RIMS-1942

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March 2021

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Abstract. In the present paper, we study combinatorial anabelian geometry. The goal is to reconstruct group-theoretically the set of generalized fiber subgroups from the associated configuration space group equipped with its collection of log-full subgroups.

0. Introduction

Mochizuki and Tamagawa gave bi-anabelian algorithm to reconstruct fiber subgroups (cf. [MzTa], Definition 2.3, (iii)):

Theorem 1 ([MzTa], **Corollary 6.3**). *Let $n \in \mathbb{Z}_{>0}$; p a prime number; $\square \in \{\dagger, \ddagger\}$; $({}^{\square}g, {}^{\square}r)$ a pair of nonnegative integers such that $2{}^{\square}g - 2 + {}^{\square}r > 1$; ${}^{\square}k$ an algebraic closed field of characteristic 0; ${}^{\square}X^{\log}$ a smooth log curve over ${}^{\square}k$ of type $({}^{\square}g, {}^{\square}r)$ (cf. Definition 4, (iv)). Write $\pi_1^p({}^{\square}X_n^{\log})$ for the maximal pro-p quotient of the fundamental group of n-th configuration space (cf. Definition 5; Definition 10, (i), (ii)). Let $\alpha\colon \pi_1^p({}^{\dagger}X_n^{\log}) \xrightarrow{\sim} \pi_1^p({}^{\ddagger}X_n^{\log})$ be an isomorphism of profinite groups. Then α induces a bijection between the set of fiber subgroups of $\pi_1^p({}^{\dagger}X_n^{\log})$ and the set of fiber subgroups of $\pi_1^p({}^{\ddagger}X_n^{\log})$.*

After that, Hoshi, Minamide, and Mochizuki gave mono-anabelian algorithm to reconstruct generalized fiber subgroups (cf. Definition 10, (iv); [HMM], Definition 2.1, (ii)):

Theorem 2 ([HMM], **Theorem A, (i), (ii)**). *Let $n \in \mathbb{Z}_{>1}$; p a prime number; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; k an algebraic closed field of characteristic 0; $X^{\log}$ a smooth log curve over k of type (g, r); $\Delta^p(g, r, n)$ a profinite group which is isomorphic to $\pi_1^p(X_n^{\log})$. Then the following hold:*

(i) One may construct (g, r, n) associated to the intrinsic structure of $\Delta^p(g, r, n)$, i.e.,

$$\Delta^p(g, r, n) \quad \rightsquigarrow \quad (g, r, n).$$

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.
Key words and phrases. anabelian geometry, configuration space.

(ii) *One may construct a set* GFS *(cf. Definition 3.1, (v)) associated to the intrinsic structure of* $\Delta^p(g,r,n)$*, i.e.,*

$$\Delta^p(g,r,n) \quad \rightsquigarrow \quad \mathrm{GFS}.$$

At the same time, the author gave bi-anabelian algorithm to reconstruct generalized fiber subgroups.

THEOREM 3 ([Hgsh], **Theorem 0.1, (v)**). *Let* $n \in \mathbb{Z}_{>1}$*;* $\square \in \{\dagger, \ddagger\}$*;* ${}^{\square}p$ *a prime number;* $({}^{\square}g, {}^{\square}r)$ *a pair of nonnegative integers such that* $2{}^{\square}g-2+{}^{\square}r>0$ *and "*$r>0$*";* ${}^{\square}k$ *an algebraic closed field of characteristic* $\neq {}^{\square}p$*;* ${}^{\square}X^{\log}$ *a smooth log curve over* ${}^{\square}k$ *of type* $({}^{\square}g, {}^{\square}r)$*;* $\alpha\colon \pi_1^{{}^{\dagger}p}({}^{\dagger}X_n^{\log}) \xrightarrow{\sim} \pi_1^{{}^{\ddagger}p}({}^{\ddagger}X_n^{\log})$ *an isomorphism of profinite groups such that* α *induces a bijection between the set of log-full subgroups of* $\pi_1^{{}^{\dagger}p}({}^{\dagger}X_n^{\log})$ *(cf. Definition 10, (iii)) and the set of log-full subgroups of* $\pi_1^{{}^{\ddagger}p}({}^{\ddagger}X_n^{\log})$*. Then* α *induces bijection between the set of generalized fiber subgroups of* $\pi_1^{{}^{\dagger}p}({}^{\dagger}X_n^{\log})$ *and the set of generalized fiber subgroups of* $\pi_1^{{}^{\ddagger}p}({}^{\ddagger}X_n^{\log})$*.*

In the present paper, we give mono-anabelian algorithm to reconstruct (g,r,n) if $r>0$ (cf. Theorem A, (ii)), and we give mono-anabelian algorithm to reconstruct generalized fiber subgroups, if $r>0$ (cf. Theorem A, (v)), i.e.,

$$(\Delta^p(g.r.n), \mathrm{LFS}) \quad \rightsquigarrow \quad (g,r,n) \qquad (\text{if } r>0),$$

$$(\Delta^p(g.r.n), \mathrm{LFS}) \quad \rightsquigarrow \quad \mathrm{GFS} \qquad (\text{if } r>0).$$

Our main result is as follows:

THEOREM A. *Let* $n \in \mathbb{Z}_{>1}$*;* (g,r) *a pair of nonnegative integers such that* $2g-2+r>0$ *and "*$r>0$*";* p *a prime number;* k *an algebraic closed field of characteristic* $\neq p$*;* $X^{\log}$ *a smooth log curve over* k *of type* (g,r)*;* $\Delta^p(g,r,n)$ *a profinite group which is isomorphic to* $\pi_1^p(X_n^{\log})$*. Write* LFS *(resp.* LD, TD, DD, GFS*) for the set of subgroups of* $\Delta^p(g,r,n)$ *such that any isomorphism* $\Delta^p(g,r,n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$ *induces a bijection*

$$\mathrm{LFS} \xrightarrow{\sim} \{\textit{log-full subgroups of } \pi_1^p(X_n^{\log})\}$$

$$(\textit{resp. } \mathrm{LD} \xrightarrow{\sim} \{\textit{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \textit{ associated to log divisors}\},$$

$$\mathrm{TD} \xrightarrow{\sim} \{\textit{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \textit{ associated to tripodal divisors}\},$$

$$\mathrm{DD} \xrightarrow{\sim} \{\textit{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \textit{ associated to drift diagonals}\},$$

$$\mathrm{GFS} \xrightarrow{\sim} \{\textit{generalized fiber subgroups of } \pi_1^p(X_n^{\log})\})$$

(cf. Definition 6, (iv); Definition 7, (ii), (iv)). Write DC *for the set of subsets of* DD *such that any isomorphism* $\Delta^p(g,r,n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$ *induces a bijection*

$$\mathrm{DC} \xrightarrow{\sim} \{\{\textit{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \textit{ associated to } V \in \Lambda\} \mid \Lambda\colon \textit{a drift collection}\}$$

(cf. Definition 7, (v)). Then the following hold:

(i) *One may construct a set* LD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS *(cf. Proposition 32, (i)), i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathrm{LD}.$$

(ii) *One may construct* (g,r,n) *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS *(cf. Proposition 32, (v)), i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow (g,r,n).$$

(iii) *One may construct a set* TD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LD *(cf. Proposition 34, (viii)), i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LD}) \rightsquigarrow \mathrm{TD}.$$

(iv) *One may construct a set* DD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS *(cf. Proposition 35, (iii)), i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathrm{DD}.$$

(v) *One may construct a set* GFS *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS *(cf. Proposition 37, (iii)), i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathrm{GFS}.$$

Remark 1. *Note that one verifies easily that Theorem 2, (i), (ii), imply Theorem A, (ii), (v). In the present paper, we do not apply Theorem 1; Theorem 2, (i), (ii), to prove Theorem A.*

This paper is organized as follows: In §1, we explain some notations. In §2, we introduce various type of log divisors and we calculate the number of various type of log divisors. In §3, we give mono-anabelian algorithm to reconstruct (g,r,n) if $r>0$, and we give mono-anabelian algorithm to reconstruct a set GFS if $r>0$.

1. Notation

Definition 1. Let a,b be nonnegative integers. Then

$$\binom{b}{a} \overset{\mathrm{def}}{=} \begin{cases} \frac{b!}{a!(b-a)!} & (a \le b) \\ 0 & (a > b), \end{cases}$$

where $n! \overset{\mathrm{def}}{=} n \times (n-1) \times \cdots \times 2 \times 1$ for $n \in \mathbb{Z}_{>0}$, and $0! \overset{\mathrm{def}}{=} 1$.

Definition 2. Let p be a prime number, and $\mathcal{G}$ a semi-graph of anabelioids of pro-p PSC-type (cf. [CmbGC], Definition 1.1, (i)) and $\mathbb{G}$ the underlying semi-graph of $\mathcal{G}$. Write

$$\mathrm{Cusp}(\mathbb{G}) \text{ (resp. } \mathrm{Node}(\mathbb{G}),\ \mathrm{Vert}(\mathbb{G}),\ \mathrm{Edge}(\mathbb{G}))$$

for the set of cusps (resp. nodes, vertices, edges) of $\mathbb{G}$ and

$$\mathrm{Cusp}(\mathcal{G}) \overset{\mathrm{def}}{=} \mathrm{Cusp}(\mathbb{G}),\ \mathrm{Node}(\mathcal{G}) \overset{\mathrm{def}}{=} \mathrm{Node}(\mathbb{G}),$$

$$\mathrm{Vert}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Vert}(\mathbb{G}),\ \mathrm{Edge}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \mathrm{Edge}(\mathbb{G}).$$

Definition 3. Let $S^{\log}$ be an fs log scheme (cf. [Nky], Definition 1.7).

(i) Write S for the underlying scheme of $S^{\log}$.
(ii) Write $\mathcal{M}_S$ for the sheaf of monoids that defines the log structure of $S^{\log}$.
(iii) Let $\overline{s}$ be a geometric point of S. Then we shall denote by $I(\overline{s}, \mathcal{M}_S)$ the ideal of $\mathcal{O}_{S,\overline{s}}$ generated by the image of $\mathcal{M}_{S,\overline{s}} \setminus \mathcal{O}^{\times}_{S,\overline{s}}$ via the homomorphism of monoids $\mathcal{M}_{S,\overline{s}} \to \mathcal{O}_{S,\overline{s}}$ induced by the morphism $\mathcal{M}_S \to \mathcal{O}_S$ which defines the log structure of $S^{\log}$.
(iv) Let $s \in S$ and $\overline{s}$ a geometric point of S which lies over s. Write $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ for the groupification of $\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}}$. Then we shall refer to the rank of the finitely generated free abelian group $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ as the *log rank* at s. Note that one verifies easily that this rank is independent of the choice of $\overline{s}$, i.e., depends only on s.
(v) Let $m \in \mathbb{Z}$. Then we shall write

$$S^{\log \leq m} \stackrel{\mathrm{def}}{=} \{s \in S \mid \text{the log rank at } s \text{ is} \leq m\}.$$

Note that since $S^{\log \leq m}$ is open in S (cf. [MzTa], Proposition 5.2, (i)), we shall also regard (by abuse of notation) $S^{\log \leq m}$ as an open subscheme of S.
(vi) We shall write $U_S \stackrel{\mathrm{def}}{=} S^{\log \leq 0}$ and refer to U_S as the *interior* of $S^{\log}$. When $U_S = S$, we shall often use the notation S to denote the log scheme $S^{\log}$.

