

First-Principles Derivation of Causal Relativistic
Dissipative Hydrodynamic Equation
in Energy Frame with Stable Equilibrium State
from Kinetic Equation
by Renormalization-Group Method

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Abstract

We construct first-order relativistic dissipative hydrodynamics as a slow dynamics that is defined on the invariant manifold spanned by the zero modes of the relativistic Boltzmann equation, on the basis of the renormalization-group (RG) method in a systematic manner with no ad-hoc assumption. The resultant equation uniquely leads to the one in the energy frame proposed by Landau and Lifshitz, provided that the macroscopic-frame vector, which defines the local rest frame of the flow velocity, is independent of the momenta of constituent particles, as it should. We argue that the relativistic hydrodynamic equations for viscous fluids must be defined on the energy frame if it is consistent with the underlying relativistic kinetic equation. Furthermore, we develop the doublet scheme, i.e., a generic framework in the RG method to extract a mesoscopic dynamics from an evolution equation by incorporating some excited (fast) modes as additional components to the invariant manifold originally spanned by the zero modes. Our equation derived from the relativistic Boltzmann equation by the doublet scheme in the RG method has the same form as Israel-Stewart's fourteen-moment equation in the energy frame, i.e., a typical equation of second-order relativistic dissipative hydrodynamics, but the microscopic formulae of the coefficients, e.g., the transport coefficients and relaxation times, are different. It is found that our theory leads to the same expressions for the transport coefficients as given by the Chapman-Enskog expansion method in contrast to Israel-Stewart's equation, and suggests novel formulae of the relaxation times expressed in terms of relaxation functions which allow a natural physical interpretation of the relaxation times.

Contents

1	Introduction	5
1.1	Fundamental three problems with relativistic dissipative hydrodynamics	6
1.2	Hydrodynamics as infrared asymptotic dynamics of kinetic theory	9
1.3	Results of this thesis	11
1.4	Organization of this thesis	13
2	Preliminaries	15
2.1	Renormalization-group method	15
2.1.1	RG method applied to van der Pol equation	15
2.1.2	Foundation of RG method	18
2.2	Relativistic Boltzmann equation	19
2.2.1	Basics of relativistic Boltzmann equation	19
2.3	Chapman-Enskog expansion and Maxwell-Grad moment methods	22
2.3.1	Ad-hoc aspect in Chapman-Enskog expansion method	23
2.3.2	Ad-hoc aspect in Maxwell-Grad moment method	25
2.3.3	Microscopic representations of transport coefficients and relaxation time	27
3	First-order relativistic dissipative hydrodynamics	29
3.1	Macroscopic-frame vector	29
3.1.1	Macroscopic-frame vector as time-like Lorentz vector	30
3.2	Reduction of relativistic Boltzmann equation to hydrodynamics with RG method	32
3.2.1	Construction of approximate solution around arbitrary initial time	33
3.2.2	Linearized collision operator and inner product	34
3.2.3	First-order solution	36
3.2.4	Second-order solution	37
3.2.5	RG improvement of perturbative expansion	38
3.2.6	Reduction of RG equation to simpler form	39
3.3	Relativistic dissipative hydrodynamic equation	42
3.3.1	Uniqueness of Landau-Lifshitz energy frame	42
3.3.2	Microscopic representations of transport coefficients	46

CONTENTS

3.3.3	Generic stability of relativistic hydrodynamic equation in energy frame	47
3.3.4	Higher-order correction to relativistic hydrodynamic equation caused by derivative terms in macroscopic frame vector . . .	50
4	Second-order relativistic dissipative hydrodynamics	53
4.1	Mesoscopic dynamics derived from generic evolution equation with RG method	53
4.1.1	Evolution equation	54
4.1.2	Construction of approximate solution around arbitrary initial time	55
4.1.3	Doublet scheme	57
4.1.4	First-order solution	59
4.1.5	Second-order solution	60
4.1.6	RG improvement of perturbative expansion	61
4.1.7	Reduction of RG equation to simpler form	62
4.1.8	Consistency of mesoscopic dynamics with slow dynamics as described with zero modes in asymptotic regime	64
4.1.9	Discussion	65
4.2	Reduction of relativistic Boltzmann equation to mesoscopic dynamics with RG method	67
4.2.1	Set up in doublet scheme	68
4.2.2	Reduced dynamics by RG method	71
4.2.3	Fourteen-moment equation in energy frame and microscopic representations of transport coefficients and relaxation times .	73
4.2.4	Causality and generic stability of fourteen-moment equation in energy frame	76
4.2.5	Novel ansatz in moment method consistent to mesoscopic dynamics of relativistic Boltzmann equation	81
5	Summary and concluding remarks	85
	Acknowledgment	87
A	Detailed derivation of formulae used in this thesis	89
A.1	Jüttner function as entropy-conserving distribution function	89
A.2	Detailed derivation of explicit form of excited modes	91
A.3	Solution to linear differential equation with time-dependent inhomogeneous term	93
A.4	Slow dynamics as described with would-be zero modes	94
A.5	Detailed derivation of relaxation equation in fourteen-moment equation in energy frame	97

B	Mesoscopic dynamics of 2D evolution equation	102
B.1	Reduction of 2D evolution equation to mesoscopic dynamics with RG method	102
B.1.1	Set up in doublet scheme	103
B.1.2	Reduced dynamics by RG method	106
C	Mesoscopic dynamics of non-relativistic Boltzmann equation	107
C.1	Main results	109
C.2	Preliminaries	110
C.2.1	Basic properties of Boltzmann equation	110
C.2.2	Thirteen-moment approximation and Grad equation	114
C.3	Reduction of Boltzmann equation to mesoscopic dynamics with RG method	117
C.3.1	Set up in doublet scheme	118
C.3.2	Reduced dynamics by RG method	122
C.3.3	Causal equation for thirteen moments	125
C.4	Detailed derivation of explicit form of excited modes	127
C.5	Detailed derivation of explicit form of causal equation for thirteen moments	129
	Bibliography	136

Chapter 1

Introduction

Today, the quantum chromodynamics (QCD) is believed to be a fundamental theory for the strong interactions among quarks and gluons which make up hadrons such as the proton, neutron, or pion. The quarks are elementary fermions that carry three kinds of the color charges (red, green, and blue) [1], while the gluons are elementary bosons that act as the exchange particles for the strong force between the color-charged quarks in an analogous way to the exchange of photons in the electromagnetic force between two charged particles. Furthermore, the gluons themselves carry the color charges, unlike the colorless photon, and therefore participate in the strong interaction in addition to mediating it. A rotation in the color space is called the gauge transformation. The QCD Lagrangian is constructed in such a way that this gauge (or color) symmetry is respected, and hence is given as that of a non-abelian $SU(3)$ gauge theory with quark field $q(x)$ (fundamental representation of $SU(3)$) and gluon field $A_\mu(x)$ (adjoint representation of $SU(3)$) [2, 3, 4, 5]. Although the color charges might be observed just like the electromagnetic charges, any single colored object has never been detected. This mysterious phenomenon is called the color confinement [6, 7, 8, 9]: Quarks and gluons are confined inside hadrons to form color singlet (white) states. Another symmetry possessed by QCD is the chiral symmetry [10]: Quarks have the quantum number of flavor besides color. In the limit of the massless u- and d- quarks, the QCD Lagrangian is invariant under $SU_L(2) \times SU_R(2)$ transformation in the flavor space. This symmetry, however, is broken spontaneously by the appearance of the chiral condensate $\langle \bar{q} q \rangle$ in the vacuum. The massless Nambu-Goldstone bosons, which should be expected by the Nambu-Goldstone theorem [11, 12], are provided by the nearly massless pions.

