 = -\tan \beta_1 (> \tan \alpha_2) \quad \text{as } p \rightarrow +\infty. \end{aligned}$$

Then it follows from (4.9) that there exist $p_1 \in (0, p_c(0))$ and $p_2 \in (p_c(0), +\infty)$ such that $\hat{G}(p_1, 0) = \hat{G}(p_2, 0) = \tan \alpha_2$. For $p \in (p_1, p_2)$, we obtain that $\hat{G}(p, 0) < \tan \alpha_2$. In view of $\partial \hat{G} / \partial q < 0$, we are led to $\hat{G}(p, q) < \tan \alpha_2$ for $p \in (p_1, p_2)$ and $q > 0$. Thus for $p \in (p_1, p_2)$ there is no solution of $\hat{G}(p, q) = \tan \alpha_2$. For $p \in (0, p_1) \cup (p_2, +\infty)$, we observe that $\hat{G}(p, 0) > \tan \alpha_2$. Since $\hat{G}(p, +\infty) = -\infty$ and $\partial \hat{G} / \partial q < 0$ for all $p > 0$, there is a unique $q = \hat{g}(p) > 0$ such that $\hat{G}(p, \hat{g}(p)) = \tan \alpha_2$ for each $p \in (0, p_1) \cup (p_2, +\infty)$. Applying the same argument as the case $\alpha_2 \leq \tan^{-1} G_c$, we see

$$\begin{cases} \hat{g}'(p) < 0 & \text{for } 0 < p < p_1, \\ \hat{g}'(p) > 0 & \text{for } p > p_2. \end{cases}$$

We also have $g(0^+) = +\infty$ and $g(+\infty) = +\infty$. Moreover, by means of $\hat{G}(p_1, 0) = \hat{G}(p_2, 0) = \tan \alpha_2$, we derive $\hat{g}(p_1) = \hat{g}(p_2) = 0$ (see Figure 5(b)).



(a) $-\beta_1 \leq \alpha_2$ (b) $\tan^{-1} G_c < \alpha_2 < -\beta_1$ (c) $\alpha_2 \leq \tan^{-1} G_c$

Figure 5: (p, q) -line satisfying $\hat{G}(p, q) = \tan \alpha_2$

Case $-\beta_1 \leq \alpha_2$: Let $p_1 \in (0, p_c(0))$ be defined as the above. Recalling $\hat{G}(p, 0) \rightarrow -\tan \beta_1$ as $p \rightarrow +\infty$ with $\partial \hat{G} / \partial p > 0$ for $p > p_c(0)$, we have $\hat{G}(p, 0) < \tan \alpha_2$ for $p > p_1$. It follows from $\partial \hat{G} / \partial q < 0$ that $\hat{G}(p, q) < \tan \alpha_2$ for $p > p_1$ and $q > 0$. Thus for $p > p_1$ there is no solution of $\hat{G}(p, q) = \tan \alpha_2$. For $p \in (0, p_1)$, we derive that $\hat{G}(p, 0) > \tan \alpha_2$. Since $\hat{G}(p, +\infty) = -\infty$ and $\partial \hat{G} / \partial q < 0$ for all $p > 0$, there is a unique $q = \hat{g}(p) > 0$ such that $\hat{G}(p, \hat{g}(p)) = \tan \alpha_2$ for each $p \in (0, p_1)$. Applying the same argument as the previous case, we derive $\hat{g}'(p) < 0$ for $p \in (0, p_1)$, $\hat{g}(0^+) = +\infty$, and $\hat{g}(p_1) = 0$ (see Figure 5(a)).

Let consider the equation $\hat{H}(p, q) = \tan \beta_2$ for a given $\beta_2 \in [0, \pi/2)$. Since $\hat{H}(p, 0^+) = +\infty$ and $\hat{H}(p, +\infty) = -\infty$ for all $p > 0$, it follows from $\partial \hat{H} / \partial q < 0$ that for each $p > 0$ there is a unique $q = \hat{h}(p) > 0$ such that $\hat{H}(p, \hat{h}(p)) = \tan \beta_2$. Now we compute that

$$\hat{H}(p, q_u(p)) = -\tan \beta_1 - \frac{1}{p^2 - 1} A_1 (< 0) \quad \text{for } p > 1, \quad (4.14)$$

$$\hat{H}(p, q_s(p)) = -\tan \beta_1 + \frac{1}{1 - p^2} A_1, \quad \text{for } p \in (0, 1). \quad (4.15)$$