Definition 4. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$ and k a field.

(i) Write $\overline{\mathcal{M}}_{g,r}$ for the moduli stack (over k) of pointed stable curves of type (g, r), and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ for the open substack corresponding to the smooth curves (cf. [Knu]). Here, we assume the marked points to be ordered.
(ii) Write

$$\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$$

for the tautological curve over $\overline{\mathcal{M}}_{g,r}$; $\overline{\mathcal{D}}_{g,r} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ for the divisor at infinity.
(iii) Write $\overline{\mathcal{M}}^{\log}_{g,r}$ for the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g,r}$ with the log structure determined by the divisors with normal crossings $\overline{\mathcal{D}}_{g,r}$.
(iv) The divisor of $\overline{\mathcal{C}}_{g,r}$ given by the union of $\overline{\mathcal{C}}_{g,r} \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{D}}_{g,r}$ with the divisor of $\overline{\mathcal{C}}_{g,r}$ determined by the marked points determines a log structure on $\overline{\mathcal{C}}_{g,r}$; we denote the resulting log stack by $\overline{\mathcal{C}}^{\log}_{g,r}$. Thus, we obtain a morphism of log stacks

$$\overline{\mathcal{C}}^{\log}_{g,r} \to \overline{\mathcal{M}}^{\log}_{g,r},$$

which we refer to as the *tautological log curve* over $\overline{\mathcal{M}}_{g,r}^{\log}$. If $S^{\log}$ is an arbitrary log scheme, then we shall refer to a morphism

$$C^{\log} \to S^{\log}$$

whose pull-back to some finite étale covering $T \to S$ is isomorphic to the pull-back of the tautological log curve via some morphism $T^{\log} \stackrel{\text{def}}{=} S^{\log} \times_S T \to \overline{\mathcal{M}}_{g,r}^{\log}$ as a *stable log curve* (of type (g,r)). If $C \to S$ is smooth, i.e., every geometric fiber of $C \to S$ is free of nodes, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve* (of type (g,r)).

Definition 5. Let k be a field; $S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$; (g,r) a pair of nonnegative integers such that $2g-2+r>0$;

$$X^{\log} \to S$$

(cf. Definition 3, (vi)) a smooth log curve of type (g,r); $n \in \mathbb{Z}_{>0}$. Suppose the marked points of $X^{\log}$ are equipped with an ordering. Then the smooth log curve $X^{\log}$ over S determines a *classifying morphism* $S \to \overline{\mathcal{M}}_{g,r}^{\log}$. Thus, by pulling back via this morphism $S \to \overline{\mathcal{M}}_{g,r}^{\log}$ the morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ given by forgetting the last n marked points, we obtain a morphism of log schemes

$$X_n^{\log} \to S.$$

We shall refer to $X_n^{\log}$ as the *n-th log configuration space associated to* $X^{\log} \to S$. Note that $X_1^{\log} = X^{\log}$. Write $X_0^{\log} \stackrel{\text{def}}{=} S$.

Definition 6. Let "$n \in \mathbb{Z}_{>0}$"; (g,r) a pair of nonnegative integers such that $2g-2+r>0$; p a prime number; k an algebraic closed field of characteristic $\neq p$; $X^{\log}$ a smooth log curve over k of type (g,r); P a point of X_n.

(i) By abuse of notation, we shall use the notation "P" *both* for the corresponding point of the scheme X_n *and* for the reduced closed subscheme of X_n determined by this point. Then we shall say that P is a *log-full point* of $X_n^{\log}$ if

$$\dim(\mathcal{O}_{X_n,P}/I(P,\mathcal{M}_{X_n})) = 0$$

(cf. Definition 3, (iii)).

(ii) P parametrizes a pointed stable curve of type $(g,r+n)$ over k. Thus, P determines a semi-graph of anabelioids of pro-p PSC-type (cf. [CmbGC], Definition 1.1, (i)), which is in fact easily verified to be independent of the choice of geometric point lying over P. We shall write $\mathcal{G}_P$ for this semi-graph of anabelioids of pro-p PSC-type.

(iii) Let us fix an ordered set

$$\mathcal{C}_{r,n} \stackrel{\text{def}}{=} \{c_1, \dots, c_{r+n}\}.$$

Thus, by definition, we have a natural bijection $\mathcal{C}_{r,n} \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_P)$ that determines a bijection between the subset $\{c_1, \dots, c_r\}$ and the set of cusps of $X^{\log}$ (cf. [Hgsh], Definition 2.2, (v)). In the following, let us identify the set $\mathrm{Cusp}(\mathcal{G}_P)$ with $\mathcal{C}_{r,n}$. Write $x_i \stackrel{\mathrm{def}}{=} c_{r+i}$ for each $i \in \{1, \ldots, n\}$.

(iv) We shall refer to an irreducible divisor of X_n contained in the complement $X_n \setminus U_{X_n}$ of the interior U_{X_n} of X_n as a *log divisor* of $X_n^{\log}$. That is to say, a log divisor of $X_n^{\log}$ is an irreducible divisor of X_n whose generic point parametrizes a pointed stable curve with precisely two irreducible components (cf. [Hgsh], Definition 2.2, (vi)).

(v) Let V be a log divisor of $X_n^{\log}$. Then we shall write $\mathcal{G}_V$ for "$\mathcal{G}_P$" in the case where we take "P" to be the generic point of V.

(vi) Let $m \in \mathbb{Z}_{>1}$; $y_1, \ldots, y_m \in \mathcal{C}_{r,n}$ distinct elements such that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$. Then one verifies immediately — by considering *clutching morphisms* (cf. [Knu], Definition 3.8) — that there exists a unique log divisor V of $X_n^{\log}$, which we shall denote by $V(y_1, \ldots, y_m)$, that satisfies the following condition: the semi-graph of anabelioids $\mathcal{G}_V$ has precisely two vertices v_1, v_2 such that v_1 is of type $(0, m{+}1)$, v_2 is of type $(g, n{+}r{-}m{+}1)$, and $y_1, \ldots, y_m$ are cusps of $\mathcal{G}_V|_{v_1}$ (cf. [CbTpI], Definition 2.1, (iii)).

(vii) For each $i \in \{1, \ldots, n\}$, write $p_i \colon X_n^{\log} \to X^{\log}$ for the projection morphism of co-profile $\{i\}$ (cf. [MzTa], Definition 2.1, (ii)). Write

$$\iota \stackrel{\mathrm{def}}{=} (p_1, \ldots, p_n) \colon X_n^{\log} \to X^{\log} \times_k \cdots \times_k X^{\log}.$$

REMARK 2. *Let V be a log divisor of $X_n^{\log}$. Then let us observe that there exists a unique collection of distinct elements $y_1, \ldots, y_m \in C_{r,n}$ such that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$ and $V = V(y_1, \ldots, y_m)$. (Note that uniqueness holds even in the case where $g = 0$ (in which case $r \geq 3$), as a consequence of the condition that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$.)*

2. Geometric description of log divisors

In the present §2, let "$n \in \mathbb{Z}_{>1}$"; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; k an algebraic closed field; $X^{\log}$ a smooth log curve over k of type (g, r). In the present §2, we introduce various type of log divisors and we calculate the number of various type of log divisors.

DEFINITION 7. (i) For positive integers $i \in \{1, \ldots, n-1\}$, $j \in \{i + 1, \ldots, n\}$, write

$$\pi_{i,j} \colon X^n \stackrel{\mathrm{def}}{=} X \times_k \cdots \times_k X \to X^2 \stackrel{\mathrm{def}}{=} X \times_k X$$

for the projection of the fiber product of n copies of $X \to \mathrm{Spec}(k)$ to the i-th and j-th factors. Write $\delta'_{i,j}$ for the inverse image via $\pi_{i,j}$ of the image of the diagonal embedding $X \hookrightarrow X^2$. Write $\delta_{i,j}$ for the uniquely determined log divisor of $X_n^{\log}$ whose generic point maps to the generic point of $\delta'_{i,j}$ via the natural morphism $X_n \to X^n$ (cf. Definition 6, (vii)). We shall refer to the log divisor $\delta_{i,j}$ as a *naive diagonal* of $X_n^{\log}$.

(ii) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *tripodal divisor* if one of the vertices of $\mathcal{G}_V$ (cf. Definition 6, (vi)) is of type $(0,3)$ (cf. Definition 6, (vii); [CbTpI], Definition 2.3, (iii)).
(iii) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *(g,r)-divisor* if one of the vertices of $\mathcal{G}_V$ is of type (g,r).
(iv) Let V be a log divisor of $X_n^{\log}$. We shall say that V is a *drift diagonal* if there exist a naive diagonal δ and an automorphism α of $X_n^{\log}$ over S such that $V = \alpha(\delta)$.
(v) Let Λ be a set of drift diagonals of $X_n^{\log}$. Then we shall say that Λ is a *drift collection* of $X_n^{\log}$ if there exists an automorphism α of $X_n^{\log}$ over S such that $\Lambda = \{\alpha(V) \mid V \text{ is a naive diagonal}\}$.

Proposition 1. *The following hold:*
(i) $\{$*log divisors of* $X^{\log}\} = \{$*log-full points of* $X^{\log}\}$.
(ii) $\sharp\{$*log-full points of* $X^{\log}\} = r$.

Proof. Assertions (i), (ii) follow from Definition 6, (i), (iv). □

Proposition 2.

$$\{\textit{naive diagonals}\} \subseteq \{\textit{drift diagonals}\} \subseteq \{\textit{tripodal divisors}\} \subseteq \{\textit{log divisors}\},$$
$$\{(g,r)\textit{-divisors}\} \subseteq \{\textit{log divisors}\}.$$

Proof. It follows from Definition 6, (iv); Definition 7, (i), (ii), (iii), (iv); [Hgsh], Proposition 3.4, (i). □

Proposition 3. *Let* $m \in \{2, \dots, n+1\}$. *Write*

$$V_{[m]}^{\mathrm{vertical}} \stackrel{\mathrm{def}}{=} \{V(y_1,\dots,y_m) \mid y_1,\dots,y_m \in C_{r,n} \textit{ distinct elements } \textit{such that } \sharp(\{y_1,\dots,y_m\} \cap \{c_1,\dots,c_r\}) = 1\},$$

$$V_{[m]}^{\mathrm{naive}} \stackrel{\mathrm{def}}{=} \{V(y_1,\dots,y_m) \mid y_1,\dots,y_m \in C_{r,n} \textit{ distinct elements } \textit{such that } \sharp(\{y_1,\dots,y_m\} \cap \{c_1,\dots,c_r\}) = 0\},$$

$$V_{[m]} \stackrel{\mathrm{def}}{=} \begin{cases} V_{[m]}^{\mathrm{vertical}} \sqcup V_{[m]}^{\mathrm{naive}} & (2 \le m \le n) \\ V_{[n+1]}^{\mathrm{vertical}} & (m = n+1) \end{cases}$$

(cf. Remark 2). Note that $V_{[n+1]} = V_{[m]}^{\mathrm{vertical}} = \emptyset$ *if* $r = 0$. *Then*

$$\sharp V_{[m]}^{\mathrm{vertical}} = \binom{n}{m-1} r, \quad \sharp V_{[m]}^{\mathrm{naive}} = \binom{n}{m}.$$

Proof. It follows from Definition 6, (vi); Remark 2. □

Proposition 4. *Let V be a log divisor of $X_n^{\log}$. Write $V^{\log}$ for the log scheme obtained by equipping V with the log structure induced by the log structure of $X_n^{\log}$. Let $T^{\log} \to \mathrm{Spec}(k)$ be a smooth log curve of type $(0,3)$. For $m \in \mathbb{Z}_{>0}$, write $T_m^{\log}$ for the m-th log configuration space associated to $T^{\log} \to \mathrm{Spec}(k)$.*

(i) *Let* $V \in V_{[2]}$, *then* $V^{\log \le 1}$ *is isomorphic to* $U_{X_{n-1}}$.
(ii) *Let* $V \in V_{[n+1]}$, *then* $V^{\log \le 1}$ *is isomorphic to* $U_{T_{n-1}}$.
(iii) *Let* $m \in \{3, \dots, n\}$ *and* $V \in V_{[m]}$. *Then* $V^{\log \le 1}$ *is isomorphic to* $U_{T_{m-2}} \times_k U_{X_{n-m+1}}$.