Numerical simulations based on the lattice QCD [13] and some model calculations suggest that at high enough temperature, there exists another phase called the quark-gluon plasma (QGP), where quarks and gluons are not confined in hadrons any longer and the chiral symmetry gets restored. The high-temperature QGP phase and the low-temperature normal phase are separated by the two phase transitions, i.e., the confinement-deconfinement transition and the chiral transition. The two phase transitions are predicted to occur at the almost same temperature, $T \sim 170$ MeV [13], which is so high that an extreme condition is needed for appearance of the QGP.

The condition could be fulfilled in the early universe, and the QGP is expected to have existed until the universe was cooled down to the transition temperature.

Experimental attempts to create the QGP in the laboratory and measure its properties have been carried out for more than 25 years. When two heavy nuclei are accelerated to very high energy and collided with each other, there will appear a high temperature strong matter. In fact, in the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory [14], the acceleration energy of 200 GeV/A is possible, which is estimated to be energetic enough to produce the QGP.

Many theorists have concluded that the RHIC data provides sufficient evidence that the QGP has indeed been observed [15]. Furthermore, they have also stated that detailed analyses of the data make it clear that the QGP has properties that are surprising, and not yet fully understood in terms of the early expectations for the QGP. For example, it was often stated, prior to the RHIC data, that the QGP should behave like an ideal gas of quarks and gluons (i.e., like a weakly-coupled plasma state). However, the dynamical evolution of the hot and/or dense QCD matter produced in the RHIC can be well described by relativistic hydrodynamics with *no* dissipative effects [16, 17]. This fact indicates that the QGP behaves more like an ideal fluid, in analogy to a strongly coupled plasma state. Some theorists have referred to this state as the strongly-coupled quark-gluon plasma, or sQGP [18, 19, 20, 21, 22, 23]. It seems to be the case also for the created matter in the Large Hadron Collider (LHC) in European Organization for Nuclear Research (CERN); see, for example, [24, 25].

The suggestion that the created matter at RHIC and LHC may have only a tiny viscosity prompted an interest in the origin of the viscosity in the created matter to be described using the relativistic quantum field theory and also the relativistic dissipative hydrodynamic equation. We note that since the created matter expands, the proper dynamics for the description may change from hydrodynamics to kinetic one and vice versa [25, 26, 27, 28, 29, 30].

The relativistic dissipative hydrodynamics is also relevant to the soft-mode dynamics [31, 32, 33] around the possible critical point(s) in QCD phase diagram [34, 35]; see [36] for the latest up date. Moreover, the relativistic dissipative hydrodynamic equation has been also applied to various high-energy astrophysical phenomena [37], e.g., the accelerated expansion of the universe caused by the bulk viscosity of dark matter and/or dark energy [38, 39].

It is, however, noteworthy that we have not necessarily reached a full understanding of the theory of relativistic hydrodynamics for viscous fluids, although there have been many important studies since Eckart's pioneering work [40].

1.1 Fundamental three problems with relativistic dissipative hydrodynamics

We may summarize the fundamental problems on the relativistic dissipative hydrodynamic equations as follows: (a) ambiguities and ad-hoc ansatz in the definition of

1.1. FUNDAMENTAL THREE PROBLEMS WITH RELATIVISTIC DISSIPATIVE HYDRODYNAMICS

the flow velocity [40, 41, 42, 43, 44]; (b) unphysical instabilities of the equilibrium state [45, 46]; (c) lack of causality [47, 48, 49, 50].

(a) The form of the relativistic dissipative hydrodynamic equation depends on the definition of the flow velocity, which is equivalent to the choice of the local rest frame of the fluid. Typical rest frames include the particle frame and the energy frame, and a phenomenological equation for the respective frame is constructed by Eckart [40] and Landau and Lifshitz [41], respectively. Here, the phenomenological construction is based on the following three ingredients: the particle-number and energy-momentum conservation laws; the law of the increase in entropy; some specific assumptions on the choice of the flow. The first and second points are reasonable and used also in the construction of the non-relativistic Navier-Stokes equation, while the third point is specific for the relativistic case. To be more explicit, let $\delta T^{\mu\nu}$ and δN^μ be the dissipative part of the energy-momentum tensor and the particle current, respectively. The point is that the forms of $\delta T^{\mu\nu}$ and δN^μ are not determined uniquely only by the particle-number and energy-momentum conservation laws and the law of the increase in entropy without some physical ansatz involving the flow velocity u_μ with $u_\mu u^\mu = g^{\mu\nu} u_\mu u_\nu = 1$ and $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$: The implicit assumptions made by Eckart [40] are

$$(i) \quad \delta e \equiv u_\mu \delta T^{\mu\nu} u_\nu = 0, \quad (1.1.1)$$

$$(ii) \quad \delta n \equiv u_\mu \delta N^\mu = 0, \quad (1.1.2)$$

$$(iii) \quad \nu_\mu \equiv \Delta_{\mu\nu} \delta N^\nu = 0, \quad (1.1.3)$$

where $\Delta_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu$. The condition (i) claims the absence of the internal energy δe of the dissipative origin, while (ii) and (iii) require no dissipative particle-number density δn and current ν^μ , respectively. On the other hand, those of Landau and Lifshitz [41] consist of (i), (ii), and

$$(iv) \quad Q_\mu \equiv \Delta_{\mu\nu} \delta T^{\nu\rho} u_\rho = 0. \quad (1.1.4)$$

In physical terms, (i) and (iv) claim the absence of the internal energy δe and current Q^μ of dissipative origin. As one sees, the first two ansatz are common for the two equations, and the third ones ((iii) and (iv)) are supposed to specify the respective local rest frame of the flow velocity. We note that the conditions (iii) and (iv) and hence the two of frames specified by these conditions can not be connected with each other by a Lorentz transformation. It is here noteworthy that there is a proposal by Stewart [42] for the condition for the particle frame, as given by (ii), (iii), and

$$(v) \quad \delta T^\mu_\mu = \delta e - 3\delta p = 0, \quad (1.1.5)$$

where $\delta p \equiv -\Delta_{\mu\nu} \delta T^{\mu\nu}/3$ is the dissipative pressure to be identified with the standard bulk pressure. Here, the condition (i) of Eckart is replaced by the different one (v), which claims a constraint between the dissipative internal energy δe and the dissipative pressure δp . One may ask if both the Eckart and Stewart ansatz make sense or not. It is noteworthy that the most general derivation [51] of the hydrodynamic

equation on the basis of the phenomenological argument gives a class of equations which can allow the existence of the dissipative internal energy δe and the dissipative particle-number density δn as well as the standard dissipative pressure δp , as shown by the present author and Kunihiro [51].

