

SYMBOLIC DYNAMICAL SYSTEMS AND ENDOMORPHISMS ON C^* -ALGEBRAS

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1. INTRODUCTION

This article is a survey of the author's recent preprint entitled "Actions of symbolic dynamical systems on C^* -algebras", that is written based on the talk at RIMS, Jan. 2004. Details are given in the preprint.

In [CK], J. Cuntz and W. Krieger have founded a close relationship between symbolic dynamics and C^* -algebras (cf.[C2]). They constructed purely infinite simple C^* -algebras from irreducible topological Markov shifts. They have proved that their stabilization with gauge action is invariant under topological conjugacy of topological Markov shifts, so that K-theoretic invariants of the C^* -algebras with gauge actions yield invariants of topological Markov shifts. The invariants are the dimension group introduced by W. Krieger [Kr] and the Bowen-Franks group [BF]. They play a crucial rôle in the classification theory of topological Markov shifts. R. F. Williams has classified topological Markov shifts in terms of an algebraic relation of underlying matrices [Wi]. The algebraic relation is called a strong shift equivalence. M. Nasu generalized Williams's classification result to sofic shifts, that are subshifts coming from finite labeled graphs [N].

In [Ma], the author introduced a notion of λ -graph system, whose matrix version is called symbolix matrix system. A λ -graph system is a generalization of a finite labeled graph and presents a subshift. Conversely any subshift is presented by a λ -graph system, and the topological conjugacy classes of the subshifts are exactly corresponding to the strong shift equivalence classes of the symbolic matrix systems of the canonical λ -graph systems. He constructed C^* -algebras from λ -graph systems [Ma3] as a generalization of the above Cuntz-Krieger algebras. It has been proved that the outer conjugacy class of the stabilized gauge action is invariant under strong shift equivalence of the symbolic matrix system of the λ -graph system [Ma4]. Hence K-theoretic invariants of the C^* -algebras with gauge actions constructed from λ -graph systems yield invariants of topological conjugacy classes of subshifts.

In this survey article, we will study and generalize the above discussions in purely C^* -algebra setting. We will introduce a notion of C^* -symbolic dynamical system, that is a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital C^* -algebra $\mathcal{A}$ indexed by symbols Σ satisfying the condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$. A finite labeled graph gives rise to a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that $\mathcal{A}$ is commutative

and finite dimensional. Conversely, if the C^* -algebra $\mathcal{A}$ is commutative and finite dimensional, the C^* -symbolic dynamical system comes from a finite labeled graph. A λ -graph system gives rise to a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that $\mathcal{A}$ is commutative and AF. Conversely, if the C^* -algebra $\mathcal{A}$ is commutative and AF, the C^* -symbolic dynamical system comes from a λ -graph system ([Theorem 3.4]). We may prove that equivalence classes of the predecessor-separated λ -graph systems exactly correspond to the isomorphism classes of the predecessor-separated C^* -symbolic dynamical systems of the commutative AF-algebras ([Corollary 3.7]).

A C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a nontrivial subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ over Σ and a Hilbert C^* -right $\mathcal{A}$ -module $\mathcal{H}_{\mathcal{A}}^{\rho}$. For $\alpha_1, \dots, \alpha_k \in \Sigma$, a word $(\alpha_1, \dots, \alpha_k)$ is admissible for the subshift if and only if $(\rho_{\alpha_k} \circ \dots \circ \rho_{\alpha_1})(1) \neq 0$. The Hilbert C^* -right $\mathcal{A}$ -module $\mathcal{H}_{\mathcal{A}}^{\rho}$ has an orthogonal finite basis $\{u_{\alpha}\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi_{\rho}: \mathcal{A} \rightarrow L(\mathcal{H}_{\mathcal{A}}^{\rho})$. It is called a Hilbert C^* -symbolic bimodule over $\mathcal{A}$, and written as $(\phi_{\rho}, \mathcal{H}_{\mathcal{A}}^{\rho}, \{u_{\alpha}\}_{\alpha \in \Sigma})$.

We will consider C^* -algebras constructed from the Hilbert C^* -symbolic bimodules $(\phi_{\rho}, \mathcal{H}_{\mathcal{A}}^{\rho}, \{u_{\alpha}\}_{\alpha \in \Sigma})$. A general construction of C^* -algebras from Hilbert C^* -bimodules has been established by M. Pimsner [Pim] (see [Ka] for the case of von Neumann algebras). The C^* -algebras are called Cuntz-Pimsner algebras. Its ideal structure and simplicity conditions have been studied by Kajiwara-Pinzari-Watatani [KPW] and Muhly-Solel [MS] (see also [KW], [Sch]). The constructed C^* -algebra from the Hilbert C^* -symbolic bimodule $(\phi_{\rho}, \mathcal{H}_{\mathcal{A}}^{\rho}, \{u_{\alpha}\}_{\alpha \in \Sigma})$ is denoted by $\mathcal{A} \rtimes_{\rho} \Lambda$, where Λ is the subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ associated with the C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$. We call the algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ the C^* -symbolic crossed product of $\mathcal{A}$ by the subshift Λ . As in [Pim] (cf. [KPW]), the gauge action, denoted by $\hat{\rho}$, on the algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ of the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is defined as a generalization of that of the Cuntz-Krieger algebras. We remark that Pimsner showed the following fact [Pim]: For every Hilbert C^* -bimodule E over a C^* -algebra $\mathcal{A}$, if $\mathcal{A}$ is commutative and finite dimensional, and if E is projective and finitely generated, the associated C^* -algebra is a Cuntz-Krieger algebra. We present the following theorem

Theorem A (Theorem 5.2). *Let $(\mathcal{A}, \rho, \Sigma)$ be a C^* -symbolic dynamical system and Λ be the associated subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$. Assume that $\mathcal{A}$ is commutative.*

- (i) *If $\mathcal{A} = \mathbb{C}$, the subshift Λ is the full shift $\Sigma^{\mathbb{Z}}$, and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$.*
- (ii) *If $\mathcal{A}$ is finite dimensional, the subshift Λ is a sofic shift $\Lambda_{\mathcal{G}}$ presented by a left-resolving labeled graph $\mathcal{G}$, and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is a Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph. Conversely, for any sofic shift, that is presented by a left-resolving labeled graph $\mathcal{G}$, there exists a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the sofic shift, the algebra $\mathcal{A}$ is finite dimensional, and the algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph.*
- (iii) *If $\mathcal{A}$ is an AF-algebra, there uniquely exists a λ -graph system $\mathcal{L}$ up to equivalence such that the subshift Λ is presented by $\mathcal{L}$ and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the C^* -algebra $\mathcal{O}_{\mathcal{L}}$ associated with the λ -graph system $\mathcal{L}$. Conversely, for any subshift, that is presented by a left-resolving λ -graph system $\mathcal{L}$, there exists a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated*

subshift is the subshift presented by $\mathcal{L}$, the algebra $\mathcal{A}$ is a commutative AF-algebra, and the algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the C^* -algebra $\mathcal{O}_{\mathcal{L}}$ associated with the λ -graph system $\mathcal{L}$.

We will introduce notions of strong shift equivalence and shift equivalence of C^* -symbolic dynamical systems, that are generalizations of those of square non-negative matrices defined by Williams [Wi], of finite symbolic square matrices defined by Nasu [N] and Boyle-Krieger [BK] and of symbolic matrix systems defined by [Ma]. They are generalizations of conjugacy of single automorphisms of C^* -algebras. Strong shift equivalence and shift equivalence of Hilbert C^* -symbolic bimodules are introduced. We know that two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent (resp. shift equivalent) if and only if their associated Hilbert C^* -symbolic bimodules $(\phi_{\rho}, \mathcal{H}_{\mathcal{A}}^{\rho}, \{u_{\alpha}\}_{\alpha \in \Sigma})$ and $(\phi_{\rho'}, \mathcal{H}_{\mathcal{A}'}^{\rho'}, \{u'_{\alpha}\}_{\alpha \in \Sigma'})$ are strong shift equivalent (resp. shift equivalent). A notion of strong shift equivalence of C^* -symbolic crossed products with gauge actions is introduced. We finally obtain the following theorem.

Theorem B(Theorem 7.5). *Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two C^* -symbolic dynamical systems. Let Λ and Λ' be their associated subshifts $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ and $\Lambda_{(\mathcal{A}', \rho', \Sigma')}$ respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, then*

- (i) *the subshifts Λ and Λ' are topologically conjugate,*
- (ii) *the C^* -symbolic crossed products $(\mathcal{A} \rtimes_{\rho} \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ with gauge actions are strong shift equivalent, and*
- (iii) *the stabilized gauge actions $(\mathcal{A} \rtimes_{\rho} \Lambda \otimes \mathcal{K}, \hat{\rho} \otimes \text{id}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda' \otimes \mathcal{K}, \hat{\rho}' \otimes \text{id}, \mathbb{T})$ are cocycle conjugate, where $\mathcal{K}$ denotes the C^* -algebra of all compact operators on a separable infinite dimensional Hilbert space.*

The result (iii) is a generalization of the main result of [Ma4] (cf.[CK:3.8. Theorem]).

We define the K -groups $K_*(\mathcal{A}, \rho, \Sigma)$, the Bowen-Franks groups $BF^*(\mathcal{A}, \rho, \Sigma)$ and the dimension groups $D_*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting for $* = 0, 1$

$$K_*(\mathcal{A}, \rho, \Sigma) = K_*(\mathcal{A} \rtimes_{\rho} \Lambda), \quad BF^*(\mathcal{A}, \rho, \Sigma) = \text{Ext}_*(\mathcal{A} \rtimes_{\rho} \Lambda),$$

$$D_*(\mathcal{A}, \rho, \Sigma) = (K_*((\mathcal{A} \rtimes_{\rho} \Lambda) \rtimes_{\hat{\rho}} \mathbb{T}), \hat{\rho}_*)$$

where $\hat{\rho}_*$ is the automorphism of $K_*((\mathcal{A} \rtimes_{\rho} \Lambda) \rtimes_{\hat{\rho}} \mathbb{T})$ induced from the dual action $\hat{\rho}$ of the gauge action $\hat{\rho}$. The dimension groups and the Bowen-Franks groups are generalizations of those groups for a finite square nonnegative matrix, that is regarded as a finite labeled graph for which labels are edges itself (cf.[BF], [Kr], [LM]). Then Theorem B implies that all the abelian groups $K_*(\mathcal{A}, \rho, \Sigma)$, $BF^*(\mathcal{A}, \rho, \Sigma)$ and $D_*(\mathcal{A}, \rho, \Sigma)$ are invariant under strong shift equivalence of C^* -symbolic dynamical systems (Proposition 7.6).