PROOF. Assertions (i), (ii), (iii) follow from Definition 3, (v); [Hgsh], Lemma 6.1, (i), (ii), (iii). □

PROPOSITION 5.

$$\begin{aligned}\{\textit{log divisors}\} &= \coprod_{m=2}^{n+1} V_{[m]} \\ &= \coprod_{m=2}^{n+1} V_{[m]}^{\text{vertical}} \sqcup \coprod_{m=2}^{n} V_{[m]}^{\text{naive}},\end{aligned}$$

$$\begin{aligned}\sharp\{\textit{log divisors}\} &= \left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right)r + \left(\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}\right) \\ &= (2^n - 1)r + (2^n - 1 - n).\end{aligned}$$

PROOF. Note that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n, \quad \binom{n}{0} = 1, \quad \binom{n}{1} = n.$$

Then it follows from Remark 2; Proposition 3. □

PROPOSITION 6. *The following hold:*

(i) *If* $(g, r) \neq (0, 3)$, *then*

$$\{\textit{tripodal divisors}\} = V_{[2]} = V_{[2]}^{\text{vertical}} \sqcup V_{[2]}^{\text{naive}},$$

$$\sharp\{\textit{tripodal divisors}\} = \binom{n}{1} r + \binom{n}{2}.$$

(ii) *If* $(g, r) = (0, 3)$, *then*

$$\begin{aligned}\{\textit{tripodal divisors}\} &= V_{[2]} \sqcup V_{[n+1]} \\ &= V_{[2]}^{\text{vertical}} \sqcup V_{[n+1]}^{\text{vertical}} \sqcup V_{[2]}^{\text{naive}},\end{aligned}$$

$$\sharp\{\textit{tripodal divisors}\} = \left(\binom{n}{1} + \binom{n}{n}\right)r + \binom{n}{2}.$$

PROOF. Assertions (i), (ii) follow from Proposition 3; [Hgsh], Proposition 3.3, (ii), (iii). □

PROPOSITION 7. *(i)* *If* $(g, r) \neq (0, 3), (1, 1)$, *then there exists an isomorphism*

$$\begin{array}{ccc}\mathrm{Aut}_k(X_n^{\log}) & \xrightarrow{\sim} & \{\beta \in \mathrm{Aut}(C_{r,n}) \mid \beta(c_i) = c_i \textit{ for } i \in \{1, \dots, r\}\} \\ \alpha & \mapsto & \beta\end{array}$$

such that

$$\alpha(V(y_1,\ldots,y_m)) = V(\beta(y_1),\ldots,\beta(y_m))$$

for each log divisor $V(y_1,\ldots,y_m)$ *(cf. Remark 2). In particular,* $\mathrm{Aut}_k(X_n^{\log})$ *is isomorphic to the symmetric group on* n *letters* S_n*.*

(ii) If $(g,r)=(0,3)$ *or* $(1,1)$*, then there exists an isomorphism*

$$\begin{array}{ccc}\mathrm{Aut}_k(X_n^{\log}) & \xrightarrow{\sim} & \mathrm{Aut}(C_{r,n}) \\ \alpha & \mapsto & \beta\end{array}$$

such that

$$\alpha(V(y_1,\ldots,y_m)) = V(\beta(y_1),\ldots,\beta(y_m))$$

for each log divisor $V(y_1,\ldots,y_m)$ *(cf. Remark 2). In particular,* $\mathrm{Aut}_k(X_n^{\log})$ *is isomorphic to the symmetric group on* $r+n$ *letters* S_{r+n}*.*

Proof. Assertions (i), (ii) follow from the proof of [Hgsh], Proposition 3.4, (ii), (iii); □

Proposition 8. *Let* Λ *be a drift collection of* $X_n^{\log}$ *(cf. Definition 7, (v)). Then the following hold:*

(i)

$$\sharp\{\textit{naive diagonals}\} = \sharp\Lambda = \sharp V_{[2]}^{\text{naive}} = \binom{n}{2}.$$

(ii) If $(g,r)\neq(0,3),(1,1)$*, then*

$$\Lambda = \{\textit{drift diagonals}\} = \{\textit{naive diagonals}\} = V_{[2]}^{\text{naive}},$$

$$\sharp\{\textit{drift diagonals}\} = \binom{n}{2},$$

$$\sharp\{\textit{drift collections}\} = \binom{n}{n} = 1.$$

(iii) If $(g,r)=(0,3)$*, then there exist distinct elements* $y_1,\ldots,y_n\in\mathcal{C}_{3,n}$ *such that*

$$\Lambda = \{V(y_i,y_j) \mid 1\le i<j\le n\},$$

$$\{\textit{drift diagonals}\} = \{\textit{tripodal divisors}\},$$

$$\sharp\{\textit{drift diagonals}\} = \left(\binom{n}{1}+\binom{n}{n}\right)r + \binom{n}{2},$$

$$\sharp\{\textit{drift collections}\} = \binom{n+3}{n}.$$

(iv) If $(g,r)=(1,1)$*, then there exist distinct elements* $y_1,\ldots,y_n\in\mathcal{C}_{1,n}$ *such that*

$$\Lambda = \{V(y_i,y_j) \mid 1\le i<j\le n\},$$

$$\{\textit{drift diagonals}\} = \{\textit{tripodal divisors}\},$$

$$\sharp\{\textit{drift diagonals}\} = \binom{n}{1}r + \binom{n}{2},$$

$$\sharp\{\textit{drift collections}\} = \binom{n+1}{n}.$$

PROOF. Assertion (i) follows from Proposition 3; [Hgsh], Proposition 3.3, (i). Assertions (ii), (iii), (iv) follow from Proposition 6, (i), (ii); Proposition 7; [Hgsh], Proposition 3.4, (ii), (iii); the proof of [Hgsh], Lemma 8.10. $\square$

PROPOSITION 9. *The following hold:*

(i) If $r > 0$ and $(g, r) \neq (0, 3)$, then

$$\{(g, r)\textit{-divisor}\} = V_{[n+1]}$$

and

$$\sharp\{(g, r)\textit{-divisor}\} = \binom{n}{n} r.$$

(ii) If $(g, r) = (0, 3)$, then

$$\{(g, r)\textit{-divisor}\} = \{\textit{tripodal divisors}\}$$

and

$$\sharp\{(g, r)\textit{-divisor}\} = \left(\binom{n}{1} + \binom{n}{n}\right) r + \binom{n}{2}.$$

(iii) If $r = 0$, then

$$\{(g, r)\textit{-divisor}\} = \emptyset.$$

PROOF. Assertions (i), (ii), (iii) follow from Definition 7, (iii); Proposition 3; Proposition 6, (ii). $\square$

PROPOSITION 10. *Let $V_1 = V(y_1, \cdots, y_s)$, $V_2 = V(z_1, \cdots, z_t)$ be log divisors of $X_n^{\log}$ (cf. Remark 2). Then the following conditions are equivalent:*

(i) $V_1 \cap V_2 \neq \emptyset$.

(ii) there exists a log-full point contained in $V_1 \cap V_2$.

(iii) $\{y_1, \ldots, y_s\} \cap \{z_1, \ldots, z_t\} = \emptyset$ or $\{y_1, \ldots, y_s\} \subseteq \{z_1, \ldots, z_t\}$ or $\{y_1, \ldots, y_s\} \supseteq \{z_1, \ldots, z_t\}$ in $\mathcal{C}_{r,n}$.

PROOF. The equivalence (i) $\Longleftrightarrow$ (ii) follows from [Hgsh], Lemma 8.4. The implication (i) $\Longrightarrow$ (iii) follows immediately (cf. the proof of [Hgsh], Lemma 8.6). The implication (iii) $\Longrightarrow$ (i) follows immediately (cf. the proof of [Hgsh], Lemma 8.5). $\square$

PROPOSITION 11. *Let P be a log-full point of $X_n^{\log}$ and V a log divisor of $X_n^{\log}$.*

(i) $P \in V \Longleftrightarrow \mathcal{G}_V$ is obtained from $\mathcal{G}_P$ by generalization (cf. [CbTpI]*, Definition 2.8).*

(ii) $\mathrm{Cusp}(\mathcal{G}_P) = \mathrm{Cusp}(\mathcal{G}_V) = \mathcal{C}_{r,n}$ *(cf. Definition 2). In particular,* $\sharp\mathrm{Cusp}(\mathcal{G}_P) = r + n$.

(iii) $\sharp\mathrm{Node}(\mathcal{G}_V) = 1$ *(cf. Definition 2).*

(iv) If $r > 0$, then $\mathrm{Node}(\mathcal{G}_P) = n$.

(v) If $r > 0$, then there exist distinct log divisors $V_1, \ldots, V_n$ of $X_n^{\log}$ such that $P = V_1 \cap \cdots \cap V_n$.

(vi) If $r = 0$, then $\sharp\mathrm{Node}(\mathcal{G}_P) = n - 1$.

(vii) If $r = 0$, then there exist distinct log divisors $V_1, \ldots, V_{n-1}$ of $X_n^{\log}$ such that $P \in V_1 \cap \cdots \cap V_{n-1}$.

Proof. Assertion (i) follows from [Hgsh], Proposition 2.9. Assertion (ii) follows from Definition 6, (iii). Assertion (iii) follows from Definition 6, (iv). Assertions (iv), (vi) follow immediately from Definition 6, (i), together with the well-known modular interpretation of the log moduli stack that appear in the definition of $X_n^{\log}$ (where we recall that the log structure of this log stack arises from a divisor with normal crossings) (cf. [Hgsh], Proposition 3.6). Assertions (v), (vii) follow from [Hgsh], Proposition 3.7, (iii); the proof of [Hgsh], Proposition 3.7, (iii). $\square$

Proposition 12. *Let $p\colon X_n^{\log} \to X_{n-1}^{\log}$ be a projection and $m \in \{2, \ldots, n+1\}$. Write ${}^{\dagger}V_{[m]}$ for the set $V_{[m]} \subseteq 2^{X_n^{\log}}$ (cf. Proposition 3) and ${}^{\ddagger}V_{[m]}$ for the set $V_{[m]} \subseteq 2^{X_{n-1}^{\log}}$. Then the following hold:*

(i) Let V be a log divisor of $X_n^{\log}$. Then $p(V)$ is a log divisor of $X_{n-1}^{\log}$ or $p(V) = X_{n-1}$. Moreover, suppose that $p\colon X_n^{\log} \to X_{n-1}^{\log}$ is a projection of co-profile $\{n\}$ (cf. [MzTa]*, Definition 2.1, (ii)). Then*

$$p(V) = X_{n-1} \iff \text{there exists } y \in \mathcal{C}_{r,n} \setminus \{x_n\} \text{ such that } V = V(y, x_n).$$

(ii) $p({}^{\dagger}V_{[m]}) \subseteq {}^{\ddagger}V_{[m]} \cup {}^{\ddagger}V_{[m-1]}$ for $m \in \{3, \ldots, n\}$.

