

Reciprocity laws of Dedekind sums in characteristic p

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1 Introduction

The purpose of this paper is to report our recent results about Dedekind sums in finite characteristic.

For two relatively prime integers $a, c \in \mathbb{Z}$ with $c \neq 0$, we define the classical Dedekind sum in the form

$$s(a, c) = \frac{1}{c} \sum_{k \in (\mathbb{Z}/c\mathbb{Z}) - \{0\}} \cot\left(\pi \frac{k}{c}\right) \cot\left(\pi \frac{ak}{c}\right).$$

As is well known, $s(a, c)$ has the following properties:

- (1) $s(-a, c) = -s(a, c)$.
- (2) If $a \equiv a' \pmod{c}$, then $s(a, c) = s(a', c)$.
- (3)(Reciprocity law) For two relatively prime integers $a, c \in \mathbb{Z} - \{0\}$,

$$s(a, c) + s(c, a) = \frac{1}{3} \left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a} \right) - \text{sign}(ac).$$

The sum $s(a, c)$ is related to the module $\mathbb{Z}$. In [6], Sczech defined the Dedekind sum for a given lattice $\mathbb{Z}w_1 + \mathbb{Z}w_2$. Okada [5] introduced the Dedekind sum for a given function field. His Dedekind sum is related to the $\mathbb{F}_q[T]$ -module L corresponding to the Carlitz module (cf. 2.1). Inspired by Okada's result, we defined in [2] the Dedekind sum for a given finite field. Our previous result is related to a given finite field itself. Observing these former results, we have noticed that it is possible to define the Dedekind sum for a given lattice in finite characteristic. In this paper, we introduce Dedekind sums for lattices, and establish the reciprocity law for them.

Our results is divided into two parts. Section 2 deals with finite fields case. In section 3, we discuss function fields case.

2 Finite Dedekind sums

In this section, we use the following notations.

$K = \mathbb{F}_q$: the finite field with q elements

$\overline{K}$: an algebraic closure of K

$\sum'$: the sum over non-zero elements

$\prod'$: the product over non-zero elements

2.1 Lattices

A *lattice* Λ in $\overline{K}$ means a linear K -subspace in $\overline{K}$ of finite dimension. For such a lattice Λ , we define the Euler product

$$e_\Lambda(z) = z \prod'_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

The product defines a map $e_\Lambda : \overline{K} \rightarrow \overline{K}$. The map e_Λ has the following properties:

- e_Λ is $\mathbb{F}_q$ -linear and Λ -periodic.
- If $\dim_K \Lambda = r$, then $e_\Lambda(z)$ has the form

$$e_\Lambda(z) = \sum_{i=0}^r \alpha_i(\Lambda) z^{q^i}, \quad (1)$$

where $\alpha_0(\Lambda) = 1$ and $\alpha_r(\Lambda) \neq 0$.

- e_Λ has simple zeros at the points of Λ , and no other zeros.
- $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence we have

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}. \quad (2)$$

We recall the Newton formula for power sums of the zeros of a polynomial.

Proposition 1 (The Newton formula cf. [1]) *Let*

$$f(X) = X^n + c_1 X^{n-1} + \cdots + c_{n-1} X + c_n$$

be a polynomial, and $\alpha_1, \dots, \alpha_n$ the roots of $f(X)$. For each positive integer k , put

$$T_k = \alpha_1^k + \cdots + \alpha_n^k.$$

Then

$$\begin{aligned} T_k + c_1 T_{k-1} + \cdots + c_{k-1} T_1 + k c_k &= 0 \quad (k \leq n), \\ T_k + c_1 T_{k-1} + \cdots + c_{n-1} T_{k-n+1} + c_n T_{k-n} &= 0 \\ &\quad (k \geq n). \end{aligned}$$

Using this formula, we have

Proposition 2 *Let Λ be a lattice in $\overline{K}$, and take a non-zero element $a \in \overline{K}$. For $m = 1, 2, \dots, q-2$, we have*

$$\frac{a^m}{e_\Lambda(az)^m} = \sum_{x \in \Lambda} \frac{1}{(z - x/a)^m}.$$

For $b \in \overline{K} - \{0\}$, set

$$R(b) = \{\lambda/b \mid \lambda \in \Lambda\} - \{0\}.$$

Lemma 3

$$\sum_{x \in R(b)} x^{-m} = \begin{cases} 0 & (m = 1, \dots, q-2) \\ \alpha_1(\Lambda) b^{q-1} & (m = q-1) \end{cases},$$

where $\alpha_1(\Lambda)$ is as in (1).