2. λ -GRAPH SYSTEMS AND ITS C^* -ALGEBRAS

Let Σ be a finite set with its discrete topology. We call it an alphabet. Each element of Σ is called a symbol or a label. Let $\Sigma^{\mathbb{Z}}$ be the infinite product spaces

$\prod_{i \in \mathbb{Z}} \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $(\sigma(x_i))_{i \in \mathbb{Z}} = (x_{i+1})_{i \in \mathbb{Z}}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_{\Lambda})$ is called a subshift. We write the subshift as Λ for brevity. A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a word of length $|\mu| = k$. For a subshift Λ , we denote by Λ^l the set of all admissible words of length l of Λ . By a symbolic matrix $\mathcal{B}$ over Σ we mean a finite matrix with entries in finite formal sums of elements of Σ . A square symbolic matrix $\mathcal{B}$ naturally gives rise to a finite labeled directed graph which we denote by $\mathcal{G}_{\mathcal{B}}$. The labeled directed graph defines a subshift over Σ which consists of all infinite labeled sequences following the labeled directed edges in $\mathcal{G}_{\mathcal{B}}$. Such a subshift is called a sofic shift presented by $\mathcal{G}_{\mathcal{B}}$ and written as $\Lambda_{\mathcal{G}_{\mathcal{B}}}$ ([Fi],[Kr2],[Kr3],[We], cf. [Kit],[LM]). Throughout this paper, a labeled graph means a labeled directed graph with finite vertices and finite directed edges such as every vertex has at least one in-coming edge and at least one out-going edge.

Let $\mathcal{B}$ and $\mathcal{B}'$ be symbolic matrices over Σ and Σ' respectively. Let ϕ be a bijection from a subset of Σ onto a subset of Σ' , that is called a specification. Following M. Nasu in [N],[N2], we say that $\mathcal{B}$ and $\mathcal{B}'$ are specified equivalent under specification ϕ if $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by replacing every symbol α appearing in $\mathcal{B}$ by $\phi(\alpha)$. We write it as $\mathcal{B} \stackrel{\phi}{\sim} \mathcal{B}'$. Let $\mathbb{Z}_+$ be the set of all nonnegative integers.

Recall that a λ -graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is a directed Bratteli diagram with a vertex set $V = \bigcup_{l \in \mathbb{Z}_+} V_l$, an edge set $E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1}$, and a map $\lambda : E \rightarrow \Sigma$, and that is supplied with a sequence of surjective maps $\iota (= \iota_{l,l+1}) : V_{l+1} \rightarrow V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets $V_l, l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l , its terminal vertex $t(e)$ in V_{l+1} and its label $\lambda(e)$ in Σ . Every vertex in V has successors and every vertex in V , except V_0 , has predecessors. It is then required that for $u \in V_{l-1}$ and $v \in V_{l+1}$, there exists a bijective correspondence between the edge set $\{e \in E_{l,l+1} | t(e) = v, \iota(s(e)) = u\}$ and the edge set $\{e \in E_{l-1,l} | s(e) = u, t(e) = \iota(v)\}$ that preserves labels. The required property is called the local property.

Two λ -graph systems $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ and $\mathfrak{L}' = (V', E', \lambda', \iota')$ over Σ' are said to be isomorphic if there exist bijections $\Phi_V : V_l \rightarrow V'_l$, $\Phi_E : E_{l,l+1} \rightarrow E'_{l,l+1}$ and a specification $\phi : \Sigma \rightarrow \Sigma'$ such that $\Phi_V(s(e)) = s(\Phi_E(e))$, $\Phi_V(t(e)) = t(\Phi_E(e))$ and $\lambda'(\Phi_E(e)) = \phi(\lambda(e))$ for $e \in E$, and $\iota'(\Phi_V(v)) = \Phi_V(\iota(v))$ for $v \in V$.

A symbolic matrix system over Σ consists of a sequence of pairs of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}$ have their entries in formal sums of Σ and the matrices $I_{l,l+1}$ have their entries in $\{0, 1\}$. The matrices $\mathcal{M}_{l,l+1}$ and $I_{l,l+1}$ have the same size for each $l \in \mathbb{Z}_+$ and satisfy the following relations

$$(2.1) \quad I_{l,l+1} \mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1} I_{l+1,l+2}, \quad l \in \mathbb{Z}_+.$$

The matrices $I_{l,l+1}, l \in \mathbb{Z}_+$ have one 1 in each column and at least one 1 in each row. We denote it by $(\mathcal{M}, I)$. A λ -graph system naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The edges from a vertex $v_i^l \in V_l$ to a vertex $v_j^{l+1} \in V_{l+1}$ are given by the (i, j) -component $\mathcal{M}_{l,l+1}(i, j)$ of the matrix $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection $\iota_{l,l+1}$ from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$.

Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if there exists a specification ϕ from Σ to Σ' and an $m(l) \times m(l)$ -square permutation matrix P_l for each $l \in \mathbb{N}$ such that

$$P_l \mathcal{M}_{l,l+1} \stackrel{\phi}{\simeq} \mathcal{M}'_{l,l+1} P_{l+1}, \quad P_l I_{l,l+1} = I'_{l,l+1} P_{l+1} \quad \text{for } l \in \mathbb{Z}_+.$$

There exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of λ -graph systems.

Let $\mathcal{G} = (G, \lambda)$ be a labeled graph with finite directed graph G and labeling λ . Let $\{v_1, \dots, v_n\}$ be the vertex set of G . Put $V_l = \{v_1, \dots, v_n\}$ for all $l \in \mathbb{Z}_+$. We regard the sets $V_l, l \in \mathbb{Z}_+$ as disjoint sets. Define $\iota : V_{l+1} \rightarrow V_l$ by $\iota(v_i) = v_i$ for $i = 1, \dots, n$. Write labeled edges from V_l to V_{l+1} for $l \in \mathbb{N}$ following the directed graph G with labeling λ . The resulting labeled Bratteli diagram with ι becomes a λ -graph system. A labeled graph and also a λ -graph system are said to be left-resolving if different edges with the same label have different terminals. Hence a labeled graph defines a λ -graph system such that if the labeled graph is left-resolving, so is the λ -graph system. We call the resulting λ -graph system the associated λ -graph system with the labeled graph. We note that any sofic shift may be presented by left-resolving labeled graph ([Kr2],[Kr3],[We]).

A λ -graph system $\mathcal{L}$ gives rise to a subshift $\Lambda_{\mathcal{L}}$ on the sequence space of labels appearing in the labeled Bratteli diagram. We say that $\mathcal{L}$ presents the subshift $\Lambda_{\mathcal{L}}$. A canonical method to construct a λ -graph system from an arbitrary subshift Λ has been introduced in [Ma]. The λ -graph system and its symbolic matrix system are said to be canonical for the subshift and written as $\mathcal{L}^{\Lambda}$ and $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ respectively.

Let $\mathcal{L} = (V, E, \lambda, \iota)$ be a λ -graph system over Σ . For a vertex $v \in V_l$, we denote by $\Gamma_{\mathcal{L},l}^-(v)$ the set of all label sequences of length l in $\mathcal{L}$ that start at vertices of V_0 and terminate at v . We say that $\mathcal{L}$ is predecessor-separated if for $u, v \in V_l$ the condition $\Gamma_{\mathcal{L},l}^-(u) = \Gamma_{\mathcal{L},l}^-(v)$ implies $u = v$. The canonical λ -graph systems are left-resolving and predecessor-separated.

We will introduce an equivalence relation of predecessor-separated λ -graph systems. Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be the symbolic matrix systems over Σ and Σ' respectively. We denote by $m(l)$ the row size of the matrix $\mathcal{M}_{l,l+1}$ and by $m'(l)$ that of $\mathcal{M}'_{l,l+1}$ respectively. We say that $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are *equivalent* if there exist $N \in \mathbb{Z}_+$ and a bijection $\pi : \Sigma \rightarrow \Sigma'$ such that for each $l \in \mathbb{Z}_+$, there exist an $m(l) \times m'(N+l)$ matrix H_l over $\{0, 1\}$ and an $m'(l) \times m(N+l)$ matrix K_l over $\{0, 1\}$ satisfying the following equations:

$$\mathcal{M}_{l,l+1} H_{l+1} \stackrel{\pi}{\simeq} H_l \mathcal{M}'_{l+N,l+N+1}, \quad \mathcal{M}'_{l,l+1} K_{l+1} \stackrel{\pi^{-1}}{\simeq} K_l \mathcal{M}_{l+N,l+N+1},$$

$$I_{l,l+1} H_{l+1} = H_l I'_{l+N,l+N+1}, \quad I'_{l,l+1} K_{l+1} = K_l I_{l+N,l+N+1}$$

and

$$H_l K_{N+l} = I_{l,2N+l}, \quad K_l H_{N+l} = I'_{l,2N+l}.$$

We write this equivalence relation as $(\mathcal{M}, I) \cong (\mathcal{M}', I')$. Two λ -graph systems are called *equivalent* if their respect symbolic matrix systems are equivalent.

In the rest of this section, we briefly review the C^* -algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ -graph system $\mathfrak{L}$. The C^* -algebras have been originally constructed in [Ma3] as groupoid C^* -algebras of certain r -discrete groupoids constructed from continuous graphs in the sense of Deaconu ([De],[De2],[De3],cf.[Re]) obtained by the λ -graph systems. They are realized as universal unique C^* -algebras as in the following way. For a λ -graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ , let $\{v_1^l, \dots, v_{m(l)}^l\}$ be the vertex set V_l . We put

(2.2)

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

(2.3)

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+1)$, $\alpha \in \Sigma$.

Lemma 2.1([Ma3; Theorem A]). *The C^* -algebra $\mathcal{O}_{\mathfrak{L}}$ is the universal concrete C^* -algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$ satisfying the following relations called $(\mathfrak{L})$:*

$$(2.4) \quad \sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1,$$

$$(2.5) \quad \sum_{k=1}^{m(l)} E_k^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1},$$

$$(2.6) \quad S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*,$$

$$(2.7) \quad S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) E_j^{l+1},$$

for $i = 1, 2, \dots, m(l)$, $l \in \mathbb{Z}_+$, $\alpha \in \Sigma$.

If $\mathfrak{L}$ satisfies condition (I), a generalized condition of condition (I) for a finite square matrix with entries in $\{0, 1\}$ defined in [CK], the algebra $\mathcal{O}_{\mathfrak{L}}$ is the unique C^* -algebra subject to the above relations $(\mathfrak{L})$. Furthermore, if $\mathfrak{L}$ is irreducible, the C^* -algebra $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([Ma3],[Ma5]). The gauge action $\alpha^{\mathfrak{L}}$ on $\mathcal{O}_{\mathfrak{L}}$ is defined by an action of $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ such that $\alpha_z^{\mathfrak{L}}(S_\alpha) = z S_\alpha$, $\alpha_z^{\mathfrak{L}}(E_i^l) = E_i^l$ for $\alpha \in \Sigma, i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+$.