(iii) Let $V \in {}^{\dagger}V_{[2]}$. Then $p(V) = X_{n-1}$ or $p(V) \in {}^{\ddagger}V_{[2]}$. In particular,

$$p({}^{\dagger}V_{[2]}) = {}^{\ddagger}V_{[2]} \sqcup \{X_{n-1}\}.$$

(iv) $p({}^{\dagger}V_{[n+1]}) = {}^{\ddagger}V_{[n]}$. In particular,

$$\begin{aligned} &p(\{(g,r)\text{-divisor of } X_n^{\log}\}) \\ &= \begin{cases} \{(g,r)\text{-divisor of } X_{n-1}^{\log}\} & (\text{if } (g,r) \neq (0,3)) \\ \{(g,r)\text{-divisor of } X_{n-1}^{\log}\} \sqcup \{X_{n-1}\} & (\text{if } (g,r) = (0,3)). \end{cases} \end{aligned}$$

Proof. Assertions (i), (ii), (iii), (iv) follow immediately from the latter portion of Definition 6 (iv), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ and $X_{n-1}^{\log}$ (cf. [Hgsh], Proposition 4.1, (i), (ii)). $\square$

Proposition 13. *Let $p\colon X_n^{\log} \to X_{n-1}^{\log}$ be a projection. Then the following hold:*

(i) Let P be a log-full point of $X_n^{\log}$. Then $p(P)$ is a log-full point of $X_{n-1}^{\log}$.

(ii) Let V be a log divisor of $X_{n-1}^{\log}$. Then there exist distinct log divisors W_1, W_2 of $X_n^{\log}$ such that $W_1 \cup W_2 = p^{-1}(V)$. Moreover, suppose that

$p\colon X_n^{\log} \to X_{n-1}^{\log}$ is a projection of co-profile $\{n\}$ and $V = V(y_1, \ldots, y_s)$, where $\{y_1, \ldots, y_s\} \subseteq \mathcal{C}_{r,n-1}$. Then

$$\{W_1, W_2\} = \{V(y_1, \ldots, y_s), V(y_1, \ldots, y_s, x_n)\},$$

where $\{y_1, \ldots, y_s, x_n\} \subseteq \mathcal{C}_{r,n}$.

(iii) Suppose that $r > 0$. Let P be a log-full point of $X_{n-1}^{\log}$. Then there exist log divisors $V_1, \ldots, V_{n(n-1)}$ of $X_n^{\log}$ such that

$$\sharp\{V_1, \ldots, V_{n(n-1)}\} = 2n - 2,$$

$$V_{1+i(n-1)}, \cdots, V_{n-1+i(n-1)} \text{ are distinct log divisors,}$$

$$p^{-1}(P) = \bigcup_{i=0}^{n-1} (V_{1+i(n-1)} \cap \cdots \cap V_{n-1+i(n-1)}).$$

PROOF. Assertion (i) follows immediately from the latter portion of Definition 6 (iv), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$ and $X_{n-1}^{\log}$ (cf. [Hgsh], Proposition 4.1, (i), (ii)). Since $\sharp\mathrm{Vert}(\mathcal{G}_V) = 2$ (cf. Definition 2; Proposition 11, (iii)), assertion (ii) follows immediately. Next, we consider assertion (iii). Let W be an irreducible component of $p^{-1}(P)$. Since $\sharp\mathrm{Node}(\mathcal{G}_P) = n - 1$ (cf. Proposition 11, (iv)), it holds that $\sharp\mathrm{Node}(\mathcal{G}_W) = n - 1$. In particular, there exists distinct log divisors $V_{1+i(n-1)}, \cdots, V_{n-1+i(n-1)}$ such that $W = V_{1+i(n-1)} \cap \cdots \cap V_{n-1+i(n-1)}$. Since $\sharp\mathrm{Vert}(\mathcal{G}_P) = \sharp\mathrm{Node}(\mathcal{G}_P) + 1 = n$, it holds that

$$\sharp\{\text{irreducible components of } p^{-1}(P)\} = n.$$

Since $\sharp\mathrm{Node}(\mathcal{G}_P) = n - 1$, it holds that

$$\sharp\{V_1, \ldots, V_{n(n-1)}\} = 2n - 2.$$

This completes the proof of assertion (iii). □

PROPOSITION 14. *Let $p\colon X_n^{\log} \to X_{n-1}^{\log}$ be a projection and P a log-full point of $X_{n-1}^{\log}$.*

(i) If $r > 0$, then

$$\sharp\{\text{log-full points of } X_n^{\log} \text{ contained in } p^{-1}(P)\} = r + 2(n-1).$$

In particular,

$$\sharp\{\text{log-full points of } X_n^{\log}\} = \prod_{i=0}^{n-1} (r + 2i).$$

(ii) If $r = 0$, then

$$\sharp\{\text{log-full points of } X_n^{\log} \text{ contained in } p^{-1}(P)\} = 2n - 3.$$

PROOF. First, suppose that $r > 0$. Note that by [Hgsh], Proposition 3.7, (i), it holds that $\sharp\mathrm{Node}(\mathcal{G}_P) = n-1$ and $\sharp\mathrm{Cusp}(\mathcal{G}_P) = \sharp\mathcal{C}_{r,n-1} = r+n-1$. Thus, if follows immediately from Proposition 1, (ii). Next, suppose that $r = 0$. Then it holds that $\sharp\mathrm{Node}(\mathcal{G}_P) = n-2$ and $\sharp\mathrm{Cusp}(\mathcal{G}_P) = \sharp\mathcal{C}_{r,n-1} = n-1$. Thus, assetions (i), (ii) follow immedaitely from the various definitions involved. □

DEFINITION 8. Suppose that $r = 0$. Then we shall say that an irreducible subset W of $X_n^{\log}$ is *log-full curve* if each element of W is a log-full point.

PROPOSITION 15. *Suppose that $r = 0$. Let W be a log-full curve of $X_n^{\log}$ and $p\colon X_n^{\log} \to X_{n-1}^{\log}$ a projection.*

(i) There exists a projection $q\colon X_n^{\log} \to X^{\log}$ such that q induces a bijection $W \to X$.

(ii) If $n > 2$, then $p(W)$ is a log-full curve of $X_{n-1}^{\log}$.

(iii) There exist distinct log divisors $V_1, \ldots, V_{n-1}$ of $X_n^{\log}$ such that $W = V_1 \cap \cdots \cap V_{n-1}$.

(iv) Suppose that $n > 2$. Let Z be a log-full curve of $X_{n-1}^{\log}$. Then there exist log divisors $V_1, \ldots, V_{(n-1)(n-2)}$ of $X_n^{\log}$ such that

$$\sharp\{V_1, \ldots, V_{(n-1)(n-2)}\} = 2n-4,$$

$$V_{1+i(n-2)}, \cdots, V_{n-2+i(n-2)} \text{ are distinct log divisors},$$

$$p^{-1}(Z) = \bigcup_{i=0}^{n-2} (V_{1+i(n-2)} \cap \cdots \cap V_{n-2+i(n-2)}).$$

(v) Suppose that $n > 2$. Let Z be a log-full curve of $X_{n-1}^{\log}$. Then

$$\sharp\{\text{log-full curves of } X_n^{\log} \text{ contained in } p^{-1}(Z)\} = 2n-3.$$

(vi)

$$\sharp\{\text{log-full curves of } X_n^{\log}\} = \prod_{i=0}^{n-2} (2i+1).$$

PROOF. Assertions (i), (ii) follow immediately from the latter portion of Definition 6 (iv), together with the well-known modular interpretation of the log moduli stacks that appear in the definition of $X_n^{\log}$, $X_{n-1}^{\log}$, and $X^{\log}$. Assertion (iii) follows from Proposition 11, (vii). Next, we consider assertions (iv), (v). Let $P \in W$ be a log-full point of $X_n^{\log}$. Since $\mathrm{Node}(\mathcal{G}_P) = n-2$, $\mathrm{Cusp}(\mathcal{G}_P) = n-1$, assertions (iv), (v) follow immediately (cf. the proof of Proposition 13, (iii); the proof of Proposition 14, (i)). Assertion (vi) follows from assertion (v); Proposition 14, (ii). □

PROPOSITION 16. *Let $m \in \{2, \ldots, n+1\}$ and $V = V(y_1, \ldots, y_m) \in V_{[m]}$ a log divisor of $X_n^{\log}$, where $y_1, \ldots, y_m \in \mathcal{C}_{r,n}$ are distinct elements. Then the following hold:*

(i)

$$
\begin{aligned}
&\sharp\{W \subseteq X_n^{\log}\colon \textit{log divisors} \mid V \cap W \neq \emptyset\}\\
&= \begin{cases} \sharp\{\textit{log divisors of } T_{m-2}^{\log}\} + \sharp\{\textit{log divisors of } X_{n-m+1}^{\log}\} & (3 \leq m \leq n)\\ \sharp\{\textit{log divisors of } X_{n-1}^{\log}\} & (m = 2)\\ \sharp\{\textit{log divisors of } T_{n-1}^{\log}\} & (m = n+1)\end{cases}\\
&= 2^m + 2^{n-m+1}(r+1) - r - n - 4 \quad (2 \leq m \leq n+1).
\end{aligned}
$$

(ii) If $r > 0$, then

$$
\begin{aligned}
&\sharp\{\textit{log-full points of } X_n^{\log} \textit{ contained in } V\}\\
&= \begin{cases} \sharp\{\textit{log-full points of } T_{m-2}^{\log}\} \cdot \sharp\{\textit{log-full points of } X_{n-m+1}^{\log}\} & (3 \leq m \leq n)\\ \sharp\{\textit{log-full points of } X_{n-1}^{\log}\} & (m = 2)\\ \sharp\{\textit{log-full points of } T_{n-1}^{\log}\} & (m = n+1).\end{cases}
\end{aligned}
$$

(iii) If $r = 0$, then

$$
\begin{aligned}
&\sharp\{\textit{log-full curves of } X_n^{\log} \textit{ contained in } V\}\\
&= \begin{cases} \sharp\{\textit{log-full points of } T_{m-2}^{\log}\} \cdot \sharp\{\textit{log-full curves of } X_{n-m+1}^{\log}\} & (3 \leq m \leq n-1)\\ \sharp\{\textit{log-full curves of } X_{n-1}^{\log}\} & (m = 2)\\ \sharp\{\textit{log-full points of } T_{n-2}^{\log}\} & (m = n).\end{cases}
\end{aligned}
$$