2.2 Finite Dedekind sums

Observing that (2) is similar to a formula for $\pi \cot \pi z$, for a lattice Λ in $\overline{K}$, we define Dedekind sum as follows.

Definition 4 Set

$$\tilde{\Lambda} = \{x \in \overline{K} \mid x\lambda \in \Lambda \text{ for some } \lambda \in \Lambda\}.$$

We choose $c, a \in \overline{K} - \{0\}$ such that $a/c \notin \tilde{\Lambda}$. For $m = 1, \dots, q-2$, define

$$s_m(a, c)_\Lambda = \frac{1}{c^m} \sum'_{\lambda \in \Lambda} \left(\frac{\lambda}{c}\right)^{-q+1+m} e_\Lambda\left(\frac{a\lambda}{c}\right)^{-m}.$$

Moreover, we define

$$s_0(c)_\Lambda = s_0(a, c)_\Lambda = \sum'_{\lambda \in \Lambda} \left(\frac{\lambda}{c}\right)^{-q+1}.$$

We call $s_m(a, c)_\Lambda$ the m -th *finite Dedekind sum* for Λ .

Remark 5 In [2], we defined the Dedekind sum for $\Lambda = K$. Our definition generalizes it.

It follows from Lemma 3 that

$$s_0(c)_\Lambda = s_0(a, c)_\Lambda = \alpha_1(\Lambda) c^{q-1},$$

where $\alpha_1(\Lambda)$ is the coefficient of z^q in $e_\Lambda(z)$ as in (1).

The following result is analogous to the properties (1), (2) of the classical Dedekind sums in section one.

Proposition 6 *Dedekind sums $s_m(a, c)_\Lambda$ ($m = 1, \dots, q-1$) satisfy the following properties:*

- (1) *For any $\alpha \in K^*$, $s_m(\alpha a, c)_\Lambda = \alpha^{-m} s_m(a, c)_\Lambda$.*
- (2) *If $a, a' \in \overline{K}$ satisfy $a - a' \in c\Lambda$, then $s_m(a, c)_\Lambda = s_m(a', c)_\Lambda$.*

2.3 Reciprocity Law

We present the reciprocity law for our Dedekind sums. Let a, c be the elements of $\overline{K} - \{0\}$ such that $a/c \notin \tilde{\Lambda}$.

Theorem 7 (Reciprocity law I) *For $m = 1, \dots, q-2$, we have*

$$\begin{aligned} s_m(a, c)_\Lambda + (-1)^{m-1} s_m(c, a)_\Lambda \\ = \sum_{r=1}^{m-1} \frac{(-1)^{m-r} s_{m-r}(c, a)_\Lambda}{a^r c^r} \cdot \binom{m+1}{r} + \frac{s_0(c)_\Lambda + m \cdot s_0(a)_\Lambda}{a^m c^m}. \end{aligned}$$

As a corollary to this result, the next theorem is obtained.

Theorem 8 (Reciprocity law II) For $m = 1, \dots, q-2$, we have

$$s_m(a, c)_\Lambda + (-1)^{m-1} s_m(c, a)_\Lambda = \sum_{r=1}^{m-1} \frac{(-1)^{r-1} (s_{m-r}(a, c)_\Lambda + (-1)^{m-1} s_{m-r}(c, a)_\Lambda) \binom{m+1}{r}}{2a^r c^r} + \frac{(m + (-1)^{m-1}) (s_0(a)_\Lambda + (-1)^{m-1} s_0(c)_\Lambda)}{2a^m c^m}.$$

Example 9 Using the notation in the previous subsection, we have

$$s_1(a, c)_\Lambda + s_1(c, a)_\Lambda = \frac{\alpha_1(\Lambda) (a^{q-1} + c^{q-1})}{ac},$$

$$s_3(a, c)_\Lambda + s_3(c, a)_\Lambda = \frac{2s_2(a, c)_\Lambda + 2s_2(c, a)_\Lambda}{ac} - \frac{\alpha_1(\Lambda) (a^{q-1} + c^{q-1})}{a^3 c^3}.$$

In particular, if $\Lambda = K$, then $e_K(z) = z - z^q$, so that

$$s_1(a, c)_K + s_1(c, a)_K = -\frac{a^{q-1} + c^{q-1}}{ac},$$

$$s_3(a, c)_K + s_3(c, a)_K = \frac{2s_2(a, c)_K + 2s_2(c, a)_K}{ac} + \frac{a^{q-1} + c^{q-1}}{a^3 c^3}.$$