It follows from (4.14), $\tan \beta_2 > 0$, and $\partial \hat{H} / \partial q < 0$ that $\hat{h}(p) \in (0, q_u(p))$ for $p > 1$. Then, in view of $\partial \hat{H} / \partial q < 0$ and (4.13), we have $\hat{h}'(p) < 0$ for $p > 1$. Note that $\hat{h}(1) \in (0, +\infty)$ and $\hat{h}(+\infty) = 0$, since $q_u(1^+) = +\infty$ and $q_u(+\infty) = 0$. Hereafter, we investigate $\hat{h}(p)$ for $p \in (0, 1)$. By (4.15), $\hat{H}(p, q_s(p)) = \tan \beta_2$ is equivalent to

$$p^2 = \frac{\tan \beta_2 - \tan \alpha_1}{\tan \beta_1 + \tan \beta_2} \in (0, 1). \quad (4.16)$$

We separate into three cases; $(\bar{a})\beta_2 < \alpha_1$, $(\bar{b})\beta_2 = \alpha_1$, $(\bar{c})\beta_2 > \alpha_1$.

Case $\beta_2 < \alpha_1$: Note that there is no $p \in (0, +\infty)$ satisfying (4.16). Since $\tan \beta_1 + \tan \beta_2 \leq A_1$, we have

$$\begin{aligned}\hat{H}(p, \hat{h}(p)) = \tan \beta_2 &\leq -\tan \beta_1 + A_1 \\ &< -\tan \beta_1 + \frac{1}{1-p^2} A_1 = \hat{H}(p, q_s(p)) \quad \text{for } p \in (0, 1).\end{aligned}$$

Recalling that $\hat{H}(p, q)$ is monotone decreasing in q , we see $\hat{h}(p) > q_s(p)$ for all $p \in (0, 1)$. It follows from $\partial \hat{H} / \partial q < 0$, (4.11), and (4.12) that $\hat{h}'(p) < 0$ for all $p \in (0, 1)$. In addition, we derive $\hat{h}(0^+) = +\infty$ (see Figure 6). Indeed, if $\hat{h}(0^+) < +\infty$, for any $q_* > \max\{2\hat{h}(0^+), r_0\}$ we have $\hat{H}(p, q_*) \rightarrow \tan \alpha_1$ as $p \rightarrow 0^+$. Then (4.11) implies that there is a $p_* \in (0, 1)$ such that $\hat{H}(p, q_*) > \tan \beta_2 = \hat{H}(p, \hat{h}(p))$ for all $p < p_*$. It follows from $\partial \hat{H} / \partial q < 0$ that $q_* < \hat{h}(p)$ for all $p < p_*$. This is a contradiction. Hence $\hat{h}(0^+) = +\infty$.

Case $\beta_2 = \alpha_1$: Applying the same argument as the previous case, we have $\hat{h}(p) > q_s(p)$ for all $p \in (0, 1)$ and $\hat{h}'(p) < 0$. Moreover, since $\hat{H}(p, r_0) < \tan \alpha_1 = \tan \beta_2 = \hat{H}(p, \hat{h}(p))$ and $\partial \hat{H} / \partial q < 0$ imply that $\hat{h}(p) \in (q_s(p), r_0)$ for all $p \in (0, 1)$, we see $\hat{h}(0^+) = r_0$ (see Figure 6).

Case $\beta_2 > \alpha_1$: There is a unique $p_\dagger \in (0, 1)$ satisfying (4.16). That is, $\hat{H}(p_\dagger, q_s(p_\dagger)) = \tan \beta_2$. Using (4.12) and $\partial \hat{H} / \partial q < 0$, it is easy to see that

$$\begin{cases} \hat{h}(p) < q_s(p) & \text{and } \hat{h}'(p) > 0 & \text{for } 0 < p < p_\dagger, \\ \hat{h}(p) > q_s(p) & \text{and } \hat{h}'(p) < 0 & \text{for } p_\dagger < p < 1. \end{cases}$$

Note that $\hat{h}(0^+) \in [0, r_0)$ (see Figure 6).

From now on, we assume $\beta_1, \beta_2 \in [0, \pi/2)$, $\alpha_1 \in (-\beta_1, \pi/2)$, and $\alpha_2 \in (-\pi/2, \alpha_1) \cap (-\pi/2, \beta_2)$. We are ready to state and prove the following theorems.