(iv) If $(g, r) \neq (0, 3)$ and $V \in V_{[m]}^{\mathrm{naive}}$. Then

$$
\begin{aligned}
&\{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}\\
=&(\{V(y_i, y_j) \mid 1 \leq i < j \leq m\} \setminus \{V\})\\
&\sqcup \{V(z_1, z_2) \mid z_1, z_2 \in \{x_1, \ldots, x_n\} \setminus \{y_1, \ldots, y_m\}\colon \textit{distinct}\ \}\\
&\sqcup \{V(x, c) \mid x \in \{x_1, \ldots, x_n\} \setminus \{y_1, \ldots, y_m\},\ c \in \{c_1, \ldots, c_r\}\}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\sharp\{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}\\
&\qquad= \begin{cases} \binom{m}{2} + \binom{n-m}{2} + (n-m)r & (3 \leq m \leq n)\\ \binom{n-2}{2} + (n-2)r & (m = 2).\end{cases}
\end{aligned}
$$

(v) If $(g,r) \neq (0,3)$ *and* $V \in V_{[m]}^{\mathrm{vertical}}$. *We may assume that* $y_m \in \{c_1, \ldots, c_r\}$. *Then*

$$\begin{aligned} &\{W \subseteq X_n^{\log} \colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ =&\{V(y_i, y_j) \mid 1 \leq i < j \leq m-1\} \\ &\sqcup \{V(z_1, z_2) \mid z_1, z_2 \in \{x_1, \ldots, x_n\} \setminus \{y_1, \ldots, y_m\} \colon \textit{distinct}\ \} \\ &\sqcup \{V(y_k, y_m) \mid 1 \leq k \leq m-1\} \\ &\sqcup \{V(x, c) \mid x \in \{x_1, \ldots, x_n\} \setminus \{y_1, \ldots, y_m\},\ c \in \{c_1, \ldots, c_r\} \setminus \{y_m\}\}. \end{aligned}$$

In particular,

$$\begin{aligned} &\sharp\{W \subseteq X_n^{\log} \colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \begin{cases} \binom{m-1}{2} + \binom{n-m+1}{2} + (m-1) + (n-m+1)(r-1) & (3 \leq m \leq n+1) \\ \binom{n-1}{2} + (n-1)(r-1) & (m = 2). \end{cases} \end{aligned}$$

(vi) For each $2 \leq m \leq n$, *it holds that*

$$\begin{aligned} \binom{m}{2} + \binom{n-m}{2} + (n-m)r &= \binom{m-1}{2} + \binom{n-m+1}{2} + (m-1) + (n-m+1)(r-1) \\ \Longleftrightarrow \binom{n-2}{2} + (n-2)r &= \binom{n-1}{2} + (n-1)(r-1) \\ \Longleftrightarrow r &= 1 \end{aligned}$$

(cf. (iv), (v)).

(vii) If $(g,r) = (0,3)$. *Then*

$$\begin{aligned} &\{W \subseteq X_n^{\log} \colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ =&(\{V(y_i, y_j) \mid 1 \leq i < j \leq m\} \setminus \{V\}) \\ &\sqcup (\{V(z_1, z_2) \mid z_1, z_2 \in \mathcal{C}_{r,n} \setminus \{y_1, \ldots, y_m\} \colon \textit{distinct}\ \} \setminus \{V\}). \end{aligned}$$

In particular,

$$\begin{aligned} &\sharp\{W \subseteq X_n^{\log} \colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ &= \begin{cases} \binom{m}{2} + \binom{n+3-m}{2} & (3 \leq m \leq n) \\ \binom{n+1}{2} & (m = 2 \textit{ or } n+1). \end{cases} \end{aligned}$$

(viii) If $m = n+1$, *i.e.,* V *is a* (g,r)*-divisor. Then*

$$\sharp\{W \subseteq X_n^{\log} \colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} = \binom{n+1}{2}.$$

PROOF. Assertions (i), (ii), (iii) follow from Proposition 4, (i), (ii), (iii); Proposition 10. Assertions (iv), (v), (vii), (viii) follow from Proposition 10. Assertion (vi) follows immediately. □

PROPOSITION 17. *Let $m \in \{2, \ldots, n+1\}$. Write $a_m \overset{\mathrm{def}}{=} 2^m + 2^{n-m+1}(r+1) - r - n - 4$ (cf. Proposition 16, (i)). Then the following hold:*

(i) $a_{m+1} - a_m = 2^m - 2^{n-m}(r+1)$.

(ii) If $16 > 2^n(r+1)$, then a_m is a monotonically increasing sequence i.e.,

$$a_2 < a_3 < \cdots < a_{n+1}.$$

(iii) If $2^n < r+1$, then a_m is a monotonically decreasing sequence i.e.,

$$a_2 > a_3 > \cdots > a_{n+1}.$$

(iv) Let $m, m' \in \{2, \ldots, n+1\}$ be distinct elements. Then $a_m = a_{m'}$ if and only if $r+1$ is a power of 2 and $m + m' = \log_2(2^{n+1}(r+1))$.

PROOF. Assertions (i), (ii), (iii), (iv) follow immediately. □

PROPOSITION 18. *Suppose that $r > 0$. Let $m \in \{2, \ldots, n\}$. Then*

$$(V_{[2]} \sqcup V_{[n+1]}) \cap \{V \mid V \cap W \neq \emptyset \text{ for each } W \in V_{[m]}^{\mathrm{naive}}\} = V_{[n+1]}.$$

PROOF. Assertion follows immediately from Proposition 10. □

PROPOSITION 19. *Suppose that $r > 0$. Then the following hold:*

(i) Let $V \in V_{[n+1]}$. Then

$$\sharp\{W \in V_{[2]} \sqcup V_{[n+1]} \mid V \cap W \neq \emptyset,\ V \neq W\} = \frac{n^2+n}{2}.$$

(ii) Let $V \in V_{[2]}$. Then

$$\sharp\{W \in V_{[2]} \sqcup V_{[n+1]} \mid V \cap W \neq \emptyset,\ V \neq W\} = \frac{n^2+2nr-2r-5n+6}{2}.$$

(iii)

$$\frac{n^2+n}{2} = \frac{n^2+2nr-2r-5n+6}{2} \quad \text{if and only if} \quad r = 3$$

(cf. (i), (ii)).

PROOF. Assertions (i), (ii), (iii) follow immediately from Proposition 10. □

PROPOSITION 20. *Suppose that $g \neq 0$ and $r = 3$. Then the following hold:*

(i) Let $V \in V_{[n+1]} \sqcup V_{[2]}^{\mathrm{vertical}}$. Then

$$\sharp\{W \in V_{[n]} \mid V \cap W \neq \emptyset,\ V \neq W\} = n+1.$$

(ii) Let $V \in V_{[2]}^{\mathrm{naive}}$. Then

$$\sharp\{W \in V_{[n]} \mid V \cap W \neq \emptyset,\ V \neq W\} = 3n-5.$$

(iii)

$$n+1 = 3n-5 \quad \textit{if and only if} \quad n = 3$$

(cf. (i), (ii)).

Proof. Assertions (i), (ii), (iii) follow immediately from Proposition 10. $\square$

Proposition 21. *Suppose that $(g,r) \neq (0,3),(1,1)$ and $r > 0$. Let $m \in \{2,\ldots,n\}$ and $V \in V_{[m]}$. Then the following hold:*

(i) If $V \in V_{[m]}^{\mathrm{naive}}$, then

$$\sharp\{\sigma(V) \mid \sigma \in \mathrm{Aut}_k(X_n^{\log})\} = \binom{n}{m}.$$

(ii) If $V \in V_{[m]}^{\mathrm{vertical}}$, then

$$\sharp\{\sigma(V) \mid \sigma \in \mathrm{Aut}_k(X_n^{\log})\} = \binom{n}{m-1}.$$

(iii)

$$\binom{n}{m} = \binom{n}{m-1} \Longleftrightarrow n+1 = 2m$$

(cf. (i), (ii)).

(iv) $V \in V_{[m]}^{\mathrm{naive}} \Longleftrightarrow$ there exists $W \in V_{[m+1]}$ such that $W \cap \sigma(V) \neq \emptyset$ for each $\sigma \in \mathrm{Aut}_k(X_n^{\log})$.

Proof. Assertions (i), (ii), (iii) follow immediately from Proposition 7, (i). Assertion (iv) follows immediately (cf. Remark 2). $\square$

Proposition 22. *Suppose that $(g,r) \neq (0,3),(1,1)$, $r > 0$, and $n > 2$. Let $V \in V_{[2]} \sqcup V_{[n+1]}$. If $\sharp\{\sigma(V) \mid \sigma \in \mathrm{Aut}_k(X_n^{\log})\} = 1$, then $V \in V_{[n+1]}$.*

Proof. Assertion follows immediately from Proposition 7, (i); Proposition 21, (i), (ii). $\square$

Definition 9. Suppose that $r > 0$. Let $\mathcal{N}$ be a set of tripodal diagonals of $X_n^{\log}$ such that $\sharp\mathcal{N} = n$. Then we shall say that $\mathcal{N}$ is a *new vertical collection* of $X_n^{\log}$ if there exist an automorphism of $X_n^{\log}$ over k and $c \in \{c_1,\ldots,c_r\} \subseteq \mathcal{C}_{r,n}$ such that $\mathcal{N} = \alpha(\{V(c,x_i)\}_{i=1}^n)$.

Proposition 23. *Suppose that $r > 0$ and $(g,r) \neq (0,3),(1,1)$. Let $\mathcal{N}$ be a set of tripodal diagonals of $X_n^{\log}$ such that $\sharp\mathcal{N} = n$. Then the following conditions are equivalent:*

(i) $\mathcal{N}$ is a new vertical collection of $X_n^{\log}$.

(ii) There exists $c \in \{c_1,\ldots,c_r\} \subseteq \mathcal{C}_{r,n}$ such that $\mathcal{N} = \{V(x_i,c)\}_{i=1}^n$.

(iii) There exists a (g,r)-divisor V such that

$$\mathcal{N} = \{W \subseteq X_n^{\log} \colon \textit{tripodal divisors} \mid V \cap W \neq \emptyset\} \setminus \{\textit{naive diagonals}\}.$$

(iv) There exists a (g,r)-divisor V such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \cap W \neq \emptyset\}$$
$$\setminus \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V(x_1, \ldots, x_n) \cap W \neq \emptyset\}.$$

PROOF. The implication (i) $\Longrightarrow$ (ii) follows from Proposition 7, (i); Definition 9. The implication (ii) $\Longrightarrow$ (i) follows immediately from Definition 9. Next, we consider the implication (ii) $\Longrightarrow$ (iii). Write $V \stackrel{\text{def}}{=} V(x_1, \ldots, x_n, c)$. Then by Proposition 10, V is a (g,r)-divisor, and

$$\begin{aligned}&\{W \subseteq X_n^{\log}\colon \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}\\ =&\{V(y_1, y_2) \mid y_1, y_2 \in \{x_1, \ldots, x_n, c\} \text{ are distinct elements}\}\\ =&\mathcal{N} \sqcup \{\text{naive diagonals}\}.\end{aligned}$$

This completes the proof of the implication (ii) $\Longrightarrow$ (iii). Next, we consider the implication (iii) $\Longrightarrow$ (ii). Let V be a (g,r)-divisor. Then by Proposition 9; Remark 2, there exist distinct elements $y_1, \ldots, y_{n+1} \in \mathcal{C}_{r,n}$ such that $V = V(y_1, \ldots, y_{n+1})$, and $\sharp(\{y_1, \ldots, y_{n+1}\} \cap \{c_1, \ldots, c_r\}) \leq 1$. Note that $n+1 \geq \sharp\{x_1, \ldots, x_n\}$. Thus, $\sharp(\{y_1, \ldots, y_{n+1}\} \cap \{c_1, \ldots, c_r\}) = 1$, and $\{y_1, \ldots, y_n\} = \{x_1, \ldots, x_n\}$. We may assume $y_{n+1} \in \{c_1, \ldots, c_r\}$. By Proposition 10, it holds that

$$\begin{aligned}&\{W \subseteq X_n^{\log}\colon \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}\\ &= \{V(x_i, y_{n+1})\}_{i=1}^n \sqcup \{V(x_i, x_j) \mid 1 \leq i < j \leq n\}\\ &= \{V(x_i, y_{n+1})\}_{i=1}^n \sqcup \{\text{naive diagonals}\}.\end{aligned}$$

In particular, $\mathcal{N} = \{V(x_i, y_{n+1})\}_{i=1}^n$. This completes the proof of the implication (iii) $\Longrightarrow$ (ii). Next, we consider the equivalence (iii) $\Longleftrightarrow$ (iv). Let W is a tripodal divisor. Since $(g,r) \neq (0,3)$, it follows immediately from Proposition 8, (ii); Proposition 10, that $V(x_1, \ldots, x_n) \cap W \neq \emptyset \Longleftrightarrow W$ is a naive diagonal. This completes the proof of the equivalence (iii) $\Longleftrightarrow$ (iv). $\square$

PROPOSITION 24. *Suppose that $(g,r) = (0,3)$. Let $\mathcal{N}$ be a set of tripodal diagonals of $X_n^{\log}$ such that $\sharp\mathcal{V} = n$. Then the following conditions are equivalent:*

(i) $\mathcal{N}$ is a new vertical collection of $X_n^{\log}$.