3. C^* -SYMBOLIC DYNAMICAL SYSTEMS

Let $\mathcal{A}$ be a unital C^* -algebra. Throughout this paper, an endomorphism of $\mathcal{A}$ means a $*$ -endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_{\mathcal{A}}$ of $\mathcal{A}$. The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. We denote by $\text{End}(\mathcal{A})$ the set of all endomorphisms of $\mathcal{A}$. Let Σ be a finite set. A finite family of endomorphisms $\rho_\alpha \in \text{End}(\mathcal{A}), \alpha \in \Sigma$ is said to be *essential* if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$.

It is said to be *faithful* if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.

Definition. A C^* -symbolic dynamical system is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^* -algebra $\mathcal{A}$ and a finite family of endomorphisms ρ_α of $\mathcal{A}$ indexed by $\alpha \in \Sigma$, that is essential and faithful.

Two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : \mathcal{A} \rightarrow \mathcal{A}'$ and a bijection $\pi : \Sigma \rightarrow \Sigma'$ such that $\Phi \circ \rho_\alpha = \rho'_{\pi(\alpha)} \circ \Phi$ for all $\alpha \in \Sigma$.

Proposition 3.1. For a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, there uniquely exists a subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for the subshift if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$.

Suppose that $\mathcal{A}$ is a commutative C^* -algebra $C(\Omega)$ of all continuous functions on a compact Hausdorff space Ω . An endomorphism of $\mathcal{A}$ bijectively corresponds to a continuous map from a clopen set of Ω to Ω . Hence a C^* -symbolic dynamical system $(C(\Omega), \rho, \Sigma)$ bijectively corresponds to a family $\{f_\alpha, E_\alpha\}_{\alpha \in \Sigma}$ of clopen sets $E_\alpha \subset \Omega$ and continuous maps $f_\alpha : E_\alpha \rightarrow \Omega$, $\alpha \in \Sigma$ such that

$$\bigcup_{\alpha \in \Sigma} E_\alpha = \Omega \quad \text{and} \quad \bigcup_{\alpha \in \Sigma} f_\alpha(E_\alpha) = \Omega.$$

We will study this situation in more graphical examples for a while.

For a left-resolving labeled graph $\mathcal{G} = (G, \lambda)$, let $v_1, \dots, v_n$ be its vertex set. Consider the n -dimensional commutative C^* -algebra $\mathcal{A}_{\mathcal{G}} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$ where each minimal projection E_i corresponds to the vertex v_i for $i = 1, \dots, n$. Then we may define an $n \times n$ -matrix for $\alpha \in \Sigma$ with entries in $\{0, 1\}$ by

$$(3.1) \quad A^{\mathcal{G}}(i, \alpha, j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \dots, n$. We set

$$\rho_\alpha^{\mathcal{G}}(E_i) = \sum_{j=1}^n A^{\mathcal{G}}(i, \alpha, j) E_j, \quad i = 1, \dots, n, \alpha \in \Sigma.$$

Then $\rho_\alpha^{\mathcal{G}}, \alpha \in \Sigma$ define endomorphisms of $\mathcal{A}_{\mathcal{G}}$ such that $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ is a C^* -symbolic dynamical system.

Conversely, let $(\mathcal{A}, \rho, \Sigma)$ be a C^* -symbolic dynamical system such that $\mathcal{A}$ is n -dimensional and commutative. Take $E_1, \dots, E_n$ the orthogonal minimal projections of $\mathcal{A}$ such that $\mathcal{A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$. Define an $n \times n$ matrix $[A(i, \alpha, j)]_{i,j=1,\dots,n}$ for $\alpha \in \Sigma$ by setting

$$(3.2) \quad A(i, \alpha, j) = \begin{cases} 1 & \text{if } \rho_\alpha(E_i) \geq E_j, \\ 0 & \text{otherwise} \end{cases}$$

so that one has

$$\rho_\alpha(E_i) = \sum_{j=1}^n A(i, \alpha, j) E_j, \quad i = 1, \dots, n, \alpha \in \Sigma.$$

Let $v_1, \dots, v_n$ be the vertex set corresponding to the projections $E_1, \dots, E_n$. Define a directed labeled edge e such as the source vertex $s(e) = v_i$, the terminal vertex $t(e) = v_j$ and the label $\lambda(e) = \alpha$ if $A(i, \alpha, j) = 1$. Then we have a left-resolving labeled graph $\mathcal{G}$ which presents the subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$. Hence we have

Proposition 3.2. *For a left-resolving labeled graph $\mathcal{G}$, there exists a C^* -symbolic dynamical system $(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)$ such that the algebra $\mathcal{A}_{\mathcal{G}}$ is commutative and finite dimensional, and the associated subshift $\Lambda_{(\mathcal{A}_{\mathcal{G}}, \rho^{\mathcal{G}}, \Sigma)}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$. Conversely, for a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, if $\mathcal{A}$ is commutative and finite dimensional, there exists a left-resolving labeled graph $\mathcal{G}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ and the associated subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by $\mathcal{G}$.*

Let us apply the above discussions to general subshifts and λ -graph systems. For a λ -graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ , let $(\mathcal{M}, I)$ be its corresponding symbolic matrix system. Let $A_{l,l+1}$ be the matrices defined by (2.2). We equip V_l with discrete topology. We denote by $\Omega_{\mathfrak{L}}$ the topological space of the projective limit

$$V_0 \xleftarrow{\iota} V_1 \xleftarrow{\lambda} V_2 \xleftarrow{\lambda} \dots,$$

that is a compact, totally disconnected, second countable topological space. We regard the algebra of all continuous functions on V_l as the direct sum

$$C(V_l) = \mathbb{C}E_1^l \oplus \mathbb{C}E_2^l \oplus \dots \oplus \mathbb{C}E_{m(l)}^l,$$

where the vertices $v_i^l \in V_l, i = 1, \dots, m(l)$ correspond to the minimal projections $E_i^l \in V_l, i = 1, \dots, m(l)$. We denote $C(V_l)$ by $\mathcal{A}_{\mathfrak{L},l}$. Let $\mathcal{A}_{\mathfrak{L}}$ be the commutative C^* -algebra of all continuous functions on $\Omega_{\mathfrak{L}}$, that is the inductive limit algebra

$$\mathcal{A}_{\mathfrak{L},0} \xrightarrow{I_{0,1}^t} \mathcal{A}_{\mathfrak{L},1} \xrightarrow{I_{1,2}^t} \mathcal{A}_{\mathfrak{L},2} \xrightarrow{I_{2,3}^t} \mathcal{A}_{\mathfrak{L},3} \xrightarrow{I_{3,4}^t} \dots$$

Hence $\mathcal{A}_{\mathfrak{L}}$ is a unital commutative AF-algebra. For a symbol $\alpha \in \Sigma$ we set

$$\rho_{\alpha}^{\mathfrak{L}}(E_i^l) = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) E_j^{l+1} \quad \text{for } i = 1, 2, \dots, m(l).$$

By the commutation relation (2.1), $\rho_{\alpha}^{\mathfrak{L}}$ defines an endomorphism of $\mathcal{A}_{\mathfrak{L}}$. Since each vertex $v_i^l \in V_l$ except $l = 0$ has an in-coming edge, the family $\{\rho_{\alpha}^{\mathfrak{L}}\}_{\alpha \in \Sigma}$ is essential. It is also faithful because each vertex $v_i^l \in V_l$ has an out-going edge. Thus we have

Proposition 3.3. *For a λ -graph system $\mathfrak{L}$ over Σ , there exists a C^* -symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that the C^* -algebra $\mathcal{A}_{\mathfrak{L}}$ is commutative and AF, and the associated subshift $\Lambda_{(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)}$ coincides with the subshift $\Lambda_{\mathfrak{L}}$ presented by $\mathfrak{L}$.*

Conversely

Theorem 3.4. *Let $(\mathcal{A}, \rho, \Sigma)$ be a C^* -symbolic dynamical system. If the algebra $\mathcal{A}$ is commutative and AF, there exists a λ -graph system $\mathfrak{L}$ over Σ such that the associated C^* -symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$.*

A C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be *predecessor-separated* if the projections $\{(\rho_{\alpha_k} \circ \dots \circ \rho_{\alpha_1})(1) \mid \alpha_1, \dots, \alpha_k \in \Sigma, k \in \mathbb{N}\}$ generate the C^* -algebra $\mathcal{A}$.

Proposition 3.5.

- (i) If a λ -graph system $\mathfrak{L}$ is predecessor-separated, the associated C^* -symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is predecessor-separated.
- (ii) Suppose that an algebra $\mathcal{A}$ is unital, commutative and AF. If a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is predecessor-separated, there exists a predecessor-separated λ -graph system $\mathfrak{L}$ over Σ such that the associated C^* -symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}, \rho, \Sigma)$.

Proposition 3.6. Let $\mathfrak{L}$ and $\mathfrak{L}'$ be predecessor-separated λ -graph systems over Σ and Σ respectively. Then $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ is isomorphic to $(\mathcal{A}_{\mathfrak{L}'}, \rho^{\mathfrak{L}'}, \Sigma')$ if and only if $\mathfrak{L}$ and $\mathfrak{L}'$ are equivalent. In this case, the presented subshifts $\Lambda_{\mathfrak{L}}$ and $\Lambda_{\mathfrak{L}'}$ are identified through a symbolic conjugacy.

Therefore we have

Corollary 3.7. The equivalence classes of the predecessor-separated λ -graph systems are identified with the isomorphism classes of the predecessor-separated C^* -symbolic dynamical systems of the commutative AF-algebras.

We formulate here an action of a subshift to a C^* -algebra. We say that a subshift Λ acts on a C^* -algebra $\mathcal{A}$ if there exists a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ coincides with Λ .