(b) There arises an unphysical instability of the equilibrium state caused by a special form of the constitutive equation involving the heat flux conventionally adopted in the Eckart (particle) frame [45, 46]. The unphysical instability might be attributed to the lack of causality, and Israel-Stewart's formalism is presently being examined in connection to this problem [52, 53, 54]. Although their equation may get rid of the instability problem with a choice of the relaxation times as shown in Ref. [45], we emphasize that there exists no connection between the unphysical instabilities and the lack of causality. In fact, the Landau-Lifshitz equation is free from the instabilities of the equilibrium state in contrast of the Eckart equation. Furthermore, one should notice that the causal equation by Israel and Stewart is an extended version of the Eckart equation and hence it can naturally exhibit unphysical instabilities depending on the values of transport coefficients and relaxation times contained in the equation [55].

(c) The relativistic dissipative hydrodynamic equations proposed by Eckart, Landau and Lifshitz, and Stewart, unfortunately, suffer from instantaneous propagation of information, which ought to be completely prohibited in a consistent relativistic theory. The origin of this deficiency is the parabolic character of the equations containing the time derivative only in first order [56, 57]. Thus, in order to solve instantaneous propagation, we should modify the form of the equations to the hyperbolic equation by additional terms which contains the second-order time-derivative. These terms are so-called relaxation terms. In 1967, Müller [58, 59] proposed a method to introduce relaxation terms to the original equations in a logically satisfying way, in the context of non-relativistic thermodynamics, where the causality problem also exists: Müller showed that instantaneous propagation lies in the conventional theory's neglect of terms of second order in heat flow and viscous stress in the expression for the entropy. Restoring these terms, Müller derived a modified version of the Navier-Stokes equation with relaxation terms. Müller's theory was rediscovered and extended to relativistic fluid by Israel [49] in 1976. The causal equations are called the *second-order* hydrodynamic equation, while the Eckart, Landau and Lifshitz, and Stewart equations the *first-order* equation. In 1996, Jou and his collaborators [60, 61] called the description by the second-order equation *mesoscopic* since it occupies an intermediate level between the descriptions by hydrodynamics and kinetic theory. In fact, in the fields of the non-relativistic fluid, Müller's second-order equation has been applied to various kinetic problems, e.g., in plasma and in photon transport, whose dynamics often cannot be described by the Navier-Stokes equation because the systems are not close to equilibrium state. Since the advent of Israel's second-order hydrodynamic equation, a number of papers on this subject have appeared [62, 63, 64, 65, 66, 67, 68], but the proper form of second-order equation has not yet been definitely determined. This is because the Müller and Israel methods inherently

cannot restrict the form of the equation to a unique form.

1.2 Hydrodynamics as infrared asymptotic dynamics of kinetic theory

We note the hierarchy of the dynamics of the time evolution of a many-body system: In the beginning of the time evolution of a prepared state, the whole dynamical evolution of the system will be governed by Hamiltonian dynamics that is time-reversal invariant. When the system becomes old, the dynamics is relaxed to the kinetic regime, where the time-evolution system is well described by a truncation of the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierarchy [69, 70, 71, 72]; the Boltzmann equation composed of one-body distribution function describes a coarse grained slower dynamics, in which time-reversal invariance is lost. Then, as the system is further relaxed, the time evolution will be described in terms of the hydrodynamic quantities, i.e., the flow velocity, the particle-number density, and the local temperature. In this sense, the hydrodynamics is the infrared asymptotic dynamics of the kinetic equation.

For obtaining the proper relativistic hydrodynamic equation, it is a legitimate and natural way to start with the relativistic Boltzmann equation which is Lorentz invariant and expected to be free from causality problem [47, 48]; moreover, apparent instability is not known for numerical simulations of the relativistic Boltzmann equation, as far as we are aware of, and the stability is proved at least for the linearized version of it [73, 74]. We note that such an approach is important also for a systematic analysis of RHIC/LHC data, because the proper dynamics for the description may change from hydrodynamics to kinetic one and vice versa, as mentioned above. Indeed, there have been some vigorous attempts to derive the phenomenological equations from the relativistic Boltzmann equation; for instance, with use of the Chapman-Enskog expansion method [47, 48, 75] and the Maxwell-Grad moment method [47, 48, 50, 76]. We would say, however, that these works in the microscopic approaches are not fully satisfactory: Although the past works certainly succeeded in identifying the assumptions and/or approximations to reproduce the known hydrodynamic equations by Eckart, Landau and Lifshitz, Stewart, and Israel, the physical meaning and foundation of these assumptions/approximations remain obscure, and thus the uniqueness of those hydrodynamic equations has never been elucidated as the long-wavelength and low-frequency limit of the underlying dynamics. In fact, the standard derivation of relativistic hydrodynamic equations based on the Chapman-Enskog expansion or Maxwell-Grad moment method [47, 48] utilizes the ansatz given by (i)-(v) as the constraints on the distribution function as the solution of the relativistic Boltzmann equation, rather than consequences of the derivation. Their validity or the fundamental compatibility with the underlying Boltzmann equation has never been questioned nor addressed. This unsatisfactory situation rather reveals the incompleteness of the Chapman-Enskog expansion method and the Maxwell-Grad

moment methods themselves as a reduction theory of the dynamics.

Thus, the form of relativistic dissipative hydrodynamic equations is still controversial and far from being established. The origins of the difficulty of the derivation are identified as the absence of an appropriate coarse-graining method that keeps the Lorentz covariance and is applicable to the relativistic Boltzmann equation. It should be mentioned here that van Kampen [77] applied his reduction theory to derive a relativistic hydrodynamic equation. The resultant equation was, unfortunately, of a non-covariant form.

In this thesis, we aim to derive the relativistic dissipative hydrodynamic equation with a definite frame, a stable equilibrium state, and causality from the relativistic Boltzmann equation in a more natural and systematic way: For this purpose, it is essential to adopt a powerful reduction theory of the dynamics [78]. As such a reduction theory, we take the “renormalization-group (RG) method” [79, 80, 81, 82, 83]. The RG method as formulated in Refs. [81, 82, 83, 84, 85, 86, 87, 88, 89, 90] is a powerful tool for reducing evolution equations based on the notion of attractive manifold or invariant manifold [91], which the dynamical variables approach to and after some time are eventually confined in. In fact, the RG method [81, 82, 83, 84, 85, 86, 87, 88, 89, 90] has been applied to reduce kinetic equations to a slower dynamics with fewer degrees of freedom, which is realized on the invariant manifold asymptotically.