(ii) There exist distinct elements $y_1, \ldots, y_{n+1} \in \mathcal{C}_{r,n}$ such that

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^n.$$

(iii) There exist a (g,r)-divisor V and a drift collection Λ such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \setminus \Lambda.$$

(iv) There exist a (g,r)-divisor ${}^{\dagger}V$ and ${}^{\ddagger}V \in V_{[3]} \sqcup V_{[n]}$ such that ${}^{\dagger}V \neq {}^{\ddagger}V$, ${}^{\dagger}V \cap {}^{\ddagger}V \neq \emptyset$, and

$$\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid {}^{\dagger}V \neq W, {}^{\dagger}V \cap W \neq \emptyset\}$$
$$\setminus\{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid {}^{\ddagger}V \cap W \neq \emptyset\}.$$

Proof. The equivalence (i) $\iff$ (ii) follows from Proposition 7, (ii); Definition 9. Next, we consider the implication (ii) $\Longrightarrow$ (iii). Write

$$V \stackrel{\text{def}}{=} V(y_1, \ldots, y_{n+1}), \text{ and } \Lambda \stackrel{\text{def}}{=} \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}.$$

Then by Proposition 8, (iii); Proposition 10, it holds that V is a (g, r)-divisor, Λ is a drift collection, and

$$\begin{aligned} &\{W \subseteq X_n^{\log} \colon \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \\ =&\{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} = \mathcal{N} \sqcup \Lambda. \end{aligned}$$

This completes the proof of the implication (ii) $\Longrightarrow$ (iii). Next, we consider the implication (iii) $\Longrightarrow$ (ii). By Proposition 9, (ii), we may assume that the (g, r)-divisor V is equal to $V(y_1, \ldots, y_{n+1})$. Then it follows from Proposition 10 that

$$\{W \colon \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} = \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\}.$$

Since

$$\sharp\{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} = \frac{(n+1)n}{2},$$

$$\sharp\Lambda = \frac{n(n-1)}{2}, \quad \sharp\mathcal{N} = n,$$

and

$$n = \frac{(n+1)n}{2} - \frac{n(n-1)}{2},$$

it holds that

$$\Lambda \subseteq \{W \colon \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}.$$

So we may assume

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$$

and

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^{n}.$$

This completes the proof of the implication (iii) $\Longrightarrow$ (ii). Next, we consider the implication (ii) $\Longrightarrow$ (iv). Write ${}^{\dagger}V \stackrel{\text{def}}{=} V(y_1, \ldots, y_{n+1})$, ${}^{\ddagger}V \stackrel{\text{def}}{=} V(y_1, \ldots, y_n)$, and $\Lambda \stackrel{\text{def}}{=} \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$. Then by Proposition 8, (iii); Proposition 10, it holds that ${}^{\dagger}V$ is a (g, r)-divisor, ${}^{\ddagger}V \in V_{[3]} \sqcup V_{[n]}$, Λ is a drift collection, and

$$\begin{aligned} &\{W \subseteq X_n^{\log} \colon \text{tripodal divisors} \mid {}^{\dagger}V \neq W, {}^{\dagger}V \cap W \neq \emptyset\} \\ =&\{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} \\ =&\mathcal{N} \sqcup \{W \subseteq X_n^{\log} \colon \text{tripodal divisors} \mid {}^{\ddagger}V \cap W \neq \emptyset\}. \end{aligned}$$

This completes the proof of the implication (ii) $\Longrightarrow$ (iv).

Next, we consider the implication (iv) $\Longrightarrow$ (ii). By Proposition 9, (ii), we may assume that the (g, r)-divisor ${}^{\dagger}V$ is equal to $V(y_1, \ldots, y_{n+1})$, and ${}^{\ddagger}V$ is equal to $V(z_1, \ldots, z_n)$. Since ${}^{\dagger}V \neq {}^{\ddagger}V$, and ${}^{\dagger}V \cap {}^{\ddagger}V \neq \emptyset$, it holds that

$\{z_1, \ldots, z_n\} \subseteq \{y_1, \ldots, y_{n+1}\}$ (cf. Proposition 10). Thus, we may assume that $z_i = y_i$ for $i \in \{1, \ldots, n\}$. Then it follows from Proposition 10 that

$$\{W\colon \text{tripodal divisors} \mid {}^{\dagger}V \neq W, {}^{\dagger}V \cap W \neq \emptyset\} = \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\},$$

$$\begin{aligned}\{W\colon \text{tripodal divisors} \mid {}^{\ddagger}V \cap W \neq \emptyset\} \cap \{V(y_i, y_j) \mid 1 \leq i < j \leq n+1\} \\ = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}.\end{aligned}$$

Thus,

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^{n}.$$

This completes the proof of the implication (iv) $\Longrightarrow$ (ii). □

PROPOSITION 25. *Suppose that $(g, r) = (1, 1)$. Let $\mathcal{N}$ be a collection of tripodal diagonals of $X_n^{\log}$ such that $\sharp\mathcal{N} = n$. Then the following conditions are equivalent:*

(i) $\mathcal{N}$ is a new vertical collection of $X_n^{\log}$.

(ii) There exist distinct elements $y_1, \ldots, y_{n+1} \in \mathcal{C}_{r,n}$ such that

$$\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^{n}.$$

(iii) There exist a (g, r)-divisor V and a drift collection Λ such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \cap W \neq \emptyset\} \setminus \Lambda.$$

(iv) There exist a (g, r)-divisor ${}^{\dagger}V \in V_{[n+1]}$ and ${}^{\ddagger}V \in V_{[n]}$ such that

$$\begin{aligned}\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid {}^{\dagger}V \cap W \neq \emptyset\} \\ \setminus \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid {}^{\ddagger}V \cap W \neq \emptyset\}.\end{aligned}$$

PROOF. This follows from the same proof of Proposition 24. □

PROPOSITION 26. *The following hold:*

(i) Let $\mathcal{N}$ be a new vertical collection. Then there exist a unique (g, r)-divisor V and a unique drift collection Λ such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \setminus \Lambda.$$

In particular, if $\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^{n}$, then

$$V = V(y_1, \ldots, y_{n+1}),$$

and

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}.$$

(ii) Let

$$\Lambda = \{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$$

be a drift collection. Then $\mathcal{N} = \{V(y_{n+1}, y_i)\}_{i=1}^{n}$ is a new vertical collection such that

$$\mathcal{N} = \{W \subseteq X_n^{\log}\colon \textit{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\} \setminus \Lambda,$$

where $y_{n+1} \in \mathcal{C}_{r,n} \setminus \{y_1, \ldots, y_n\}$, and $V = V(y_1, \ldots, y_{n+1})$.

PROOF. First, we consider assertion (i). The existence and final portion follows from by the proof of the implication (ii) $\Longrightarrow$ (iii) of Proposition 23; the proof of the implication (ii) $\Longrightarrow$ (iii) of Proposition 24; the proof of the implication (ii) $\Longrightarrow$ (iii) of Proposition 24. Next, we consider the uniqueness of assertion (i). By the implication (i) $\Longrightarrow$ (ii) of Proposition 23; the implication (i) $\Longrightarrow$ (ii) of Proposition 24; the implication (i) $\Longrightarrow$ (ii) of Proposition 25, we may assume the new vertical collection $\mathcal{N}$ is equal to $\{V(y_{n+1}, y_i)\}_{i=1}^{n}$. Since

$$\mathcal{N} \subseteq \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\},$$

by Proposition 10, it holds that V is equal to $V(y_1, \ldots, y_{n+1})$. By the proof of the implication (iii) $\Longrightarrow$ (ii) of Proposition 23; the proof of the implication (iii) $\Longrightarrow$ (ii) of Proposition 24; Proposition 10, it holds that

$$\Lambda \subseteq \{W \subseteq X_n^{\log} : \text{tripodal divisors} \mid V \neq W, V \cap W \neq \emptyset\}$$

and Λ is equal to $\{V(y_i, y_j) \mid 1 \leq i < j \leq n\}$. This completes the proof of assertion (i). Assertion (ii) follows immediately. □

PROPOSITION 27.

$$\sharp\{\textit{new vertical collections}\} = r\sharp\{\textit{drift collections}\}.$$

PROOF. Since $\sharp(\mathcal{C}_{r,n} \setminus \{y_1, \ldots, y_n\}) = r$, this follows from Proposition 26, (i), (ii). □

3. Mono-anabelian reconstruction

In the present §3, let $n \in \mathbb{Z}_{>0}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$ and "$r > 0$"; p a prime number; k an algebraic closed field of characteristic $\neq p$; $X^{\log}$ a smooth log curve over k of type (g, r). In the present §3, we give mono-anabelian algorithm to reconstruct (g, r, n) associated to intrinsic structure of the profinite group $\Delta^p(g, r, n)$ which is isomorphic to $\pi_1^p(X_n^{\log})$ (cf. Proposition 32, (v), below), and we give mono-anabelian algorithm to reconstruct a set GFS associated to intrinsic structure of $\Delta^p(g, r, n)$ and LFS (cf. Proposition 37, (iii), below).

DEFINITION 10. (i) Write $\pi_1(X_n^{\log})$ for the fundamental group of the log scheme $X_n^{\log}$ (for a suitable choice of basepoint). We refer to [Hsh], Theorem B.1, B.2, for more datails on fundamental groups of log schemes.

(ii) Write $\pi_1^p(X_n^{\log})$ for the maximal pro-p quotient of $\pi_1(X_n^{\log})$.

(iii) Let P be a log-full point of $X_n^{\log}$ (cf. Definition 6, (i)) and $P^{\log}$ the log scheme obtained by restricting the log structure of $X_n^{\log}$ to the reduced closed subscheme of X_n determined by P. Then we obtain an outer homomorphism $\pi_1(P^{\log}) \to \pi_1^p(X_n^{\log})$ (for suitable choices of basepoints). We shall refer to the subgroup $\mathrm{Im}(\pi_1(P^{\log}) \to \pi_1^p(X_n^{\log}))$, which is well-defined up to $\pi_1^p(X_n^{\log})$-conjugation, as a *log-full subgroup* at P.