4. HILBERT C^* -SYMBOLIC BIMODULES

In this section we will construct a Hilbert C^* -bimodule from a C^* -symbolic dynamical system. Let $(\mathcal{A}, \rho, \Sigma)$ be a C^* -symbolic dynamical system. We put the projections $P_{\alpha} = \rho_{\alpha}(1)$ in $\mathcal{A}$ for $\alpha \in \Sigma$. Let $\{e_{\alpha}\}_{\alpha \in \Sigma}$ denote the standard basis of the $|\Sigma|$ -dimensional vector space $\mathbb{C}^{|\Sigma|}$, where $|\Sigma|$ denotes the cardinal number of the set Σ . Set

$$\mathcal{H}_{\mathcal{A}}^{\rho} := \sum_{\alpha \in \Sigma} \mathbb{C} e_{\alpha} \otimes P_{\alpha} \mathcal{A}.$$

Define a right $\mathcal{A}$ -action and an $\mathcal{A}$ -valued inner product on $\mathcal{H}_{\mathcal{A}}^{\rho}$ by setting

$$\begin{aligned} (e_{\alpha} \otimes P_{\alpha} x) y &:= e_{\alpha} \otimes P_{\alpha} x y, \\ \langle e_{\alpha} \otimes P_{\alpha} x \mid e_{\beta} \otimes P_{\beta} y \rangle &:= \begin{cases} x^* P_{\alpha} y & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $\alpha, \beta \in \Sigma$ and $x, y \in \mathcal{A}$. Then $\mathcal{H}_{\mathcal{A}}^{\rho}$ forms a Hilbert C^* -right $\mathcal{A}$ -module. We put

$$u_{\alpha} := e_{\alpha} \otimes P_{\alpha}, \quad \alpha \in \Sigma.$$

Lemma 4.1. The family $u_{\alpha}, \alpha \in \Sigma$ forms an orthogonal finite basis of $\mathcal{H}_{\mathcal{A}}^{\rho}$ in the sense of [KPW] such that

$$(4.1) \quad \sum_{\alpha \in \Sigma} \langle u_{\alpha} \mid u_{\alpha} \rangle \geq 1.$$

We say that a finite basis of a Hilbert C^* -module is *essential* if the basis satisfies the condition (4.1). We will next define a diagonal left action ϕ_ρ of $\mathcal{A}$ to the set of all adjointable bounded $\mathcal{A}$ -module maps $L(\mathcal{H}_\mathcal{A}^\rho)$ on $\mathcal{H}_\mathcal{A}^\rho$ as follows:

$$\phi_\rho(a)u_\alpha x := u_\alpha \rho_\alpha(a)x, \quad a, x \in \mathcal{A}, \alpha \in \Sigma.$$

The above definition is well-defined. If $u_\alpha x = u_\alpha y$, then $P_\alpha x = P_\alpha y$ so that $\rho_\alpha(a1)x = \rho_\alpha(a1)y$ for $a \in \mathcal{A}$. Hence one has that $u_\alpha \rho_\alpha(a)x = u_\alpha \rho_\alpha(a)y$. Since the family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ is faithful, the left action ϕ_ρ of $\mathcal{A}$ on $\mathcal{H}_\mathcal{A}^\rho$ is faithful, that is, the element $\phi_\rho(x)$ is nonzero for any nonzero $x \in \mathcal{A}$. Therefore we have

Proposition 4.2. *For a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, there exists a Hilbert C^* -right $\mathcal{A}$ -module $\mathcal{H}_\mathcal{A}^\rho$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi_\rho : \mathcal{A} \rightarrow L(\mathcal{H}_\mathcal{A}^\rho)$ such that*

$$(4.2) \quad \phi_\rho(a)u_\alpha = u_\alpha \rho_\alpha(a),$$

$$(4.3) \quad \langle u_\alpha | u_\alpha \rangle = \rho_\alpha(1), \quad a \in \mathcal{A}, \alpha \in \Sigma.$$

We note that the above two conditions imply

$$(4.4) \quad \langle u_\alpha | \phi_\rho(a)u_\alpha \rangle = \rho_\alpha(a), \quad a \in \mathcal{A}, \alpha \in \Sigma.$$

Conversely

Proposition 4.3. *For a Hilbert C^* -right $\mathcal{A}$ -module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi : \mathcal{A} \rightarrow L(\mathcal{H}_\mathcal{A})$, define ρ_α for $\alpha \in \Sigma$ by setting*

$$\rho_\alpha(a) = \langle u_\alpha | \phi(a)u_\alpha \rangle, \quad a \in \mathcal{A}.$$

Then ρ_α gives rise to an endomorphism of $\mathcal{A}$ such that $(\mathcal{A}, \rho, \Sigma)$ yields a C^ -symbolic dynamical system.*

A Hilbert C^* -right $\mathcal{A}$ -module $\mathcal{H}_\mathcal{A}$ with a left action $\phi : \mathcal{A} \rightarrow L(\mathcal{H}_\mathcal{A})$ is called a Hilbert C^* -bimodule over $\mathcal{A}$ ([Pim], cf.[KW], [KPW], [MS]). Two Hilbert C^* -bimodules $(\phi, \mathcal{H}_\mathcal{A})$ and $(\phi', \mathcal{H}'_\mathcal{A})$ over $\mathcal{A}$ are said to be unitary equivalent if there exists a bimodule isomorphism $\Phi : \mathcal{H}_\mathcal{A} \rightarrow \mathcal{H}'_\mathcal{A}$ such that Φ is unitary with respect to their respect inner products.

Definition. A Hilbert C^* -right $\mathcal{A}$ -module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi : \mathcal{A} \rightarrow L(\mathcal{H}_\mathcal{A})$ is called a *Hilbert C^* -symbolic bimodule over $\mathcal{A}$* . It is written as $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$.

A Hilbert C^* -symbolic bimodule $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$ over $\mathcal{A}$ bijectively corresponds to a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ by the above discussions. Two Hilbert C^* -symbolic bimodules $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$ and $(\phi', \mathcal{H}'_\mathcal{A}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})$ over $\mathcal{A}$ are said to be unitary equivalent if there exists a bimodule isomorphism $\Phi : \mathcal{H}_\mathcal{A} \rightarrow \mathcal{H}'_\mathcal{A}$ and a bijection $\pi : \Sigma \rightarrow \Sigma'$ such that Φ is unitary with respect to their respect inner products and satisfies $\Phi(u_\alpha) = u'_{\pi(\alpha)}$, $\alpha \in \Sigma$. Let $\rho_\alpha, \alpha \in \Sigma$ and $\rho'_{\alpha'}, \alpha' \in \Sigma'$ be their respect endomorphisms of $\mathcal{A}$. In this case, we have

$\rho_\alpha(a) = \rho'_{\pi(\alpha)}(a)$, $a \in \mathcal{A}$ because the equality (4.2) implies $\phi'(a)\Phi(u_\alpha) = \Phi(u_\alpha)\rho_\alpha(a)$ and hence $\phi'(a)u'_{\pi(\alpha)} = u'_{\pi(\alpha)}\rho_\alpha(a)$. This means that $\rho_\alpha(a) = \rho'_{\pi(\alpha)}(a)$, $a \in \mathcal{A}$.

Two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are said to be *inner conjugate* if there exists an element $U_{\alpha, \beta} \in \mathcal{A}$ for $\alpha \in \Sigma, \beta \in \Sigma'$ such that

- (i) $\rho_\alpha(a)U_{\alpha, \beta} = U_{\alpha, \beta}\rho'_\beta(a)$,
- (ii) $\sum_{\epsilon \in \Sigma'} U_{\alpha, \epsilon} U_{\gamma, \epsilon}^* = \delta_{\alpha, \gamma} \rho_\alpha(1)$, $\sum_{\gamma \in \Sigma} U_{\gamma, \beta}^* U_{\gamma, \epsilon} = \delta_{\beta, \epsilon} \rho'_\beta(1)$ and
- (iii) $\rho_\alpha(1)U_{\alpha, \beta} = U_{\alpha, \beta} = U_{\alpha, \beta}\rho'_\beta(1)$

for $\alpha, \gamma \in \Sigma, \beta, \epsilon \in \Sigma'$ and $a \in \mathcal{A}$. The family $\{U_{\alpha, \beta}\}_{\alpha \in \Sigma, \beta \in \Sigma'}$ is called an *intertwiner* between $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$.

Proposition 4.4. *Two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are inner conjugate if and only if their associated Hilbert C^* -bimodules $(\phi_\rho, \mathcal{H}_\mathcal{A}^\rho)$ and $(\phi_{\rho'}, \mathcal{H}_\mathcal{A}^{\rho'})$ are unitary equivalent as a Hilbert C^* -bimodule.*

We note that if $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are inner conjugate with intertwiner $\{U_{\alpha, \beta}\}_{\alpha \in \Sigma, \beta \in \Sigma'}$, then the equalities for $\alpha \in \Sigma, \beta \in \Sigma'$ and $a \in \mathcal{A}$

$$\rho_\alpha(a) = \sum_{\epsilon \in \Sigma'} U_{\alpha, \epsilon} \rho'_\epsilon(a) U_{\alpha, \epsilon}^*, \quad \rho'_\beta(a) = \sum_{\gamma \in \Sigma} U_{\gamma, \beta}^* \rho_\gamma(a) U_{\gamma, \beta},$$

hold. For $(\mathcal{A}, \rho, \Sigma)$, let $D_\rho(a)$ for $a \in \mathcal{A}$ be the $|\Sigma| \times |\Sigma|$ -diagonal matrix $D_\rho(a)$ with diagonal entries $[\rho_\alpha(a)]_{\alpha \in \Sigma}$. One knows $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \rho', \Sigma')$ are inner conjugate if and only if there exists an $|\Sigma| \times |\Sigma'|$ -matrix U over $\mathcal{A}$ such that

$$(4.5) \quad D_\rho(a) = U D_{\rho'}(a) U^* \quad \text{for } a \in \mathcal{A}, \text{ and}$$

$$(4.6) \quad U U^* = D_\rho(1), \quad U^* U = D_{\rho'}(1).$$

Let $\mathcal{A}$ be an n -dimensional commutative C^* -algebra. By Proposition 3.2, a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ defines a left-resolving labeled graph $\mathcal{G}^\rho = (G^\rho, \lambda^\rho)$ over Σ with underlying finite directed graph G^ρ . Let $v_1, \dots, v_n$ denote the vertex set of G^ρ . We denote by $A^\rho(i, j)$ the cardinal number of the edges $E^\rho(i, j)$ whose source vertex is v_i and terminal vertex is v_j . In this case, inner conjugacy is completely characterized as in the following way.

Proposition 4.5. *Let $\mathcal{A}$ be the n -dimensional commutative C^* -algebra. Then C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}, \eta, \Sigma)$ are inner conjugate if and only if $A^\rho(i, j) = A^\eta(i, j)$ for all $i, j = 1, 2, \dots, n$. That is, the directed graphs G^ρ and G^η are isomorphic.*

5. CROSSED PRODUCTS BY SYMBOLIC DYNAMICAL SYSTEMS

We will study C^* -algebras constructed from Hilbert C^* -symbolic bimodules. A general construction of C^* -algebras from Hilbert C^* -bimodules has been established by Pimsner [Pim] (cf. [Ka]). The C^* -algebras are called Cuntz-Pimsner algebras. Its ideal structure and simplicity conditions have been studied by Kajiwara-Pinzari-Watatani [KPW] and Muhly-Solel [MS], see also [KW], [PWY], [Sch]. For a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, we have a C^* -algebra from the Hilbert C^* -symbolic bimodule $(\phi_\rho, \mathcal{H}_\mathcal{A}^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$ by using Pimsner's general construction of C^* -algebras from Hilbert C^* -bimodules. We denote the C^* -algebra by $\mathcal{A} \rtimes_\rho \Lambda$, where Λ is the subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$ associated with $(\mathcal{A}, \rho, \Sigma)$. We call the algebra $\mathcal{A} \rtimes_\rho \Lambda$ the C^* -symbolic crossed product of $\mathcal{A}$ by the subshift Λ .