3 Dedekind sums for A -lattices

In this section we use the following notations. Let $\mathbb{F}_q$ be the finite field with q elements, $A = \mathbb{F}_q[T]$ the ring of polynomials in an indeterminate T , $K = \mathbb{F}_q(T)$ the quotient field of A , $||$ the normalized absolute value on K such that $|T| = q$, K_∞ the completion of K with respect to $||$, $\overline{K_\infty}$ a fixed algebraic extension of K_∞ , and C the completion of $\overline{K_\infty}$. We denote by $\sum'$, $\prod'$ the sum over non-zero elements, the product over non-zero elements, respectively.

3.1 A -lattices

A rank r A -lattice Λ in C means a finitely generated A -submodule of rank r in C that is discrete in the topology of C . For such an A -lattice Λ , define the Euler product

$$e_\Lambda(z) = z \prod'_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

The product converges uniformly on bounded sets in C , and defines a map $e_\Lambda : C \rightarrow C$. The map e_Λ has the following properties:

- e_Λ is entire in the rigid analytic sense, and surjective;
- e_Λ is $\mathbb{F}_q$ -linear and Λ -periodic;
- e_Λ has simple zeros at the points of Λ , and no other zeros;

- $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence we have

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}. \quad (3)$$

An $\mathbb{F}_q$ -linear ring homomorphism

$$\phi : A \rightarrow \text{End}_C(\mathbb{G}_a), \quad a \mapsto \phi_a$$

is said to be a *Drinfeld module* of rank r over C if ϕ satisfies

$$\phi_T = T + a_1\tau + \cdots + a_r\tau^r, \quad a_r \neq 0$$

for some $a_i \in C$, where τ denotes the q -th power morphism in $\text{End}_C(\mathbb{G}_a)$. Given a rank r A -lattice Λ , there exists a unique rank r Drinfeld module ϕ^Λ with the condition $e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$ for all $a \in A$. The association $\Lambda \mapsto \phi^\Lambda$ yields a bijection of the set of A -lattices of rank r in C with the set of Drinfeld modules of rank r over C . The rank one Drinfeld module ρ defined by $\rho_T = T + \tau$ is said to be the *Carlitz module*. We denote the A -lattice associated to ρ by L .

Using the Newton formula, we have

Proposition 10 *Let Λ be a rank r A -lattice in C , and take a non-zero element $a \in A$. For $m = 1, 2, \dots, q-2$, we have*

$$\frac{a^m}{e_\Lambda(az)^m} = \sum_{\lambda \in \Lambda/a\Lambda} \frac{1}{e_\Lambda(z - \lambda/a)^m}.$$

For any non-zero element $c \in A$, set

$$R(c) = \{e_\Lambda(\lambda/c) \mid \lambda \in \Lambda/c\Lambda\} - \{0\}.$$

In other words, $R(c)$ consists of the non-zero roots of $\phi_c(z)$. Let Λ be a rank r A -lattice in C corresponding to the Drinfeld module ϕ with

$$\phi_c(z) = \sum_{i=0}^n l_i(c)z^{q^i}, \quad (4)$$

where $n = r \deg c$, $l_n(c) \neq 0$, and $l_0(c) = c$.

Proposition 11

$$\sum_{\alpha \in R(c)} \alpha^{-m} = \begin{cases} 0 & (m = 1, \dots, q-2) \\ l_1(c)/c & (m = q-1) \end{cases}.$$

In particular, if $\phi = \rho$, the Carlitz module, then

$$\sum_{\alpha \in R(c)} \alpha^{-q+1} = \frac{c^{q-1} - 1}{T^q - T}.$$

3.2 Dedekind sums for A -lattices

Observing that (3) is similar to a formula for $\pi \cot \pi z$, for an A -lattice Λ of finite rank in C , let us define Dedekind sum as follows.