(iv) Let H be a closed subgroup of $\pi_1^p(X_n^{\log})$. We shall say that H is a *generalized fiber subgroup* if there exist an automorphism α of $X_n^{\log}$ over k and a fiber subgroup $F \subseteq \pi_1^p(X_n^{\log})$ (cf. [MzTa], Definition 2.3, (iii)) such that $H = \beta(F)$, where β is an automorphism of $\pi_1^p(X_n^{\log})$ which arises from α (cf. [Hgsh], Definition 9.1; [HMM], Definition 2.1, (ii)).

(v) Let $\Delta^p(g,r,n)$ be a profinite group which is isomorphic to $\pi_1^p(X_n^{\log})$ (cf. Definition 10, (i), (ii)). Write LFS (resp. LD, $\mathcal{V}_{[m]}$, $\mathcal{V}_{[m]}^{\mathrm{naive}}$, $\mathcal{V}_{[m]}^{\mathrm{vertical}}$, TD, DD, GFS) for the set of subgroups of $\Delta^p(g,r,n)$ such that any isomorphism $\Delta^p(g,r,n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$ induces a bijection

$$\mathrm{LFS} \xrightarrow{\sim} \{\text{log-full subgroups of } \pi_1^p(X_n^{\log})\}$$

$$(\text{resp. } \mathrm{LD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to log divisors}\},$$

$$\mathcal{V}_{[m]} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in V_{[m]}\},$$

$$\mathcal{V}_{[m]}^{\mathrm{naive}} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in V_{[m]}^{\mathrm{naive}}\},$$

$$\mathcal{V}_{[m]}^{\mathrm{vertical}} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in V_{[m]}^{\mathrm{vertical}}\},$$

$$\mathrm{TD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to tripodal divisors}\},$$

$$\mathrm{DD} \xrightarrow{\sim} \{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to drift diagonals}\},$$

$$\mathrm{GFS} \xrightarrow{\sim} \{\text{generalized fiber subgroups of } \pi_1^p(X_n^{\log})\}).$$

Write DC for the set of subsets of DD such that any isomorphism

$$\Delta^p(g,r,n) \xrightarrow{\sim} \pi_1^p(X_n^{\log})$$

induces a bijection

$$\mathrm{DC} \xrightarrow{\sim} \{\{\text{inertia subgroups } \subseteq \pi_1^p(X_n^{\log}) \text{ associated to } V \in \Lambda\} \\ \mid \Lambda\text{: a drift collection}\}.$$

PROPOSITION 28. *The following hold:*

(i) $\pi_1^p(X_1^{\log})$ *is elastic, i.e., every topologically fnitely generated closed normal subgroup* $N \subseteq H$ *of an open subgroup* $H \subseteq \pi_1^p(X_1^{\log})$ *is either trivial or of fnite index in* $\pi_1^p(X_1^{\log})$.

(ii) If $n > 1$, *then* $\pi_1^p(X_n^{\log})$ *is not elastic.*

(iii) Let V *be a log divisor of* $X_n^{\log}$. *Then the inertia group associated to* V *is isomorphic to* $\mathbb{Z}_p$.

(iv) Let P *be a log-full point of* $X_n^{\log}$. *Then the log-full subgroup at* P *is isomorphic to* $\mathbb{Z}_p^{\oplus n}$.

(v)

$$\sharp\{\textit{conjugacy class of log-full subgroups} \subseteq \pi_1^p(X_n^{\log})\} = \prod_{i=0}^{n-1}(r+2i).$$

(vi)

$$\sharp\{\textit{conjugacy class of inertia groups associated to log divisors}\} \\ = (2^n - 1)r + (2^n - 1 - n).$$

PROOF. Assertion (i) follows from [MzTa], Theorem 1.5. Next, we consider assertion (ii). Let $F \subseteq \pi_1^p(X_n^{\log})$ be a generalized fiber subgroup (cf. Definition 10, (iv)). Since F is topologically finitely generated closed normal subgroup of $\pi_1^p(X_n^{\log})$ and F is of infinite index in $\pi_1^p(X_n^{\log})$ (cf. [MzTa], Remark 2.4.1), $\pi_1^p(X_n^{\log})$ is not elastic. This completes the proof of assertion (ii). Assertions (iii), (iv) follow from [Hgsh], Proposition 3.7, (iii). Assertion (v) follows from Proposition 14, (i). Assertion (vi) follows from Proposition 5. $\square$

PROPOSITION 29. *Suppose that $n > 1$. Then the following hold:*

(i) One may construct a p associated to the intrinsic structure of $\Delta^p(g, r, n)$, i.e.,

$$\Delta^p(g, r, n) \rightsquigarrow p.$$

(ii) One may construct an n associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS*, i.e.,*

$$(\Delta^p(g, r, n), \mathrm{LFS}) \rightsquigarrow n.$$

(iii) One may construct an r associated to the intrinsic structure of $\Delta^p(g, r, n)$ and LFS*, i.e.,*

$$(\Delta^p(g, r, n), \mathrm{LFS}) \rightsquigarrow r.$$

PROOF. Since $\Delta^p(g, r, n)$ is a pro-p group, assertion (i) follows immediate. Assertion (ii) follows from Proposition 28, (iv). Assertion (iii) follows from assertion (ii); Proposition 28, (v). $\square$

DEFINITION 11. Let $m \in \mathbb{Z}_{>0}$ and G a profinite group. Then we shall say that G is *unique factorization-like* if G satisfies the following properties:

(i) There exist nontrivial profinite subgroups $G_1, \ldots, G_m \subseteq G$ which are slim (cf. [MzTa], §0) and strongly indecomposable (cf. [MzTa], Definition 3.1) such that $G = G_1 \times \cdots \times G_m$.
(ii) Let $H_1, \ldots, H_m \subseteq G$ be nontrivial profinite subgroups which are slim and strongly indecomposable. If $G = H_1 \times \cdots \times H_m$, then there exists $\sigma \in S_m$ such that $G_i = H_{\sigma(i)}$ for each $i \in \{1, \ldots, m\}$.

PROPOSITION 30. *The following hold:*

(i) Let G be a unique factorization-like profinite group. Then one may construct a set $\{G_1, \ldots, G_m\}$ (cf. Definition 11) associated to the intrinsic structure of G, i.e.,

$$G \rightsquigarrow \{G_1, \ldots, G_m\}.$$

(ii) Let $G_1, \ldots, G_m$ are nontrivial profinite groups which are slim and strongly indecomposable. Then $G_1 \times \cdots \times G_m$ is unique factorization-like, and one

may construct a set $\{G_1, \ldots, G_m\}$ *associated to the intrinsic structure of* $G_1 \times \cdots \times G_m$*, i.e.,*

$$G_1 \times \cdots \times G_m \rightsquigarrow \{G_1, \ldots, G_m\}.$$

(iii) $\pi_1^p(X_n^{\log})$ *is slim and strongly indecomposable.*

Proof. Assertion (i) follows immediately. Assertion (ii) follows from assertion (i); [MzTa], Corollary 3.4. Assertion (iii) follows from [MzTa], Proposition 1.4; [MzTa], Proposition 3.2; [Ind], Theorem C, (i). □

Proposition 31. *Suppose that* $n > 1$*. Let* $m \in \{2, \ldots, n+1\}$*;* $V \in V_{[m]}$ *a log divisor of* $X_n^{\log}$*;* $I_V \subseteq \pi_1^p(X_n^{\log})$ *an inertia group associated to* V*;* $T^{\log}$ *a smooth log curve over* k *of type* $(0,3)$*. Then the following hold:*

(i) If $m = 2$*, then* $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ *is isomorphic to* $\pi_1^p(X_{n-1}^{\log})$*.*

(ii) If $m = n+1$*, then* $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ *is isomorphic to* $\pi_1^p(T_{n-1}^{\log})$*.*

(iii) If $m \in \{3, \ldots, n\}$*, then* $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ *is isomorphic to*

$$\pi_1^p(T_{m-2}^{\log}) \times \pi_1^p(X_{n-m+1}^{\log}).$$

(iv) $Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V$ *is unique factorization-like.*

(v)

$$\begin{aligned}&\{I_V \subseteq \pi_1^p(X_n^{\log}) \mid \textit{ inertia subgroup associated to some log divisor } V \\ &\qquad \textit{such that } Z_{\pi_1^p(X_n^{\log})}(I_V)/I_V \textit{ is strongly indecomposable}\} \\ &= \{I_V \subseteq \pi_1^p(X_n^{\log}) \mid \textit{ inertia subgroup associated to } V \in V_{[2]} \sqcup V_{[n+1]}\}.\end{aligned}$$

Proof. Assertion (i), (ii), (iii) follow from [Hgsh], Lemma 6.1, (i), (ii), (iii); [Hgsh], Remark 6.4. Assertion (iv) follows from assertions (i), (ii), (iii); Proposition 30, (ii), (iii). Assertion (v) follows from assertion (i), (ii), (iii); Proposition 30, (iii). □

Proposition 32. *Suppose that* $n > 1$*. Then the following hold;*

(i) One may construct a set LD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS*, i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathrm{LD}.$$

(ii) One may construct a set $\{\Delta^p(g,r,m), \Delta^p(0,3,m) \mid 1 \leq m \leq n-1\}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LD*, i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LD}) \rightsquigarrow \{\Delta^p(g,r,m), \Delta^p(0,3,m) \mid 1 \leq m \leq n-1\}.$$

(iii) One may construct a set $\{\Delta^p(g,r,1), \Delta^p(0,3,1)\}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* $\{\Delta^p(g,r,m), \Delta^p(0,3,m) \mid 1 \leq m \leq n-1\}$*, i.e.,*

$$(\Delta^p(g,r,n), \{\Delta^p(g,r,m), \Delta^p(0,3,m) \mid 1 \leq m \leq n-1\}) \rightsquigarrow \{\Delta^p(g,r,1), \Delta^p(0,3,1)\}.$$

(iv) One may construct a g *associated to the intrinsic structure of* $\Delta^p(g,r,n)$*,* $\{\Delta^p(g,r,1), \Delta^p(0,3,1)\}$*, and* r*, i.e.,*

$$(\Delta^p(g,r,n), \{\Delta^p(g,r,1), \Delta^p(0,3,1)\}, r) \rightsquigarrow g.$$

(v) One may construct (g, r, n) *associated to the intrinsic structure of* $\Delta^p(g, r, n)$ *and* LFS*, i.e.,*

$$(\Delta^p(g, r, n), \mathrm{LFS}) \rightsquigarrow (g, r, n).$$

Proof. Assertion (i) follows from [Hgsh], Theorem 4.7; [Hgsh], Lemma 5.1 (i), (ii), (iii). Assertion (ii) follows from Proposition 30, (ii), (iii); Proposition 31, (i), (ii), (iii), (iv). Assertion (iii) follows from Proposition 28, (i), (ii). Next, we consider assertion (iv). Write $N(g, r)$ for the number of generators of $\Delta^p(g, r, 1)$. Since $2g-2+r > 0$, it holds that $N(g, r) = 2g+r-1 \geq N(0, 3) = 2$ (cf. [MzTa], Remark 1.2.2). Thus,

$$\frac{\max(N(g, r), N(0, 3)) - r + 1}{2} = g.$$

This completes the proof of assertion (iv). Assertion (v) follows from assertions (i), (ii), (iii), (iv); Proposition 29, (ii), (iii). $\square$

Now, we consider a conjecture.