Proposition 5.1. *The C^* -symbolic crossed product $\mathcal{A} \rtimes_{\rho} \Lambda$ is the universal unital C^* -algebra $C^*(\mathcal{A}, S_{\alpha}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_{\alpha}, \alpha \in \Sigma$ subject to the following operator relations:*

$$(5.1) \quad \sum_{\beta \in \Sigma} S_{\beta} S_{\beta}^* = 1, \quad S_{\alpha}^* x S_{\alpha} = \rho_{\alpha}(x), \quad x S_{\alpha} S_{\alpha}^* = S_{\alpha} S_{\alpha}^* x$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. Furthermore for $\alpha_1, \dots, \alpha_k \in \Sigma$, a word $(\alpha_1, \dots, \alpha_k)$ is admissible for the subshift $\Lambda = \Lambda_{(\mathcal{A}, \rho, \Sigma)}$ if and only if $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$.

As in [Pim] (cf. [KPW]), the gauge action, denoted by $\hat{\rho}$, on the algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ of the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is defined by

$$\hat{\rho}_z(x) = x, \quad \hat{\rho}_z(S_{\alpha}) = z S_{\alpha}, \quad x \in \mathcal{A}, \alpha \in \Sigma, z \in \mathbb{T}.$$

We have the following theorem.

Theorem 5.2. *Let $(\mathcal{A}, \rho, \Sigma)$ be a C^* -symbolic dynamical system and Λ be the associated subshift $\Lambda_{(\mathcal{A}, \rho, \Sigma)}$. Assume that $\mathcal{A}$ is commutative.*

- (i) *If $\mathcal{A} = \mathbb{C}$, the subshift Λ is the full shift $\Sigma^{\mathbb{Z}}$, and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the Cuntz algebra $\mathcal{O}_{|\Sigma|}$ of order $|\Sigma|$.*
- (ii) *If $\mathcal{A}$ is finite dimensional, the subshift Λ is a sofic shift $\Lambda_{\mathcal{G}}$ presented by a left-resolving labeled graph $\mathcal{G}$, and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is a Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph. Conversely, for any sofic shift $\Lambda_{\mathcal{G}}$, that is presented by a left-resolving labeled graph $\mathcal{G}$, there exists a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the sofic shift $\Lambda_{\mathcal{G}}$, the algebra $\mathcal{A}$ is finite dimensional, and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the Cuntz-Krieger algebra $\mathcal{O}_{\mathcal{G}}$ associated with the labeled graph.*
- (iii) *If $\mathcal{A}$ is an AF-algebra, there uniquely exists a λ -graph system $\mathcal{L}$ up to equivalence such that the subshift Λ is presented by $\mathcal{L}$ and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the C^* -algebra $\mathcal{O}_{\mathcal{L}}$ associated with the λ -graph system $\mathcal{L}$. Conversely, for any subshift $\Lambda_{\mathcal{L}}$, that is presented by a left-resolving λ -graph system $\mathcal{L}$, there exists a C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift is the subshift $\Lambda_{\mathcal{L}}$, the algebra $\mathcal{A}$ is a commutative AF-algebra, and the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ is the C^* -algebra $\mathcal{O}_{\mathcal{L}}$ associated with the λ -graph system $\mathcal{L}$.*

We remark that Pimsner showed the following fact [Pim]: For every Hilbert C^* -bimodule E over a C^* -algebra $\mathcal{A}$, if $\mathcal{A}$ is commutative and finite dimensional, and if E is projective and finitely generated, the associated C^* -algebra is a Cuntz-Krieger algebra.

We will give some examples

- (i) Let $\alpha_1, \dots, \alpha_m \in \text{Aut}(\mathcal{B})$ be automorphisms of a unital C^* -algebra $\mathcal{B}$. Let $\mathcal{G} = (G, \lambda)$ be a left-resolving labeled graph with symbols $\Sigma = \{\alpha_1, \dots, \alpha_m\}$. Let $V = \{v_1, \dots, v_n\}$ be the vertex set. Let $[A^{\mathcal{G}}(i, \alpha_k, j)]_{i,j=1, \dots, n}$ be the $n \times n$ -matrix for $\alpha_k \in \Sigma$ with entries in $\{0, 1\}$ defined by (3.1). We put $\mathcal{A} = \mathcal{B} \oplus \cdots \oplus \mathcal{B}$ the

direct sum of the n -copies of $\mathcal{B}$. For $\alpha_k \in \Sigma$, define $\rho_{\alpha_k}^{\mathcal{G}} \in \text{End}(\mathcal{A})$ by setting

$$\begin{aligned} & \rho_{\alpha_k}^{\mathcal{G}}(b_1, \dots, b_n) \\ &= \left(\sum_{i=1}^n A^{\mathcal{G}}(i, \alpha_k, 1) \alpha_k(b_i), \dots, \sum_{i=1}^n A^{\mathcal{G}}(i, \alpha_k, n) \alpha_k(b_i) \right), \quad (b_1, \dots, b_n) \in \mathcal{A}. \end{aligned}$$

Since we assume that every vertex of G has an in-coming edge, one has $\sum_{k=1}^n \rho_{\alpha_k}^{\mathcal{G}}(1) \geq 1$. Since we also assume that every vertex of G has an out-going edge, the family $\{\rho_{\alpha_k}^{\mathcal{G}}\}_{k=1}^n$ is faithful. Hence we have a C^* -symbolic dynamical system $(\mathcal{A}, \rho^{\mathcal{G}}, \Sigma)$. The associated subshift $\Lambda_{(\mathcal{A}, \rho^{\mathcal{G}}, \Sigma)}$ is the sofic shift $\Lambda_{\mathcal{G}}$ presented by the labeled graph $\mathcal{G}$. If the underlying directed graph G is irreducible with condition (I) in the sense of [CK] and each automorphism α_k has no nontrivial invariant ideal of $\mathcal{B}$, the associated crossed product $\mathcal{A} \rtimes_{\rho} \Lambda_{\mathcal{G}}$ is simple and purely infinite.

The following example is a special case of this example.

(ii) Let $A = C(\mathbb{T})$ and $\Sigma = \{1, 2, \dots, n\}$, $n > 1$. Take irrational numbers $\theta_1, \dots, \theta_n \in \mathbb{R} \setminus \mathbb{Q}$. Define $\rho_i(f)(z) = f(e^{2\pi\sqrt{-1}\theta_i} z)$ for $f \in C(\mathbb{T})$, $z \in \mathbb{T}$. We have a C^* -symbolic dynamical system $(C(\mathbb{T}), \rho, \Sigma)$. Since the endomorphisms ρ_i , $i = 1, \dots, n$ are automorphisms and hence the associated subshift is the full shift $\Sigma^{\mathbb{Z}}$. We denote by $\mathcal{O}_{\theta_1, \dots, \theta_n}$ the C^* -symbolic crossed product $C(\mathbb{T}) \rtimes_{\theta_1, \dots, \theta_n} \Sigma^{\mathbb{Z}}$. As the algebra $\mathcal{O}_{\theta_1, \dots, \theta_n}$ is the universal unital C^* -algebra generated by n isometries and one unitary U satisfying the following relations:

$$\sum_{j=1}^n S_j S_j^* = 1, \quad S_i^* S_i = 1, \quad U S_i = e^{2\pi\sqrt{-1}\theta_i} S_i U, \quad i = 1, \dots, n.$$

Hence $\mathcal{O}_{\theta_1, \dots, \theta_n}$ is realized as the ordinary crossd product $\mathcal{O}_n \rtimes_{\alpha_{\theta_1, \dots, \theta_n}} \mathbb{Z}$ of the Cuntz algebra $\mathcal{O}_n$ by the automorphism $\alpha_{\theta_1, \dots, \theta_n}$ defined by $\alpha_{\theta_1, \dots, \theta_n}(S_i) = e^{2\pi\sqrt{-1}\theta_i} S_i$. It is simple and purely infinite whose K-groups are

$$K_0(\mathcal{O}_{\theta_1, \dots, \theta_n}) = K_1(\mathcal{O}_{\theta_1, \dots, \theta_n}) \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

(iii) Let $A = [A(i, j)]_{i, j=1, \dots, n}$ be an $n \times n$ matrix with entries in $\{0, 1\}$. We denote by Λ_A^+ the compact Hausdorff space

$$\Lambda_A^+ = \{(x_i)_{i \in \mathbb{N}} \in \{1, \dots, n\}^{\mathbb{N}} \mid A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{N}\}$$

of the right one-sided topological Markov shift associated with the matrix A . Let S_i , $i = 1, \dots, n$ be the generating partial isometries of the Cuntz-Krieger algebra $\mathcal{O}_A$ such that $\sum_{j=1}^n S_j S_j^* = 1$, $S_i^* S_i = \sum_{j=1}^n A(i, j) S_j S_j^*$. The algebra $\mathcal{A}_A = C(\Lambda_A^+)$ of all continuous functions on Λ_A^+ is identified with the subalgebra of $\mathcal{O}_A$ generated by the projections $S_{\mu} S_{\mu}^*$ for $\mu = \mu_1 \cdots \mu_k$, where $S_{\mu} = S_{\mu_1} \cdots S_{\mu_k}$ for $\mu_1, \dots, \mu_k \in \{1, \dots, n\}$. Let $\Sigma = \{\langle 1, \langle 2, \dots, \langle n, \rangle_1, \rangle_2, \dots, \rangle_n \}$ be $2n$ -brackets. We define $2n$ -endomorphisms of $\mathcal{A}_A$ by setting

$$\rho_{\langle i}^A(a) = S_i^* a S_i, \quad \rho_{\rangle i}^A(a) = S_i a S_i^*, \quad i = 1, \dots, n, \quad a \in \mathcal{A}_A.$$

We have a C^* -symbolic dynamical system $(\mathcal{A}_A, \rho^A, \Sigma)$. If in particular all entries $A(i, j), i, j = 1, \dots, n$ of A are 1, then Λ_A^+ is the right one-sided full shift $\{1, \dots, n\}^{\mathbb{N}}$ and the associated subshift is the Dyck shift D_n of the $2n$ -brackets. Let $\mathfrak{L}^{Ch(D_n)}$ be the corresponding λ -graph system for $(\mathcal{A}_A, \rho^A, \Sigma)$. It is called the Cantor horizon λ -graph system of the Dyck shift D_n , that has been studied in [KM]. The C^* -symbolic crossed product $C(\{1, \dots, n\}^{\mathbb{N}}) \rtimes_{\rho^A} D_n$ is a simple purely infinite C^* -algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_n)}}$ that is the C^* -algebra associated with $\mathfrak{L}^{Ch(D_n)}$. Its K -groups have been computed so that

$$\begin{aligned} K_0(C(\{1, \dots, n\}^{\mathbb{N}}) \rtimes_{\rho^A} D_n) &= \mathbb{Z}/n\mathbb{Z} \oplus C(\mathcal{C}, \mathbb{Z}), \\ K_1(C(\{1, \dots, n\}^{\mathbb{N}}) \rtimes_{\rho^A} D_n) &= 0 \end{aligned}$$

where $C(\mathcal{C}, \mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$ -valued continuous functions on the Cantor set $\mathcal{C}$ ([KM]).