Theorem 4.1 *Assume that $-\beta_1 \leq \alpha_2 < \alpha_1 \leq \beta_2$. Then there is at least one solution to the problem (1.4). Assume that $\tan^{-1} G_c < \alpha_2 < -\beta_1 < \alpha_1 \leq \beta_2$, Then there are at least two solutions to the problem (1.4).*

Theorem 4.2 *Assume that $\tan^{-1} G_c < \alpha_2 < -\beta_1$ and $\beta_2 < \alpha_1$. Then there is at least one solution to the problem (1.4).*

Proof of Theorem 4.1 and 4.2. For the first half of Theorem 4.1, combine Figure 5(a) and Figure 6(b), (c). For the second half of Theorem 4.1, combine Figure 5(b) and Figure 6(b), (c). For Theorem 4.2, combine Figure 5(b) and Figure 6(a). $\square$

Theorem 4.3 *Assume that $\alpha_2 \leq \tan^{-1} G_c$. Then there exists $\alpha_* \in (-\pi/2, \alpha_1)$ depending only on α_1, β_1 , and β_2 such that the problem (1.4) has at least one solution if $\alpha_2 \geq \alpha_*$.*

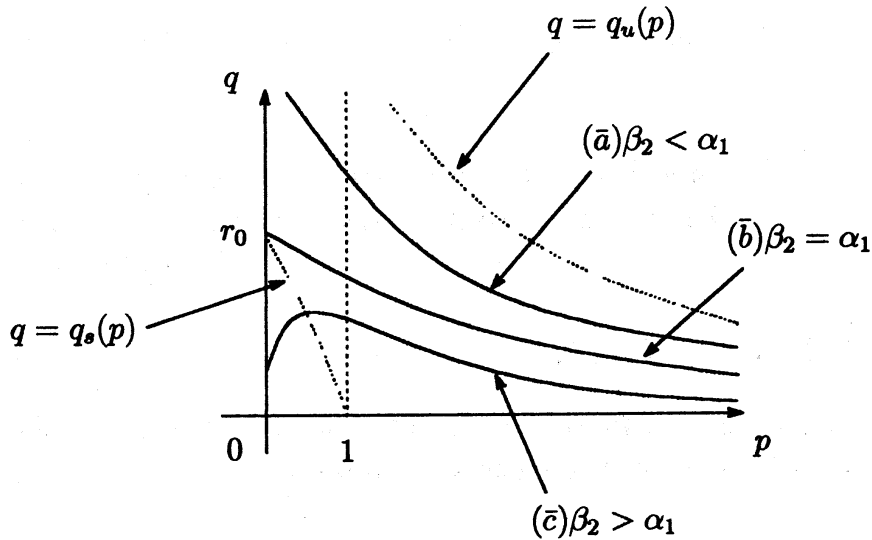


Figure 6: (p, q) -line satisfying $\hat{H}(p, q) = \tan \beta_2$

Proof. Recall that for $(\tilde{p}, \tilde{q})$ on the line $p = p_c(q)$ we have $\tilde{q} = \hat{g}(\tilde{p}) \in (0, +\infty)$ satisfying $\hat{G}(\tilde{p}, \hat{g}(\tilde{p})) = \tan \alpha_2$. This is also written as $\hat{G}(p_c(\tilde{q}), \tilde{q}) = \tan \alpha_2$. Since

$$\hat{G}(p_c(\tilde{q}), \tilde{q}) = -\tan \beta_1 - \frac{1}{p_c^2(\tilde{q}) - 1} A_1$$

is monotone decreasing in $\tilde{q}$, the function $\tilde{q} = \tilde{q}(\alpha_2)$ is monotone decreasing as α_2 increases. Note that $\tilde{q}(\alpha_2) \rightarrow +\infty$ as $\alpha_2 \rightarrow -\pi/2$ and $\tilde{q}(\alpha_2) \rightarrow 0$ as $\alpha_2 \rightarrow \tan^{-1} G_c$. Combining this fact and Figure 6, we see that there exists $\alpha_* \in (-\pi/2, \alpha_1)$ such that the problem (1.4) has at least one solution if $\alpha_2 \geq \alpha_*$. $\square$

References

- [1] J. D. Buckmaster and G. S. S. Ludford, *Theory of Laminar Flames*, Cambridge University Press, Cambridge, 1982.
- [2] V. A. Galaktionov, J. Hulshof and J. L. Vazquez, Extinction and focusing behaviour of spherical and annular flames described by a free boundary problem, *J. Math. Pures Appl.*, 76 (1997), 563-608.
- [3] D. Hilhorst and J. Hulshof, A free boundary focusing problem, *Proc. Amer. Math. Soc.*, 121 (1994), 1193-1202.
- [4] Y. Kohsaka, Free boundary problem for quasilinear parabolic equation with fixed angle of contact to a boundary, *Nonlinear Analysis*, 45 (2001), 865-894.
- [5] J. L. Vazquez, The free boundary problem for the heat equation with fixed gradient condition, *Free boundary problems, theory and applications*, Zakopane, Poland (1995), Pitman Res. Notes in Math. Ser. 363, 277-302.