It is important to note that the RG method applied to the non-relativistic Boltzmann equation successfully leads to the Navier-Stokes equation [87, 88] and the causal hydrodynamic equation similar to Müller’s equation [90, 95]: Hatta and Kunihiro [87] and Kunihiro and the author [88] used the RG method to derive the Navier-Stokes equation from the non-relativistic Boltzmann equation. An essential point in the derivation of the Navier-Stokes equation was to utilize five zero modes of the linearized collision operator, which form the invariant manifold on which hydrodynamics is defined; the would-be constant five zero modes, corresponding to temperature, density, and fluid velocity, acquire the time dependence on the manifold by the RG equation. Furthermore, Kunihiro and the author [90, 95] applied the RG method to derive the causal hydrodynamic equation similar to Müller’s equation as the mesoscopic dynamics of the non-relativistic Boltzmann equation. Namely, we constructed the invariant/attractive manifold incorporating some excited (fast) modes as well as the zero modes of the linearized collision operator, because the mesoscopic dynamics is faster than hydrodynamics that is defined on the invariant manifold originally spanned by the five zero modes. We showed that the excited modes that should be adopted are automatically determined from the non-relativistic Boltzmann equation on the basis of the *doublet scheme* in the RG method, which is developed based on the following consistency condition and general principle of the reduction theory of the dynamics [78]: (A) the resultant dynamics should be consistent with the slow dynamics obtained by employing only the zero modes in the asymptotic regime; (B) the resultant dynamics should be as simple as possible because we are interested to reduce the dynamics to a simpler one. Here, we note that the principle (B) is the very spirit

of the reduction theory of the dynamics [78], and here the term “simple” is used to express that the resultant dynamics is described with a fewer number of dynamical variables and is given by an equation composed of a fewer number of terms.

Thus, the RG method should be most suitable for the present purpose to derive relativistic dissipative hydrodynamic equations from the relativistic Boltzmann equation. The present work is simply a relativistic extension of the non-relativistic case [87, 88, 90, 95]. It is also expected that the physical meanings and the validity of the ansatz posed in the phenomenological derivation will be elucidated in the process of the reduction in the RG method.

Although the relativistic Boltzmann equation which we adopt as the kinetic equation is admittedly suitable only for a dilute gas, it is expected that the derived hydrodynamic equation itself can be valid even for dense systems; this is found plausible if one recalls the universal nature of the Navier-Stokes equation beyond dilute systems, although it can be also derived [75, 87, 88] from the non-relativistic Boltzmann equation.

1.3 Results of this thesis

We demonstrate [92, 93, 94, 95, 96, 97] that the relativistic dissipative hydrodynamic equation derived from the relativistic Boltzmann equation with the RG method has a manifest Lorentz covariance and does not show any pathological behavior such as the instability and acausality seen in existing hydrodynamic equations. Furthermore, we show that the equation satisfies the conditions of fit defining the Landau-Lifshitz energy frame, i.e., (i), (ii), and (iv): An explicit form of the equation is given as the set of the continuity equations $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu N^\mu = 0$ with

$$T^{\mu\nu} = e u^\mu u^\nu - (p + \zeta \Pi) \Delta^{\mu\nu} + 2\eta \pi^{\mu\nu}, \quad (1.3.6)$$

$$N^\mu = n u^\mu - \lambda \frac{nT}{e+p} J^\mu, \quad (1.3.7)$$

and the relaxation equations

$$\Pi + \tau_\Pi \frac{\partial}{\partial \tau} \Pi = -\nabla \cdot u, \quad (1.3.8)$$

$$J^\mu + \tau_J \Delta^{\mu\nu} \frac{\partial}{\partial \tau} J_\nu = -\frac{nT}{e+p} \nabla^\mu \frac{\mu}{T}, \quad (1.3.9)$$

$$\pi^{\mu\nu} + \tau_\pi \Delta^{\mu\nu\rho\sigma} \frac{\partial}{\partial \tau} \pi_{\rho\sigma} = \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (1.3.10)$$

with $\partial/\partial\tau \equiv u^\mu \partial_\mu$, $\nabla^\mu \equiv \Delta^{\mu\nu} \partial_\nu$, and $\Delta^{\mu\nu\rho\sigma} \equiv 1/2 (\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - 2/3 \Delta^{\mu\nu} \Delta^{\rho\sigma})$. Here, the dynamical variables are fourteen fields consisting of the temperature T , chemical potential μ , flow velocity u^μ , bulk pressure Π , heat flux J^μ , and viscous pressure $\pi^{\mu\nu}$, where J^μ and $\pi^{\mu\nu}$ are constrained by $u_\mu J^\mu = u_\mu \pi^{\mu\nu} = \pi^\mu_\mu = 0$ and $\pi^{\mu\nu} = \pi^{\nu\mu}$, and n , e , and p denote the equations of state, i.e., the particle-number density, internal energy, and pressure, respectively, which depend on T and μ .

The coefficients ζ , λ , η , τ_Π , τ_J , and τ_π in Eqs. (1.3.6)-(1.3.10) denote the transport coefficients, i.e., the bulk viscosity, heat conductivity, and shear viscosity, and relaxation times corresponding to the bulk pressure, heat flux, and viscous pressure, respectively. The microscopic formulae of the transport coefficients ζ , λ , and η read

$$\zeta = -\frac{1}{T} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \Pi_p \mathcal{L}_{pq}^{-1} \Pi_q = \int_0^\infty ds R_\Pi(s), \quad (1.3.11)$$

$$\lambda = -\frac{1}{3T^2} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} J_p^\mu \mathcal{L}_{pq}^{-1} J_{\mu q} = \int_0^\infty ds R_J(s), \quad (1.3.12)$$

$$\eta = -\frac{1}{10T} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \pi_p^{\mu\nu} \mathcal{L}_{pq}^{-1} \pi_{\mu\nu q} = \int_0^\infty ds R_\pi(s), \quad (1.3.13)$$

and those of the relaxation times τ_Π , τ_J , and τ_π are given by

$$\tau_\Pi = -\frac{\sum_{pqr} \frac{1}{p^0} f_p^{\text{eq}} \Pi_p \mathcal{L}_{pq}^{-1} (q \cdot u) \mathcal{L}_{qr}^{-1} \Pi_r}{\sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \Pi_p \mathcal{L}_{pq}^{-1} \Pi_q} = \frac{\int_0^\infty ds s R_\Pi(s)}{\int_0^\infty ds R_\Pi(s)}, \quad (1.3.14)$$