For a general matrix A with entries in $\{0, 1\}$, let $\mathfrak{L}^{Ch(D_A)}$ be the corresponding λ -graph system to $(\mathcal{A}_A, \rho^A, \Sigma)$. It is easy to see that the associated subshift is a subshift of Dyck shift D_n that has some forbidden words coming from the forbidden words of the topological Markov shift Λ_A . The subshift is a version of topological Markov shift of the Dyck shifts, and appeared in [HIK], [KM2]. We call it the topological Markov Dyck shift associated with the matrix A and write it as D_A . We then see that the C^* -symbolic crossed product $C(\Lambda_A^+) \rtimes_{\rho^A} D_A$ is a simple purely infinite C^* -algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_A)}}$ if the matrix A is irreducible.

6. STRONG SHIFT EQUIVALENCE OF C^* -SYMBOLIC DYNAMICAL SYSTEMS AND HILBERT C^* -BIMODULES

As in the preceding section, we may regard a λ -graph system as a C^* -symbolic dynamical system. The matrix interpretation of a λ -graph system is called a symbolic matrix system. In [Ma], we have formulated strong shift equivalence of symbolic matrix systems, as a generalization of nonnegative square matrices ([Wi]) and symbolic square matrices ([N]). Strong shift equivalence of symbolic matrix systems is a basic equivalence relation related to topological conjugacy of subshifts. It has been proved that two subshifts Λ and Λ' are topologically conjugate if and only if their canonical symbolic matrix systems $(\mathcal{M}^\Lambda, I^\Lambda)$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are strong shift equivalent ([Ma]).

In this section, we will formulate strong shift equivalences and shift equivalences of C^* -symbolic dynamical systems and of Hilbert C^* -symbolic bimodules as generalizations of those of λ -graph systems.

Definition. Two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be *strong shift equivalent in 1-step* if there exist finite sets C and D , two families of homomorphisms $\eta_c : \mathcal{A} \rightarrow \mathcal{A}', c \in C$ and $\zeta_d : \mathcal{A}' \rightarrow \mathcal{A}, d \in D$ and two into bijections $\kappa : \Sigma \rightarrow CD$ and $\kappa' : \Sigma' \rightarrow DC$ such that

$$\rho_\alpha = \zeta_{d_\alpha} \circ \eta_{c_\alpha} \text{ if } \kappa(\alpha) = c_\alpha d_\alpha, \quad \text{and} \quad \rho'_{\alpha'} = \eta_{c_{\alpha'}} \circ \zeta_{d_{\alpha'}} \text{ if } \kappa'(\alpha') = d_{\alpha'} c_{\alpha'}$$

and

$$\zeta_d \circ \eta_c = 0 \text{ if } cd \notin \kappa(\Sigma), \quad \text{and} \quad \eta_c \circ \zeta_d = 0 \text{ if } dc \notin \kappa'(\Sigma').$$

We write this situation as $(\mathcal{A}, \rho, \Sigma) \approx_1 (\mathcal{A}', \rho', \Sigma')$.

We set $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A}'$ and $\tilde{\Sigma} = C \sqcup D$ disjoint union of C and D . Define $\tilde{\rho}_{\tilde{\alpha}} \in \text{End}(\tilde{\mathcal{A}})$ for $\tilde{\alpha} \in \tilde{\Sigma}$ by setting

$$\tilde{\rho}_{\tilde{\alpha}}(x, y) = \begin{cases} (0, \eta_c(x)) & \text{if } \tilde{\alpha} = c \in C, \\ (\zeta_d(y), 0) & \text{if } \tilde{\alpha} = d \in D \end{cases}$$

for $(x, y) \in \mathcal{A} \oplus \mathcal{A}'$. Then we have

Lemma 6.1. $(\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{\Sigma})$ is a C^* -symbolic dynamical system.

We call $(\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{\Sigma})$ the *bipartite C^* -symbolic dynamical system* related to $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$. If there exists an N -chain of strong shift equivalences in 1-step between $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$, they are said to be *strong shift equivalent in N -step* and written as $(\mathcal{A}, \rho, \Sigma) \approx_N (\mathcal{A}', \rho', \Sigma')$. They are simply said to be strong shift equivalent.

Recall that two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exists an isomorphism $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ of C^* -algebras and a bijection $\pi : \Sigma \rightarrow \Sigma'$ such that $\rho_\alpha = \phi^{-1} \circ \rho'_{\pi(\alpha)} \circ \phi$ for all $\alpha \in \Sigma$.

Lemma 6.2.

- (i) If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are isomorphic, they are strong shift equivalent in 1-step.
- (ii) Suppose that both sets Σ and Σ' are one points $\{\alpha\}$ and $\{\alpha'\}$ respectively and both ρ_α and $\rho'_{\alpha'}$ are automorphisms. Then $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are isomorphic if and only if they are strong shift equivalent in 1-step.

We next formulate shift equivalence of C^* -symbolic dynamical systems

Definition. C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be *shift equivalent of lag N* if there exist two finite sets C and D , two families $\eta_c : \mathcal{A} \rightarrow \mathcal{A}', c \in C$ and $\zeta_d : \mathcal{A}' \rightarrow \mathcal{A}, d \in D$ of homomorphisms and four specifications $\kappa_C : \Sigma C \rightarrow C \Sigma', \kappa_D : \Sigma' D \rightarrow D \Sigma, \kappa_\Sigma : \Sigma^N \rightarrow CD$ and $\kappa_{\Sigma'} : \Sigma'^N \rightarrow DC$ such that

$$\begin{aligned} \eta_c \circ \rho_\alpha &= \rho'_{\alpha'} \circ \eta_{c'} & \text{if } \kappa_C(\alpha c) &= c' \alpha', \\ \zeta_{d'} \circ \rho'_{\alpha'} &= \rho_\alpha \circ \zeta_d & \text{if } \kappa_D(\alpha' d') &= d \alpha, \end{aligned}$$

and

$$\begin{aligned} \rho_{\alpha_N} \circ \cdots \circ \rho_{\alpha_2} \circ \rho_{\alpha_1} &= \zeta_d \circ \eta_c & \text{if } \kappa_\Sigma(\alpha_1 \alpha_2 \cdots \alpha_N) &= cd, \\ \rho'_{\alpha'_N} \circ \cdots \circ \rho'_{\alpha'_2} \circ \rho'_{\alpha'_1} &= \eta_{c'} \circ \zeta_{d'} & \text{if } \kappa_{\Sigma'}(\alpha'_1 \alpha'_2 \cdots \alpha'_N) &= d' c'. \end{aligned}$$

We write this situation as $(\mathcal{A}, \rho, \Sigma) \sim_N (\mathcal{A}', \rho', \Sigma')$.

The following proposition is proved by similar ideas to the case of matrices ([Wi], cf. [LM]).

Proposition 6.3. *Let $(\mathcal{A}, \rho, \Sigma)$, $(\mathcal{A}', \rho', \Sigma')$ and $(\mathcal{A}'', \rho'', \Sigma'')$ be C^* -symbolic dynamical systems.*

- (i) $(\mathcal{A}, \rho, \Sigma) \approx_N (\mathcal{A}', \rho', \Sigma')$ implies $(\mathcal{A}, \rho, \Sigma) \sim_N (\mathcal{A}', \rho', \Sigma')$.
- (ii) $(\mathcal{A}, \rho, \Sigma) \sim_N (\mathcal{A}', \rho', \Sigma')$ implies $(\mathcal{A}, \rho, \Sigma) \sim_{N'} (\mathcal{A}', \rho', \Sigma')$ for all $N' \geq N$.
- (iii) $(\mathcal{A}, \rho, \Sigma) \sim_N (\mathcal{A}', \rho', \Sigma')$ and $(\mathcal{A}', \rho', \Sigma') \sim_L (\mathcal{A}'', \rho'', \Sigma'')$ imply $(\mathcal{A}, \rho, \Sigma) \sim_{N+L} (\mathcal{A}'', \rho'', \Sigma'')$.

Thus shift equivalence of C^* -symbolic dynamical systems is an equivalence relation.

We will next formulate strong shift equivalence and shift equivalence of Hilbert C^* -bimodules. Let $\mathcal{A}$ and $\mathcal{A}'$ be C^* -algebras. We define a *Hilbert C^* -symbolic right $\mathcal{A}'$ -module* $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_\alpha\}_{\alpha \in \Sigma})$ over Σ with left $\mathcal{A}$ -action by a Hilbert C^* -right $\mathcal{A}'$ -module with orthogonal essential finite basis $\{w_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action φ of $\mathcal{A}$ on ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}$. Let $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_\alpha\}_{\alpha \in \Sigma})$ be a Hilbert C^* -symbolic right $\mathcal{A}'$ -module over Σ with left $\mathcal{A}$ -action and $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w'_{\alpha'}\}_{\alpha' \in \Sigma'})$ a Hilbert C^* -symbolic right $\mathcal{A}''$ -module over Σ' with left $\mathcal{A}'$ -action. Define the relative tensor product

$$\begin{aligned} & (\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_\alpha\}_{\alpha \in \Sigma}) \otimes_{\mathcal{A}'} (\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w'_{\alpha'}\}_{\alpha' \in \Sigma'}) \\ & := (\varphi \otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w_\alpha \otimes_{\mathcal{A}'} w'_{\alpha'}\}_{(\alpha, \alpha') \in \Sigma \otimes_{\mathcal{A}'} \Sigma'}) \end{aligned}$$

where ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}$ is the tensor product Hilbert C^* -right $\mathcal{A}''$ -module relative to $\mathcal{A}'$, and $\varphi \otimes 1$ is the natural left $\mathcal{A}$ -action on it. The finite set $\Sigma \otimes_{\mathcal{A}'} \Sigma'$ is defined as follows: As both the left action φ and ψ are diagonal with respect to the bases $\{w_\alpha\}_{\alpha \in \Sigma}$ and $\{w'_{\alpha'}\}_{\alpha' \in \Sigma'}$ respectively, there exist $\eta_\alpha(a) \in \mathcal{A}'$ for $a \in \mathcal{A}$ and $\zeta_{\alpha'}(b) \in \mathcal{A}''$ for $b \in \mathcal{A}'$ such that

$$\varphi(a)w_\alpha = w_\alpha \eta_\alpha(a), \quad \psi(b)w'_{\alpha'} = w'_{\alpha'} \zeta_{\alpha'}(b).$$

The finite set $\Sigma \otimes_{\mathcal{A}'} \Sigma'$ is defined by

$$\Sigma \otimes_{\mathcal{A}'} \Sigma' = \{(\alpha, \alpha') \in \Sigma \times \Sigma' \mid \zeta_{\alpha'}(\eta_\alpha(1_{\mathcal{A}})) \neq 0\}.$$

It is easy to check that

$$(\varphi \otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}''}, \{w_\alpha \otimes_{\mathcal{A}'} w'_{\alpha'}\}_{(\alpha, \alpha') \in \Sigma \otimes_{\mathcal{A}'} \Sigma'})$$

is a Hilbert C^* -symbolic right $\mathcal{A}''$ -module over $\Sigma \otimes_{\mathcal{A}'} \Sigma'$ with left $\mathcal{A}$ -action.