Definition 12 Let $a, c \in A - \mathbb{F}_q$ be relatively prime elements. In other words, assume $Aa + Ac = A$. For $m = 1, \dots, q - 2$, define

$$s_m(a, c)_\Lambda = \frac{1}{c^m} \sum_{\lambda \in \Lambda/c\Lambda} ' e_\Lambda \left(\frac{\lambda}{c} \right)^{-q+1+m} e_\Lambda \left(\frac{a\lambda}{c} \right)^{-m}.$$

Moreover, we define

$$s_0(c)_\Lambda = s_0(a, c)_\Lambda = \sum_{\lambda \in \Lambda/c\Lambda} ' e_\Lambda \left(\frac{\lambda}{c} \right)^{-q+1}.$$

We call $s_m(a, c)_\Lambda$ the m -th *Dedekind-Drinfeld sum* for Λ . In particular, if L is the rank one A -lattice associated to the Carlitz module ρ , then $s_m(a, c)_L$ is called the m -th *Dedekind-Carlitz sum*.

Remark 13 (1) In [5], Okada defines the Dedekind-Carlitz sum. Our definition generalizes it.

(2) It is possible to define Dedekind-Drinfeld sums in the same way for arbitrary function field instead of $K = \mathbb{F}_q(T)$.

It follows from Proposition 11 that

$$s_0(c)_\Lambda = s_0(a, c)_\Lambda = \frac{l_1(c)}{c},$$

where $l_1(c)$ is the coefficient of z^q in $\phi_c(z)$ as in (4). In particular, regarding the lattice L associated to the Carlitz module ρ ,

$$s_0(c)_L = s_0(a, c)_L = \frac{c^{q-1} - 1}{T^q - T}.$$

The following result is analogous to the properties (1), (2) of the classical Dedekind sums in section one.

Proposition 14 *Dedekind sums $s_m(a, c)_\Lambda$ ($m = 1, \dots, q - 2$) satisfy the following properties:*

- (1) For any $\alpha \in \mathbb{F}_q^*$, $s_m(\alpha a, c)_\Lambda = \alpha^{-m} s_m(a, c)_\Lambda$.
- (2) If $a, a' \in A$ satisfy $a - a' \in cA$, then $s_m(a, c)_\Lambda = s_m(a', c)_\Lambda$.
- (3) Take $b \in A$ with $ab - 1 \in cA$. Then $s_m(b, c)_\Lambda = c^{q-1-2m} s_{q-1-m}(a, c)_\Lambda$.

3.3 Reciprocity Law

We present the reciprocity law for our Dedekind sums. Let $a, c \in A - \mathbb{F}_q$ be relatively prime elements, and $m = 1, \dots, q - 2$.

Theorem 15 (Reciprocity law I)

$$\begin{aligned} s_m(a, c)_\Lambda + (-1)^{m-1} s_m(c, a)_\Lambda \\ = \sum_{r=1}^{m-1} \frac{(-1)^{m-r} s_{m-r}(c, a)_\Lambda}{a^r c^r} \cdot \binom{m+1}{r} + \frac{s_0(c)_\Lambda + m \cdot s_0(a)_\Lambda}{a^m c^m}. \end{aligned}$$

As a corollary to this result, the next theorem is obtained.

Theorem 16 (Reciprocity law II)

$$\begin{aligned} s_m(a, c)_\Lambda + (-1)^{m-1} s_m(c, a)_\Lambda = \\ \sum_{r=1}^{m-1} \frac{(-1)^{r-1} (s_{m-r}(a, c)_\Lambda + (-1)^{m-1} s_{m-r}(c, a)_\Lambda) \binom{m+1}{r}}{2a^r c^r} \\ + \frac{(m + (-1)^{m-1}) (s_0(a)_\Lambda + (-1)^{m-1} s_0(c)_\Lambda)}{2a^m c^m}. \end{aligned}$$

Example 17 Using the notation in the previous subsection, we have

$$\begin{aligned} s_1(a, c)_\Lambda + s_1(c, a)_\Lambda &= \frac{al_1(c) + cl_1(a)}{a^2 c^2}, \\ s_3(a, c)_\Lambda + s_3(c, a)_\Lambda &= \frac{2s_2(a, c)_\Lambda + 2s_2(c, a)_\Lambda}{ac} - \frac{al_1(c) + cl_1(a)}{a^4 c^4}. \end{aligned}$$

In particular, if $\Lambda = L$, then

$$\begin{aligned} s_1(a, c)_L + s_1(c, a)_L &= \frac{a^{q-1} + c^{q-1} - 2}{ac(T^q - T)}, \\ s_3(a, c)_L + s_3(c, a)_L &= \frac{2s_2(a, c)_L + 2s_2(c, a)_L}{ac} - \frac{a^{q-1} + c^{q-1} - 2}{a^3 c^3 (T^q - T)}. \end{aligned}$$

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