$$\tau_J = -\frac{\sum_{pqr} \frac{1}{p^0} f_p^{\text{eq}} J_p^\mu \mathcal{L}_{pq}^{-1} (q \cdot u) \mathcal{L}_{qr}^{-1} J_{\mu r}}{\sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} J_p^\mu \mathcal{L}_{pq}^{-1} J_{\mu q}} = \frac{\int_0^\infty ds s R_J(s)}{\int_0^\infty ds R_J(s)}, \quad (1.3.15)$$

$$\tau_\pi = -\frac{\sum_{pqr} \frac{1}{p^0} f_p^{\text{eq}} \pi_p^{\mu\nu} \mathcal{L}_{pq}^{-1} (q \cdot u) \mathcal{L}_{qr}^{-1} \pi_{\mu\nu r}}{\sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \pi_p^{\rho\sigma} \mathcal{L}_{pq}^{-1} \pi_{\rho\sigma q}} = \frac{\int_0^\infty ds s R_\pi(s)}{\int_0^\infty ds R_\pi(s)}, \quad (1.3.16)$$

where p , q , and r denote the indexes of the momentum, f_p^{eq} the local equilibrium distribution function, $\mathcal{L}_{pq}^{-1}$ the inverse matrix of the linearized collision operator $\mathcal{L}_{pq}$, Π_p , J_p^μ , and $\pi_p^{\mu\nu}$ the microscopic representations of the bulk pressure, heat flux, and viscous pressure, and $R_\Pi(s)$, $R_J(s)$, and $R_\pi(s)$ the relaxation functions of them, respectively. It is found that our theory leads to the same expressions for the transport coefficients as given by the Chapman-Enskog expansion method [47] and also novel formulae of the relaxation times in terms of relaxation functions, which allow a natural physical interpretation of the relaxation times. We note that ζ , λ , η , τ_Π , τ_J , and τ_π are definite Lorentz-scalar functions with respect to T and μ , as well as n , e , and p . The explicit definitions of f_p^{eq} , $\mathcal{L}_{pq}$, n , e , p , Π_p , J_p^μ , $\pi_p^{\mu\nu}$, $R_\Pi(s)$, $R_J(s)$, and $R_\pi(s)$ will be shown in Eqs. (2.2.13), (2.3.7), (3.3.2)-(3.3.4), (3.3.8)-(3.3.10), and (3.3.46)-(3.3.48), respectively.

The distribution function which is explicitly constructed in our theory provides an ansatz for the functional form of the distribution function which can reproduce Eqs. (1.3.6)-(1.3.10) in the Maxwell-Grad moment method: The obtained ansatz reads $f_p = f_p^{\text{eq}} (1 + \Phi_p)$, where Φ_p denote the deviation given by

$$\Phi_p = \Phi_p^{\text{TK}} \equiv \frac{1}{T} \sum_q \mathcal{L}_{pq}^{-1} (\Pi_q \Pi + J_q^\mu J_\mu + \pi_q^{\mu\nu} \pi_{\mu\nu}). \quad (1.3.17)$$

We note that Φ_p^{TK} has a nontrivial p dependence due to the existence of $\mathcal{L}_{pq}^{-1}$. Thus, it is obvious that our deviation Φ_p^{TK} differs from the existing deviations, for instance,

that proposed by Israel and Stewart [50] which is at most bilinear with respect to p , i.e.,

$$\Phi_p^{\text{TK}} \neq \Phi_p^{\text{IS}} \equiv a + p^\mu b_\mu + p^\mu p^\nu c_{\mu\nu}, \quad (1.3.18)$$

where a , b^μ , and $c^{\mu\nu}$ denote expansion coefficients.

1.4 Organization of this thesis

This thesis is organized as follows. We shall devote Chap. 2 to preliminaries: In Sec. 2.1, we make an account of the RG method using a simple non-linear equation, and briefly present a foundation of the RG method. In Sec. 2.2, we shall summarize a brief but self-contained account of basic properties of the relativistic Boltzmann equation. In Sec. 2.3, we clarify the ad-hoc aspects of the ansatz made in the standard Chapman-Enskog expansion and Maxwell-Grad moment methods for the derivation of the relativistic dissipative hydrodynamic equations, and present the microscopic representation of the transport coefficients obtained by the Chapman-Enskog expansion and Maxwell-Grad moment methods, respectively, which are not in accord with each others.

In Chap. 3, for analyzing the problems (a) and (b) first, we obtain the first-order relativistic dissipative hydrodynamics by constructing the distribution function in the asymptotic regime as the invariant manifold of the dynamical system: In Sec. 3.1, we introduce a time-like four vector which is called the macroscopic frame vector defining a Lorentz-covariant coordinate system, prove that the macroscopic frame vector is nothing but the fluid velocity, and derive the relativistic Boltzmann equation defined in the covariant coordinate system. In Sec. 3.2, the perturbative expansion of the distribution function with respect to the spatial derivative is performed with the zeroth-order being the local equilibrium distribution function; the dissipative effect is taken into account as a deformation of the distribution function made by the spatial inhomogeneity as the perturbation. In Sec. 3.3, we examine some properties of the resultant equation, concerning the frame, the stability of the equilibrium state, and the transport coefficients.

In Chap. 4, we report on our attempt to examine the problem (c) by constructing the second-order relativistic dissipative hydrodynamics as the mesoscopic dynamics of the relativistic Boltzmann equation: In Sec. 4.1, we develop the doublet scheme, i.e., a generic framework in the RG method to extract the mesoscopic dynamics from an evolution equation by constructing the invariant/attractive manifold incorporating some excited modes as well as the zero modes. It turns out that the number and form of the excited modes that should be included in the invariant/attractive manifold are uniquely determined by the doublet scheme. In Sec. 4.2, we apply the doublet scheme to extract the mesoscopic dynamics of the relativistic Boltzmann equation. Then, we examine some properties of the resultant equation, concerning the causality and the relaxation times as well as the frame, the stability of the equilibrium state, and the transport coefficients.

We devote Chap. 5 to summary and concluding remarks for Chap. 3 and 4.

In Appendix A, we summarize a detailed derivation of formulae used in this thesis. We devote Appendix B to the presentation of extracting the mesoscopic dynamics from a two-dimensional differential equation. Furthermore, in Appendix C, we apply the RG method to the derivation of the causal non-relativistic dissipative hydrodynamics as the mesoscopic dynamics of the non-relativistic Boltzmann equation.

In this thesis, we use the natural unit, i.e., $\hbar = c = k_B = 1$.

Chapter 2

Preliminaries

In this chapter, we shall explain the followings as preliminaries: In Sec. 2.1, we make an account of the RG method. In Sec. 2.2, we shall summarize a brief but self-contained account of basic properties of the relativistic Boltzmann equation. In Sec. 2.3, we clarify the ad-hoc aspects of the ansatz made in the standard Chapman-Enskog expansion and Maxwell-Grad moment methods for the derivation of the relativistic dissipative hydrodynamic equations.