Definition. Let $(\phi, \mathcal{H}_{\mathcal{A}})$ be a Hilbert C^* -bimodule over $\mathcal{A}$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$ a Hilbert C^* -bimodule over $\mathcal{A}'$. They are said to be *strong shift equivalent in 1-step* and written as $(\phi, \mathcal{H}_{\mathcal{A}}) \approx_1 (\phi', \mathcal{H}_{\mathcal{A}'})$ if there exist a Hilbert C^* -right $\mathcal{A}'$ -module $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'})$ with left $\mathcal{A}$ -action and a Hilbert C^* -right $\mathcal{A}$ -module $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$ with left $\mathcal{A}'$ -action such that

$$(6.1) \quad \begin{cases} (\varphi \otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}) = (\phi, \mathcal{H}_{\mathcal{A}}) \text{ as a Hilbert } C^*\text{-bimodule over } \mathcal{A}, \\ (\psi \otimes 1, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}) = (\phi', \mathcal{H}_{\mathcal{A}'}) \text{ as a Hilbert } C^*\text{-bimodule over } \mathcal{A}'. \end{cases}$$

The above all equalities of Hilbert C^* -bimodules mean unitary equivalences as Hilbert C^* -bimodules. In this situation, we say that $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'})$ and $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$ satisfy the strong shift equivalence relation between $(\phi, \mathcal{H}_{\mathcal{A}})$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$. Consider the direct sum

$$(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}) \oplus (\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}) := (\varphi \oplus \psi, {}_{\mathcal{A} \oplus \mathcal{A}'}\mathcal{H}_X)$$

that is a Hilbert C^* -right $\mathcal{A}' \oplus \mathcal{A}$ -module with left $\mathcal{A} \oplus \mathcal{A}'$ -action. It is denoted by $(\xi, \mathcal{H}_X)$ and satisfies

$${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} = \xi(\mathcal{A})\mathcal{H}_X = \mathcal{H}_X\mathcal{A}', \quad {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} = \xi(\mathcal{A}')\mathcal{H}_X = \mathcal{H}_X\mathcal{A}.$$

As $\mathcal{H}_X$ is regarded as a Hilbert C^* -right $\mathcal{A} \oplus \mathcal{A}'$ -module, $(\xi, \mathcal{H}_X)$ is considered to be a Hilbert C^* -bimodule over $\mathcal{A} \oplus \mathcal{A}'$, that is called a *bipartite* Hilbert C^* -bimodule related to $(\phi, \mathcal{H}_{\mathcal{A}})$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$. We note that the condition (6.1) is equivalent to the condition:

$$(\xi \otimes 1, \mathcal{H}_X \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_X) = (\phi, \mathcal{H}_{\mathcal{A}}) \oplus (\phi', \mathcal{H}_{\mathcal{A}'}) \text{ as a Hilbert } C^*\text{-bimodule over } \mathcal{A} \oplus \mathcal{A}'.$$

If there exists an N -chain of strong shift equivalences in 1-step between $(\phi, \mathcal{H}_{\mathcal{A}})$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$, they are said to be *strong shift equivalent in N -step* and we write it as $(\phi, \mathcal{H}_{\mathcal{A}}) \approx_N (\phi', \mathcal{H}_{\mathcal{A}'})$. They are simply said to be strong shift equivalent.

In particular, Hilbert C^* -symbolic bimodules $(\phi, \mathcal{H}_{\mathcal{A}}, \{u_{\alpha}\}_{\alpha \in \Sigma})$ and $(\phi', \mathcal{H}_{\mathcal{A}'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})$ are said to be strong shift equivalent in 1-step if there exist a Hilbert C^* -symbolic right $\mathcal{A}'$ -module $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}, \{w_c\}_{c \in C})$ with left $\mathcal{A}$ -action and a Hilbert C^* -right $\mathcal{A}$ -module $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}, \{w'_d\}_{d \in D})$ with left $\mathcal{A}'$ -action such that the qualities (6.1) are taken to be unitary equivalent as Hilbert C^* -symbolic bimodules.

Definition. Let $(\phi, \mathcal{H}_{\mathcal{A}})$ be a Hilbert C^* -bimodule over $\mathcal{A}$ and $(\phi', \mathcal{H}_{\mathcal{A}'})$ a Hilbert C^* -bimodule over $\mathcal{A}'$. They are said to be *shift equivalent of lag N* if there exist a Hilbert C^* -right $\mathcal{A}'$ -module $(\varphi, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'})$ with left $\mathcal{A}$ -action and a Hilbert C^* -right $\mathcal{A}$ -module $(\psi, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$ with left $\mathcal{A}'$ -action such that

$$\begin{aligned} (\phi, \underbrace{\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{H}_{\mathcal{A}}}_N) &= (\varphi \otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}}), \\ (\phi', \underbrace{\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} \cdots \otimes_{\mathcal{A}'} \mathcal{H}_{\mathcal{A}'} }_N) &= (\psi \otimes 1, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}), \end{aligned}$$

and

$$(\varphi \otimes 1, {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} \mathcal{H}_{\mathcal{A}'}) = (\phi, \mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}'}), \quad (\psi \otimes 1, {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{H}_{\mathcal{A}}) = (\phi', \mathcal{H}_{\mathcal{A}'} \otimes_{\mathcal{A}'} {}_{\mathcal{A}'}\mathcal{H}_{\mathcal{A}})$$

We write this situation as $(\phi, \mathcal{H}_{\mathcal{A}}) \approx_N (\phi', \mathcal{H}_{\mathcal{A}'})$.

We similarly define a shift equivalence between Hilbert C^* -symbolic bimodules by equipping with finite bases.

The above formulations of a strong shift equivalence and a shift equivalence of Hilbert C^* -bimodules are generalizations of those of nonnegative square matrices defined by Williams (cf.[N],[Ma6]). The following proposition is parallel to Proposition 6.3. ([Wi], cf.[LM]).

Proposition 6.4. *Let $(\phi, \mathcal{H}_{\mathcal{A}})$, $(\phi', \mathcal{H}_{\mathcal{A}'})$ and $(\phi'', \mathcal{H}_{\mathcal{A}''})$ be Hilbert C^* -bimodules.*

- (i) $(\phi, \mathcal{H}_{\mathcal{A}}) \approx_N (\phi', \mathcal{H}_{\mathcal{A}'})$ implies $(\phi, \mathcal{H}_{\mathcal{A}}) \sim_N (\phi', \mathcal{H}_{\mathcal{A}'})$.
- (ii) $(\phi, \mathcal{H}_{\mathcal{A}}) \sim_N (\phi', \mathcal{H}_{\mathcal{A}'})$ implies $(\phi, \mathcal{H}_{\mathcal{A}}) \sim_{N'} (\phi', \mathcal{H}_{\mathcal{A}'})$ for all $N' \geq N$.
- (iii) $(\phi, \mathcal{H}_{\mathcal{A}}) \sim_N (\phi', \mathcal{H}_{\mathcal{A}'})$ and $(\phi', \mathcal{H}_{\mathcal{A}'}) \sim_L (\phi'', \mathcal{H}_{\mathcal{A}'})$ imply $(\phi, \mathcal{H}_{\mathcal{A}}) \sim_{N+L} (\phi'', \mathcal{H}_{\mathcal{A}'})$.

The similar statements hold for Hilbert C^ -symbolic bimodules.*

Therefore shift equivalence of Hilbert C^* -bimodules and similarly shift equivalence of Hilbert C^* -symbolic bimodules are equivalence relations.

Proposition 6.5. *If C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent in 1-step, their associated Hilbert C^* -symbolic bimodules $(\phi_\rho, \mathcal{H}_{\mathcal{A}}^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$ and $(\phi_{\rho'}, \mathcal{H}_{\mathcal{A}'}^{\rho'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})$ are strong shift equivalent in 1-step.*

Its converse implication holds.

Proposition 6.6. *If Hilbert C^* -symbolic bimodules $(\phi, \mathcal{H}_{\mathcal{A}}, \{u_\alpha\}_{\alpha \in \Sigma})$ and $(\phi', \mathcal{H}_{\mathcal{A}'}^{\rho'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})$ are strong shift equivalent in 1-step, their associated C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent in 1-step.*

We may similarly see that two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are shift equivalent of lag N if and only if their associated Hilbert C^* -symbolic bimodules $(\phi_\rho, \mathcal{H}_{\mathcal{A}}^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$ and $(\phi_{\rho'}, \mathcal{H}_{\mathcal{A}'}^{\rho'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})$ are shift equivalent of lag N .

7. STRONG SHIFT EQUIVALENCE OF GAUGE ACTIONS

In this section we introduce the notion of strong shift equivalence of C^* -symbolic crossed products with gauge actions.

Definition. Two C^* -symbolic crossed products $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ with gauge actions are said to be strong shift equivalent in 1-step if there exists a C^* -symbolic dynamical system $(\mathcal{A}_0, \rho_0, \Sigma_0)$ and full projections $p, p' \in \mathcal{A}_0 \rtimes_{\rho_0} \Lambda_0$ satisfying $p + p' = 1$ and $\hat{\rho}_{0,z}(p) = p, \hat{\rho}_{0,z}(p') = p'$ for $z \in \mathbb{T}$ where Λ_0 is the subshift associated with $(\mathcal{A}_0, \rho_0, \Sigma_0)$, and

$$\begin{aligned} (p(\mathcal{A}_0 \rtimes_{\rho_0} \Lambda_0)p, \hat{\rho}_0, \mathbb{T}) &= (\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}^2, \mathbb{T}), \\ (p'(\mathcal{A}_0 \rtimes_{\rho_0} \Lambda_0)p', \hat{\rho}_0, \mathbb{T}) &= (\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}'^2, \mathbb{T}). \end{aligned}$$

We write this situation as $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T}) \approx_1 (\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$. If there exists an N -chain of strong shift equivalences in 1-step, they are said to be strong shift equivalent in N -step and written as $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T}) \approx_N (\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$. It is simply said to be strong shift equivalent.