Conjecture 1. *Suppose that* $n > 1$*. Let* $m \in \{2, \ldots, n+1\}$*. Then the following hold:*

(i) One may construct a set $\mathcal{V}_{[m]}$ *associated to the intrinsic structure of* $\Delta^p(g, r, n)$ *and* LFS*, i.e.,*

$$(\Delta^p(g, r, n), \mathrm{LFS}) \rightsquigarrow \mathcal{V}_{[m]}.$$

(ii) One may construct a set $\mathcal{V}_{[m]}^{\mathrm{naive}}$*,* $\mathcal{V}_{[m]}^{\mathrm{vertical}}$ *associated to the intrinsic structure of* $\Delta^p(g, r, n)$ *and* LFS*, i.e.,*

$$(\Delta^p(g, r, n), \mathrm{LFS}) \rightsquigarrow \mathcal{V}_{[m]}^{\mathrm{naive}} \text{ and } \mathcal{V}_{[m]}^{\mathrm{vertical}}.$$

Remark 3. *Suppose that* $n > 1$*.*

(i) If $(g, r) \neq (0, 3)$*, then Conjecture 1, (i), follows immediately from Theorem 2, (i); Proposition 31, (i), (ii), (iii);* [Hgsh]*, Lemma 6.5, (iii), (iv).*

(ii) If $(g, r) \neq (0, 3), (1, 1)$*, then Conjecture 1, (ii), follows immediately from Conjecture 1, (i); Proposition 21, (iv).*

In this paper, we do not apply Theorem 2; Remark 3, (i), (ii).

Proposition 33. *Suppose that* $n > 1$*. Let* V_1, V_2 *be log divisors of* $X_n^{\log}$*. Then the following conditions are equivalent:*

(i) $V_1 \cap V_2 \neq \emptyset$*.*

(ii) There exists a log-full subgroup $A \subseteq \pi_1^p(X_n^{\log})$ *which contains inertia groups* I_{V_1}*,* I_{V_2} *associated to* V_1, V_2*.*

Proof. It follows immediately from Proposition 10; [Hgsh], Proposition 4.3; [Hgsh], Lemma 8.4. $\square$

Proposition 34. *Suppose that* $n > 1$*. Then the following hold:*

(i) *One may construct a set* $\mathcal{V}_{[2]} \sqcup \mathcal{V}_{[n+1]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LD, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LD}) \rightsquigarrow \mathcal{V}_{[2]} \sqcup \mathcal{V}_{[n+1]}.$$

(ii) *One may construct a set* $\mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LD, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LD}) \rightsquigarrow \mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}.$$

(iii) *Suppose that* $(g,r) \neq (0,3),(1,1)$. *Then one may construct sets* $\mathcal{V}_{[3]}, \mathcal{V}_{[n]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* $\mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}$, *i.e.,*

$$(\Delta^p(g,r,n), \mathcal{V}_{[3]} \sqcup \mathcal{V}_{[n]}) \rightsquigarrow \mathcal{V}_{[3]}, \mathcal{V}_{[n]}.$$

(iv) *Suppose that* $r \neq 3$. *Then one may construct sets* $\mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}.$$

(v) *Suppose that* $(g,r) \neq (0,3),(1,1)$ *and* $n > 2$. *Then one may construct sets* $\mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}.$$

(vi) *Suppose that* $g \neq 0$, $r = 3$, *and* $n \neq 3$. *Then one may construct sets* $\mathcal{V}_{[2]}^{\mathrm{naive}}, \mathcal{V}_{[2]}^{\mathrm{vertical}}, \mathcal{V}_{[n+1]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathcal{V}_{[2]}^{\mathrm{naive}}, \mathcal{V}_{[2]}^{\mathrm{vertical}}, \mathcal{V}_{[n+1]}.$$

(vii) *Suppose that* $(g,r) \neq (0,3)$. *Then one may construct sets* $\mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathcal{V}_{[2]}, \mathcal{V}_{[n+1]}.$$

(viii) *One may construct a set* TD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LD, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LD}) \rightsquigarrow \mathrm{TD}.$$

Proof. Assertion (i) follows from Proposition 31, (v). Assertion (ii) follows from Proposition 28, (i), (ii); Proposition 30, (ii); Proposition 31, (iii) (cf. also Proposition 32, (iii)). Next, we consider assertion (iii). Since $(g,r) \neq (0,3),(1,1)$, it holds that $N(g,r) > N(0,3) = N(1,1) = 2$ (cf. the proof of Proposition 32, (iv)). Thus, assertion (iii) follows immediately. Assertion (iv) follows from assertion (i); Proposition 19, (i), (ii), (iii); Proposition 32, (i); Proposition 33. Assertion (v) follows from assertions (i); Proposition 7, (i); Proposition 22; Proposition 32, (i). Assertion (vi) follows from assertions (i), (ii), (iii); Proposition 18; Proposition 20, (i), (ii), (iii); Proposition 32, (i); Proposition 33. Assertion (vii) follows from assertions (iv), (v), (vi). Assertion (viii) follows from assertions (i), (vii); Proposition 6, (i), (ii). □

Proposition 35. *Suppose that* $n > 1$. *Then the following hold:*

(i) *Suppose that* $(g,r) = (0,3), (1,1)$. *Then one may construct a set* DD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathrm{DD}.$$

(ii) *Suppose that* $(g,r) \neq (0,3)$ *and* $r > 1$. *Let* $m \in \{2, \ldots, n\}$. *Then one may construct sets* $\mathcal{V}_{[m]}^{\mathrm{naive}}$, $\mathcal{V}_{[m]}^{\mathrm{vertical}}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$, LFS, *and* $\mathcal{V}_{[m]}$, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}, \mathcal{V}_{[m]}) \rightsquigarrow \mathcal{V}_{[m]}^{\mathrm{naive}}, \mathcal{V}_{[m]}^{\mathrm{vertical}}.$$

(iii) *Suppose that* $(g,r) \neq (0,3), (1,1)$. *Let* $m \in \{2, \ldots, n\}$ *such that* $n+1 \neq m$. *Then one may construct sets* $\mathcal{V}_{[m]}^{\mathrm{naive}}$, $\mathcal{V}_{[m]}^{\mathrm{vertical}}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$, LFS, *and* $\mathcal{V}_{[m]}$, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}, \mathcal{V}_{[m]}) \rightsquigarrow \mathcal{V}_{[m]}^{\mathrm{naive}}, \mathcal{V}_{[m]}^{\mathrm{vertical}}.$$

(iv) *Suppose that* $(g,r) \neq (0,3), (1,1)$. *Let* $m \in \{2, \ldots, n\}$. *Then one may construct sets* $\mathcal{V}_{[m]}^{\mathrm{naive}}$, $\mathcal{V}_{[m]}^{\mathrm{vertical}}$ *associated to the intrinsic structure of* $\Delta^p(g,r,n)$, LFS, $\mathcal{V}_{[m]}$, *and* $\mathcal{V}_{[m+1]}$, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}, \mathcal{V}_{[m]}, \mathcal{V}_{[m+1]}) \rightsquigarrow \mathcal{V}_{[m]}^{\mathrm{naive}}, \mathcal{V}_{[m]}^{\mathrm{vertical}}.$$

(v) *One may construct a set* DD *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS, *i.e.,*

$$(\Delta^p(g,r,n), \mathrm{LFS}) \rightsquigarrow \mathrm{DD}.$$

Proof. Assertion (i) follows from Proposition 8, (iii), (iv); Proposition 34, (viii). Assertion (ii) follows from Proposition 16, (iv), (v), (vi); Proposition 33; Proposition 34, (viii). Assertion (iii) follows from Proposition 21, (i), (ii), (iii). Assertion (iv) follows from Proposition 21, (iv). Assertion (v) follows from assertion (i), (ii), (iii), (iv); Proposition 8, (ii) Proposition 34, (iii), (vii). □

Proposition 36. *Suppose that* $n > 1$. *Then the following hold:*

(i) $\iota \colon X_n^{\log} \to X^{\log} \times_k \cdots \times_k X^{\log}$ *(cf. Definition 6, (vii)) induces the outer surjective homomorphism*

$$\iota_\Delta \colon \pi_1^p(X_n^{\log}) \to \pi_1^p(X^{\log}) \times \cdots \times \pi_1^p(X^{\log}).$$

(ii) $\mathrm{Ker}\,\iota_\Delta$ *is topologically generated by the inertia groups associated to the naive digonals.*

(iii) *Let* V *be a log divisor of* $X_n^{\log}$ *then there exists an inertia group* I_V *associated to* V *which is contained in* $\mathrm{Ker}\,\iota_\Delta$ *if and only if*

$$V \in \coprod_{m=2}^{n} V_{[m]}^{\mathrm{naive}}.$$

(iv) *Let* Λ *be a drift collection. Write* I_Λ *for the subgroup of* $\pi_1^p(X_n^{\log})$ *which is topologically generated by the inertia groups associated to* $V \in \Lambda$. *Then* $\pi_1^p(X_n^{\log})/I_\Lambda$ *is isomorphic to* $\pi_1^p(X^{\log}) \times \cdots \times \pi_1^p(X^{\log})$.

(v) *One may construct a surjection* $\Delta^p(g,r,1)\times\cdots\times\Delta^p(g,r,1)\twoheadrightarrow\Delta^p(g,r,1)$ *associated to the intrinsic structure of* $\Delta^p(g,r,1)\times\cdots\times\Delta^p(g,r,1)$*, i.e.,*

$$\Delta^p(g,r,1)\times\cdots\times\Delta^p(g,r,1)\rightsquigarrow\Delta^p(g,r,1)\times\cdots\times\Delta^p(g,r,1)\twoheadrightarrow\Delta^p(g,r,1).$$

PROOF. Assertion (i) follows immediately. Assertion (ii) follows from [Hgsh], Lemma 7.1. Assertion (iii) follows from [Hgsh], Lemma 7.2. Assertion (iv) follows from assertion (i), (ii). Assertion (v) follows from Proposition 30, (ii), (iii). $\square$

PROPOSITION 37. *Suppose that* $n>1$*. Then the following hold:*

(i) *One may construct a set* DC *associated to the intrinsic structure of* $\Delta^p(g,r,n)$, LFS*, and* DD*, i.e.,*

$$(\Delta^p(g,r,n),\mathrm{LFS},\mathrm{DD})\rightsquigarrow\mathrm{DC}.$$

(ii) *One may construct a set* GFS *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* DC*, i.e.,*

$$(\Delta^p(g,r,n),\mathrm{DC})\rightsquigarrow\mathrm{GFS}.$$

(iii) *One may construct a set* GFS *associated to the intrinsic structure of* $\Delta^p(g,r,n)$ *and* LFS*, i.e.,*

$$(\Delta^p(g,r,n),\mathrm{LFS})\rightsquigarrow\mathrm{GFS}.$$

PROOF. Assertion (i) follows from [Hgsh], Proposition 8.12. Assertion (i) follows from Proposition 36, (iv), (v). Assertion (iii) follows from assertions (i), (ii); Proposition 35, (v). $\square$

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