2.1 Renormalization-group method

Our approach is heavily based on the reduction theory of dynamics called the renormalization-group (RG) method [79, 80, 81, 82, 83, 86], and the reliability of our theory is assured by the reliability of the method. It is nice [83, 86] that the RG method can be formulated as an elementary way of construction of the invariant/attractive manifold of dynamical systems; it not only leads to asymptotic dynamics of a given equation but also extract explicitly the differential equations governing the would-be constants appearing in the solution to the differential equation.

In this section, we first make an account of the RG method using a simple non-linear equation, i.e., van der Pol equation with a limit cycle. Then, we briefly present a foundation of the RG method. A detailed account of the RG method including its foundation may be seen in Refs. [81, 83, 85, 86] or review articles [88, 89].

2.1.1 RG method applied to van der Pol equation

Let us take the van der Pol equation which admits a limit cycle:

$$\frac{d^2x}{dt^2} + x = \epsilon(1 - x^2) \frac{dx}{dt}, \quad (2.1.1)$$

where ϵ is supposed to be small.

Let $\tilde{x}(t; t_0)$ be a local solution around $t \sim \forall t_0$, and represent it as a perturbation series;

$$\tilde{x}(t; t_0) = \tilde{x}_0(t; t_0) + \epsilon \tilde{x}_1(t; t_0) + \epsilon^2 \tilde{x}_2(t; t_0) + \cdots \quad (2.1.2)$$

In the RG method, the initial value $W(t_0)$ matters: We suppose that an exact solution is given by $x(t)$ and the initial value of $\tilde{x}(t; t_0)$ at $t = t_0$ is set up to be $x(t_0)$; i.e.,

$$W(t_0) \equiv \tilde{x}(t_0; t_0) = x(t_0). \quad (2.1.3)$$

The initial value as the exact solution should be also expanded as

$$W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \dots \quad (2.1.4)$$

The RG method is actually the method to obtain the initial value $W(t)$ as an exact solution or approximate solution valid in a global domain asymptotically.

The zeroth-order equation reads

$$\mathcal{L} \tilde{x}_0(t; t_0) \equiv \left(\frac{d^2}{dt^2} + 1 \right) \tilde{x}_0(t; t_0) = 0, \quad (2.1.5)$$

and its solution can be written as

$$\tilde{x}_0(t; t_0) = A(t_0) \cos(t + \theta(t_0)). \quad (2.1.6)$$

Here, we have made it explicit that the integral constants A and θ may depend on the initial time t_0 . The initial value reads

$$W_0(t_0) = \tilde{x}_0(t_0; t_0) = A(t_0) \cos(t_0 + \theta(t_0)). \quad (2.1.7)$$

The equation for $\tilde{x}_1$ reads

$$\mathcal{L} \tilde{x}_1(t; t_0) = -A(t_0) \left(1 - \frac{A^2(t_0)}{4} \right) \sin(t + \theta(t_0)) + \frac{A^3(t_0)}{4} \sin(3(t + \theta(t_0))). \quad (2.1.8)$$

Notice that the first term in the right-hand side is a zero mode of the linear operator $\mathcal{L}$ appearing in the left-hand side. Thus, the special solution to this equation necessarily contains a secular term which is given by t times a zero mode of $\mathcal{L}$. Since we have supposed that the initial value at $t = t_0$ is on an exact solution, the corrections from the zeroth-order solution should be as small as possible. This condition is realized by setting the secular terms appearing in the higher orders vanish at $t = t_0$, which is possible because we can add freely zero mode solutions to a special solution. Thus, the first-order solution is uniquely written as

$$\tilde{x}_1(t; t_0) = (t - t_0) \frac{A(t_0)}{2} \left(1 - \frac{A^2(t_0)}{4} \right) \sin(t + \theta(t_0)) - \frac{A^3(t_0)}{32} \sin(3(t + \theta(t_0))). \quad (2.1.9)$$

Notice that the secular term surely vanishes at $t = t_0$ in Eq. (2.1.9), implying that its initial value at $t = t_0$ reads

$$W_1(t_0) = \tilde{x}_1(t_0; t_0) = -\frac{A^3(t_0)}{32} \sin(3(t_0 + \theta(t_0))). \quad (2.1.10)$$

2.1. RENORMALIZATION-GROUP METHOD

The perturbative solution up to this order reads $\tilde{x}(t; t_0) = \tilde{x}_0(t; t_0) + \epsilon \tilde{x}_1(t; t_0)$, which becomes, however, invalid when $|t - t_0|$ becomes large, because of the secular term.

Now notice that the function $\tilde{x}(t; t_0)$ corresponds to a curve drawn in the (t, x) plane for each t_0 ; in other words, we have a family of curves represented by $\tilde{x}(t; t_0)$ in the (t, x) plane, a member of which is parametrized by t_0 . An important observation is that each curve is close to the exact solution in the neighborhood of $t = t_0$. Thus, an idea is that the envelope curve of the family of curves should give a global solution. The envelope curve can be constructed by solving the following equation

$$\left. \frac{\partial}{\partial t_0} \tilde{x}(t; t_0) \right|_{t_0=t} = 0, \quad (2.1.11)$$

which leads to the following equations for $A(t)$ and $\theta(t)$,

$$\frac{dA}{dt} = \epsilon \frac{A}{2} \left(1 - \frac{A^2}{4} \right), \quad (2.1.12)$$

$$\frac{d\theta}{dt} = 0. \quad (2.1.13)$$

These equation are readily solved, and one sees that as $t \rightarrow \infty$, $A(t) \rightarrow 2$ asymptotically, meaning the existence of a limit cycle with a radius 2.

The resultant envelope function as a global solution is given by

$$x_E(t) \equiv \tilde{x}(t; t) = W(t) = A(t) \cos(t + \theta(t)) - \epsilon \frac{A^3(t)}{32} \sin(3(t + \theta(t))), \quad (2.1.14)$$

with $A(t)$ and $\theta(t)$ being the solution of Eqs. (2.1.12) and (2.1.13), respectively. Thus, we have succeeded in not only obtaining the asymptotic solution as whole but also extracting the slow variables $A(t)$ and $\theta(t)$ explicitly and their governing equations.