Theorem 7.1. *Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two C^* -symbolic dynamical systems whose associated subshifts are denoted by Λ and Λ' respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, the C^* -symbolic crossed products $(\mathcal{A} \rtimes_\rho \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ with gauge actions are strong shift equivalent.*

This theorem and its proof are generalizations of [Ma4:Theorem 3.15].

Suppose that $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent in 1-step. There exist finite sets C and D , two families of homomorphisms $\eta_c : \mathcal{A} \rightarrow \mathcal{A}'$, $c \in C$ and $\zeta_d : \mathcal{A}' \rightarrow \mathcal{A}$, $d \in D$ and two into bijections $\kappa : \Sigma \rightarrow CD$ and $\kappa' : \Sigma' \rightarrow DC$ that give rise to the strong shift equivalence between $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$. Let $(\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{\Sigma})$ be the bipartite C^* -symbolic dynamical system related to $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$. As $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A}'$, we identify $\mathcal{A}$ and $\mathcal{A}'$ with the subalgebras of $\tilde{\mathcal{A}}$ by regarding $a \in \mathcal{A}$ as $(a, 0) \in \tilde{\mathcal{A}}$ and $a' \in \mathcal{A}'$ as $(0, a') \in \tilde{\mathcal{A}}$ respectively. The symbolic crossed product

$$\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda} = C^*(S_{\tilde{\alpha}}, x \mid \tilde{\alpha} \in \tilde{\Sigma}, x \in \tilde{\mathcal{A}})$$

of $(\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{\Sigma})$ is the universal C^* -algebra generated by partial isometries $S_{\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Sigma} = C \sqcup D$ and elements $x \in \tilde{\mathcal{A}}$ that satisfy the relations (5.1). Let $C^*(S_{CD}, \mathcal{A})$ and $C^*(S_{DC}, \mathcal{A}')$ be the C^* -subalgebras of $\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda}$ defined by setting

$$\begin{aligned} C^*(S_{CD}, \mathcal{A}) &= C^*(S_{c_\alpha d_\alpha}, (a, 0) \mid c_\alpha d_\alpha = \kappa(\alpha), \alpha \in \Sigma, a \in \mathcal{A}) \quad \text{and} \\ C^*(S_{DC}, \mathcal{A}') &= C^*(S_{d_{\alpha'} c_{\alpha'}}, (0, a') \mid d_{\alpha'} c_{\alpha'} = \kappa'(\alpha'), \alpha' \in \Sigma', a' \in \mathcal{A}') \end{aligned}$$

respectively, where $S_{c_\alpha d_\alpha} = S_{c_\alpha} S_{d_\alpha}$ and $S_{d_{\alpha'} c_{\alpha'}} = S_{d_{\alpha'}} S_{c_{\alpha'}}$. Put the projections

$$P_C = \sum_{c \in C} S_c S_c^*, \quad P_D = \sum_{d \in D} S_d S_d^* \quad \text{in } \tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda}.$$

Hence $P_C + P_D = 1$.

We see that the following propositions hold.

Proposition 7.2.

$$C^*(S_{CD}, \mathcal{A}) = P_C(\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda})P_C, \quad C^*(S_{DC}, \mathcal{A}') = P_D(\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda})P_D.$$

Proposition 7.3. *The C^* -symbolic crossed products $\mathcal{A} \rtimes_{\rho} \Lambda$ and $\mathcal{A}' \rtimes_{\rho'} \Lambda'$ are canonically isomorphic to the algebras $C^*(S_{CD}, \mathcal{A})$ and $C^*(S_{DC}, \mathcal{A}')$ respectively.*

The following lemma shows that the subalgebras $P_C(\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda})P_C$ and $P_D(\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda})P_D$ are complementary full corners in $\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda}$.

Lemma 7.4. *The projections P_C, P_D are full in the algebra $\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda}$.*

Proof of sketch of Theorem 7.1. By Proposition 7.2 and Proposition 7.3, we may identify the algebras $\mathcal{A} \rtimes_{\rho} \Lambda$ with $P_C(\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda})P_C$, and $\mathcal{A}' \rtimes_{\rho'} \Lambda'$ with $P_D(\tilde{\mathcal{A}} \rtimes_{\tilde{\rho}} \tilde{\Lambda})P_D$. By these identifications, one has

$$\hat{\rho}_z^2(s_\alpha) = \hat{\tilde{\rho}}_z(S_c S_d), \quad \hat{\rho}'_z(s'_{\alpha'}) = \hat{\tilde{\rho}}_z(S_d S_c)$$

for $\kappa(\alpha) = cd \in CD$, $\kappa'(\alpha') = dc \in DC$. Thus $(\mathcal{A} \rtimes_{\rho} \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ are strong shift equivalent in 1-step. $\square$

Remark. It is possible to generalize the above discussions such as strong shift equivalent Hilbert C^* -bimodules give rise to strong shift equivalent C^* -algebras of the Hilbert C^* -bimodules. We will discuss this generalization in a forth coming paper [Ma6].

We present the following theorem.

Theorem 7.5. *Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two C^* -symbolic dynamical systems whose associated subshifts are denoted by Λ and Λ' respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, then we have*

- (i) *the subshifts Λ and Λ' are topologically conjugate,*
- (ii) *the C^* -symbolic crossed products $(\mathcal{A} \rtimes_{\rho} \Lambda, \hat{\rho}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda', \hat{\rho}', \mathbb{T})$ with gauge actions are strong shift equivalent, and*
- (iii) *the stabilized gauge actions $(\mathcal{A} \rtimes_{\rho} \Lambda \otimes \mathcal{K}, \hat{\rho} \otimes \text{id}, \mathbb{T})$ and $(\mathcal{A}' \rtimes_{\rho'} \Lambda' \otimes \mathcal{K}, \hat{\rho}' \otimes \text{id}, \mathbb{T})$ are cocycle conjugate, where $\mathcal{K}$ denotes the C^* -algebra of all compact operators on a separable infinite dimensional Hilbert space.*

In the rest of this section, we will concern K-theory for the C^* -algebra $\mathcal{A} \rtimes_{\rho} \Lambda$ constructed from a C^* -dynamical system $(\mathcal{A}, \rho, \Sigma)$. The endomorphisms $\rho_{\alpha} : \mathcal{A} \rightarrow \mathcal{A}$ for $\alpha \in \Sigma$ yield endomorphisms $\rho_{\alpha*} : K_*(\mathcal{A}) \rightarrow K_*(\mathcal{A})$ for $\alpha \in \Sigma$ on the K-theory groups of $\mathcal{A}$. Define an endomorphism

$$\rho_* : K_*(\mathcal{A}) \rightarrow K_*(\mathcal{A}), \quad * = 0, 1$$

by setting $\rho_*(g) = \sum_{\alpha \in \Sigma} \rho_{\alpha*}(g)$, $g \in K_*(\mathcal{A})$. By [Pim] (cf. [KPW]), one has the following six term exact sequence of K-theory:

$$\begin{array}{ccccc} K_0(\mathcal{A}) & \xrightarrow{\text{id} - \rho_*} & K_0(\mathcal{A}) & \xrightarrow{\iota_*} & K_0(\mathcal{A} \rtimes_{\rho} \Lambda) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{A} \rtimes_{\rho} \Lambda) & \xleftarrow{\iota_*} & K_1(\mathcal{A}) & \xleftarrow{\text{id} - \rho_*} & K_1(\mathcal{A}). \end{array}$$

Hence if in particular $K_1(\mathcal{A}) = 0$, one has

$$\begin{aligned} K_0(\mathcal{A} \rtimes_{\rho} \Lambda) &= K_0(\mathcal{A}) / (\text{id} - \rho_*)K_0(\mathcal{A}), \\ K_1(\mathcal{A} \rtimes_{\rho} \Lambda) &= \text{Ker}(\text{id} - \rho_*) \text{ in } K_0(\mathcal{A}). \end{aligned}$$

This formula is a generalization of K-theory formulae proved in [C2] and [Ma3]. As in [Ma3:Lemma 5.2], one sees that the fixed point algebra $\mathcal{F}_{(\mathcal{A}, \rho, \Sigma)}$ of $\mathcal{A} \rtimes_{\rho} \Lambda$ under the gauge action $\hat{\rho}$ is stably isomorphic to $(\mathcal{A} \rtimes_{\rho} \Lambda) \rtimes_{\hat{\rho}} \mathbb{T}$. We define the K-groups $K_*(\mathcal{A}, \rho, \Sigma)$ and the dimension groups $D_*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting

$$\begin{aligned} K_*(\mathcal{A}, \rho, \Sigma) &= K_*(\mathcal{A} \rtimes_{\rho} \Lambda) \\ D_*(\mathcal{A}, \rho, \Sigma) &= (K_*(\mathcal{F}_{(\mathcal{A}, \rho, \Sigma)}), \hat{\rho}_*) \quad * = 0, 1 \end{aligned}$$

where $\hat{\rho}_*$ is the automorphism on the abelian group $K_*(\mathcal{F}_{(\mathcal{A}, \rho, \Sigma)})$ induced by the dual action $\hat{\rho}$ of the gauge action $\hat{\rho}$. We also define the Bowen-Franks groups $BF^*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting

$$BF^*(\mathcal{A}, \rho, \Sigma) = \text{Ext}_*(\mathcal{A} \rtimes_{\rho} \Lambda), \quad * = 0, 1$$

Then Theorem 7.5 (iii) implies

Proposition 7.6. *The abelian groups $K_*(\mathcal{A}, \rho, \Sigma)$, $BF^*(\mathcal{A}, \rho, \Sigma)$ and the abelian group with automorphisms $D_*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ are invariant under strong shift equivalence of C^* -symbolic dynamical systems.*

The above results are generalization of [Ma4] see also [C2], [CK], [Ma2].

In [Ma8], dynamical property of a "subshift"

$$\mathcal{S}_{(\mathcal{A}, \rho, \Sigma)} = \{(\rho_{\alpha_i})_{i \in \mathbb{Z}} \mid (\rho_{\alpha_i} \circ \cdots \circ \rho_{\alpha_{i+k}})(1) \neq 0, i \in \mathbb{Z}, k \in \mathbb{Z}_+\}$$

will be studied.

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