However, there is a problem left: Does $x_E(t) = \tilde{x}(t; t) = W(t)$ indeed satisfy the original differential equation (2.1.1)? We give here a proof for that [81, 83]. First let us rewrite Eq. (2.1.1) into a coupled equation of the first order:

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}; \epsilon), \quad (2.1.15)$$

where

$$\mathbf{y} = {}^t(q_1 = x, q_2 = dx/dt), \quad (2.1.16)$$

$$\mathbf{F} = {}^t(q_2, -q_1 + \epsilon(1 - q_1^2)q_2). \quad (2.1.17)$$

We have an approximate local solution to Eq. (2.1.15) $\tilde{\mathbf{y}}(t; t_0)$ around $t = t_0$ up to $O(\epsilon^n)$, corresponding to $\tilde{x}(t; t_0)$;

$$\frac{\partial}{\partial t} \tilde{\mathbf{y}}(t; t_0) = \mathbf{F}(\tilde{\mathbf{y}}(t; t_0); \epsilon) + O(\epsilon^{n+1}). \quad (2.1.18)$$

Now, the RG/envelope equation implies that

$$\left. \frac{\partial}{\partial t_0} \tilde{\mathbf{y}}(t; t_0) \right|_{t_0=t} = 0. \quad (2.1.19)$$

The envelope function $\mathbf{y}_E(t)$ corresponding to $x_E(t)$ is defined by

$$\mathbf{y}_E(t) = \tilde{\mathbf{y}}(t; t). \quad (2.1.20)$$

It is now easy to show that $\mathbf{y}_E(t)$ satisfies Eq. (2.1.15) for arbitrary t up to the same order as $\tilde{\mathbf{y}}(t; t_0)$ does locally: In fact, for an arbitrary t ,

$$\begin{aligned} \frac{d\mathbf{y}_E(t)}{dt} &= \left. \frac{\partial}{\partial t} \tilde{\mathbf{y}}(t; t_0) \right|_{t_0=t} + \left. \frac{\partial}{\partial t_0} \tilde{\mathbf{y}}(t; t_0) \right|_{t_0=t} \\ &= \left. \frac{\partial}{\partial t} \tilde{\mathbf{y}}(t; t_0) \right|_{t_0=t} \\ &= \mathbf{F}(\tilde{\mathbf{y}}(t; t); \epsilon) + O(\epsilon^{n+1}) \\ &= \mathbf{F}(\mathbf{y}_E(t); \epsilon) + O(\epsilon^{n+1}). \end{aligned} \quad (2.1.21)$$

This completes the proof. Here, Eqs. (2.1.18) and (2.1.19) have been used together with the definition of $\mathbf{y}_E(t)$ shown in Eq. (2.1.20). It should be stressed that Eq. (2.1.21) is valid uniformly $\forall t$, i.e., in the global domain of t , in contrast to Eq. (2.1.18) which is in a local domain around $t = t_0$.

We can summarize what we have done as follows: When there exist zero modes of the unperturbed operator, the higher-order corrections may cause secular terms, which are renormalized into the integral constants in the zeroth-order solution by the RG/envelope equation (2.1.11), and thereby the would-be integral constants are lifted to dynamical but slow variables.

2.1.2 Foundation of RG method

Now we present a foundation to the RG method using a Wilsonian equation [98, 99, 100] or flow equation by Wegner [101].

Let us take the following n -dimensional equation;

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, t), \quad (2.1.22)$$

where n may be infinity. Let $\mathbf{X}(t) = \mathbf{W}(t)$ be an yet unknown exact solution to Eq. (2.1.22), and we try to solve the equation with the initial condition at $t = \forall t_0$;

$$\mathbf{X}(t = t_0) = \mathbf{W}(t_0). \quad (2.1.23)$$

Then, the solution may be written as $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$.

Now the basis of the RG method lies in the fact that $\mathbf{W}(t_0)$ can be determined on the basis of a simple fact of differential equations. We notice that when the initial

2.2. RELATIVISTIC BOLTZMANN EQUATION

point is shifted to t'_0 , the resultant solution should be the same as long as $\mathbf{W}(t)$ is an exact solution, i.e.,

$$\mathbf{X}(t; t_0, \mathbf{W}(t_0)) = \mathbf{X}(t; t'_0, \mathbf{W}(t'_0)). \quad (2.1.24)$$

Taking the limit $t'_0 \rightarrow t_0$, we have

$$\frac{d\mathbf{X}}{dt_0} = \frac{\partial \mathbf{X}}{\partial t_0} + \frac{\partial \mathbf{X}}{\partial \mathbf{W}} \cdot \frac{d\mathbf{W}}{dt_0} = 0. \quad (2.1.25)$$

This equation gives an evolution equation or the flow equation of the initial value $\mathbf{W}(t_0)$. This equation has the same form as and corresponds to the non-perturbative RG equations (flow equations) by Wilson [98, 99, 100], Wegner-Houghton [101], and so on, in quantum field theory and statistical physics: The 'initial time' t_0 corresponds to (the logarithm of) the renormalization point. We emphasize that the equation (2.1.25) is exact; we have no recourse to any perturbation theory so far.

The problem is to construct the seed of the RG equation (2.1.25), i.e., $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$. For that, let us take the perturbation theory. In this case, $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$ and $\mathbf{X}(t; t'_0, \mathbf{W}(t'_0))$ may be valid only for $t \sim t_0$ and $t \sim t'_0$, which condition is naturally satisfied when $t_0 < t < t'_0$ (or $t'_0 < t < t_0$) because the limit $t'_0 \rightarrow t_0$ is taken eventually. Thus, when a perturbative expansion is employed for constructing $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$, it is necessary to demand

$$\left. \frac{d\mathbf{X}}{dt_0} \right|_{t_0=t} = \left. \frac{\partial \mathbf{X}}{\partial t_0} \right|_{t_0=t} + \left. \frac{\partial \mathbf{X}}{\partial \mathbf{W}} \cdot \frac{d\mathbf{W}}{dt_0} \right|_{t_0=t} = 0. \quad (2.1.26)$$

which automatically implies the condition that $t_0 = t$.

We have already suggested an interpretation that this equation can be identified as a condition to construct an envelope of the curves represented by the unperturbed solutions with different initial times t_0 's [81]: When t_0 is varied, $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$ gives a family of curves with t_0 being a parameter characterizing curves. Then, Eq. (2.1.26) is a condition to construct the envelope of the family of curves which are valid only locally around $t \sim t_0$. The envelope is given by $\mathbf{X}_E(t) \equiv \mathbf{X}(t; t_0 = t, \mathbf{W}(t_0 = t)) = \mathbf{W}(t)$, i.e, the initial value. Furthermore, $\mathbf{X}_E(t)$ satisfies the original equation (2.1.22) in a global domain up to the order with which $\mathbf{X}(t; t_0, \mathbf{W}(t_0))$ satisfies around $t \sim t_0$.

2.2 Relativistic Boltzmann equation

In this section, we summarize the basic facts about the relativistic Boltzmann equation [47] very briefly.

2.2.1 Basics of relativistic Boltzmann equation

The relativistic Boltzmann equation reads [47, 48]

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x), \quad (2.2.1)$$

where $f_p(x)$ denotes the one-particle distribution function with p^μ being the four-momentum of the on-shell particle, i.e., $p^\mu p_\mu = p^2 = m^2$ and $p^0 > 0$, while $C[f]_p(x)$ in the right-hand side of the collision integral

$$C[f]_p(x) \equiv \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1|p_2, p_3) (f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x)), \quad (2.2.2)$$

with $\omega(p, p_1|p_2, p_3)$ being the transition probability due to the microscopic two-particle interaction with the symmetry property

$$\omega(p, p_1|p_2, p_3) = \omega(p_2, p_3|p, p_1) = \omega(p_1, p|p_3, p_2) = \omega(p_3, p_2|p_1, p), \quad (2.2.3)$$

and the energy-momentum conservation

$$\omega(p, p_1|p_2, p_3) \propto \delta^4(p + p_1 - p_2 - p_3). \quad (2.2.4)$$

To make explicit the correspondence to the general formulation of the reduction theory given in Refs. [78, 86], we treat the momentum as a discrete variable; apart from such a formal reasoning, the summation with respect to the momentum may be interpreted as the integration in practical use as follows:

$$\sum_q \equiv \int d^3 \mathbf{q}, \quad (2.2.5)$$

with $\mathbf{q}$ being the spatial components of the four momentum q^μ .

For an arbitrary vector $\varphi_p(x)$, the collision operator satisfies the following identity thanks to the above-mentioned symmetry properties,

$$\begin{aligned} \sum_p \frac{1}{p^0} \varphi_p(x) C[f]_p(x) &= \frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1|p_2, p_3) \\ &\times (\varphi_p(x) + \varphi_{p_1}(x) - \varphi_{p_2}(x) - \varphi_{p_3}(x)) \\ &\times (f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x)). \end{aligned} \quad (2.2.6)$$

Substituting $(1, p^\mu)$ into $\varphi_p(x)$ in Eq. (2.2.6), we find that $(1, p^\mu)$ are collision invariants satisfying

$$\sum_p \frac{1}{p^0} C[f]_p(x) = 0, \quad (2.2.7)$$

$$\sum_p \frac{1}{p^0} p^\mu C[f]_p(x) = 0, \quad (2.2.8)$$

due to the particle-number and energy-momentum conservation in the collision process, respectively. We note that the function $\varphi_{0p}(x) \equiv a(x) + p^\mu b_\mu(x)$ is also a collision invariant where $a(x)$ and $b_\mu(x)$ are arbitrary functions of x . This form is, in fact, the most general form of a collision invariant [47]; see [48] for a proof.

2.2. RELATIVISTIC BOLTZMANN EQUATION

Owing to the particle-number and energy-momentum conservation in the collision process leading to Eqs. (2.2.7) and (2.2.8), we have the balance equations for the particle current $N^\mu(x)$ and the energy-momentum tensor $T^{\mu\nu}(x)$,

$$\partial_\mu N^\mu(x) \equiv \partial_\mu \left[\sum_p \frac{1}{p^0} p^\mu f_p(x) \right] = 0, \quad (2.2.9)$$

$$\partial_\nu T^{\mu\nu}(x) \equiv \partial_\nu \left[\sum_p \frac{1}{p^0} p^\mu p^\nu f_p(x) \right] = 0, \quad (2.2.10)$$

respectively. It should be noted that any dynamical properties are not contained in these equations unless the evolution of $f_p(x)$ has been obtained as a solution to Eq. (2.2.1).

In the Boltzmann theory, the entropy current may be defined [47] by

$$S^\mu(x) \equiv - \sum_p \frac{1}{p^0} p^\mu f_p(x) \left[\ln((2\pi)^3 f_p(x)) - 1 \right], \quad (2.2.11)$$

where the factor $(2\pi)^3$ is necessary owing to our convention (2.2.5) [47]. The entropy current $S^\mu(x)$ satisfies

$$\partial_\mu S^\mu(x) = - \sum_p \frac{1}{p^0} C[f]_p(x) \ln((2\pi)^3 f_p(x)), \quad (2.2.12)$$

due to Eq. (2.2.1). One sees that $S^\mu(x)$ is conserved only if $\ln((2\pi)^3 f_p(x))$ is a collision invariant, i.e., $\ln((2\pi)^3 f_p(x)) = \varphi_{0p}(x) = a(x) + p^\mu b_\mu(x)$. One thus finds [47, 48] that entropy-conserving distribution function may be parametrized as

$$f_p(x) = \frac{1}{(2\pi)^3} \exp \left[\frac{\mu(x) - p^\mu u_\mu(x)}{T(x)} \right] \equiv f_p^{\text{eq}}(x), \quad (2.2.13)$$

with $u^\mu(x) u_\mu(x) = 1$. The function (2.2.13) is identified with the local equilibrium distribution function called the Jüttner function [102], where $T(x)$, $\mu(x)$, and $u^\mu(x)$ in Eq. (2.2.13) are the local temperature, chemical potential, and flow velocity, respectively: We recapitulate the proof of it in Appendix A.1 for completeness, although a lucid and detailed proof may be found in Ref. [48], where the Gibbs-Duhem relation as given by

$$ds = \frac{1}{T} de - \frac{\mu}{T} dn, \quad (2.2.14)$$

is taken for granted, where s , e , and n denote the entropy, internal energy, and particle-number density in the equilibrium state, respectively. The five variables $T(x)$, $\mu(x)$, and $u^\mu(x)$ ($u^\mu(x) u_\mu(x) = 1$) are called hydrodynamic variables. Owing to the energy-momentum conservation in the collision process, we see that the collision integral identically vanishes for the local equilibrium distribution $f_p^{\text{eq}}(x)$ as

$$C[f^{\text{eq}}]_p(x) = 0. \quad (2.2.15)$$

Some remarks are in order here. Van and Biro [43] have recently argued that the conventional Gibbs-Duhem relation (2.2.14) may be modified so as to contain the contribution from the thermal flow in the local equilibrium state of a relativistic system, and given a different interpretation for $T(x)$, $\mu(x)$, and $u^\mu(x)$ in Eq. (2.2.13); this modified definition of the local equilibrium state, they claim, leads to the relativistic hydrodynamic equation in the particle frame with the stable equilibrium state. Although this is certainly an interesting possibility, we will not follow this novel interpretation in this thesis: We shall make some comments on some related problem below.

Substituting the Jüttner function $f_p^{\text{eq}}(x)$ into the particle current $N^\mu(x)$ and the energy-momentum tensor $T^{\mu\nu}(x)$ in the balance equations (2.2.9) and (2.2.10), we have

$$N^\mu(x) = n(x) u^\mu(x), \quad (2.2.16)$$

$$T^{\mu\nu}(x) = e(x) u^\mu(x) u^\nu(x) - p(x) \Delta^{\mu\nu}(x), \quad (2.2.17)$$

with

$$\Delta^{\mu\nu}(x) \equiv g^{\mu\nu} - u^\mu(x) u^\nu(x). \quad (2.2.18)$$

Here, $n(x)$, $e(x)$, and $p(x)$ have been given by

$$\begin{aligned} n(x) &\equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}}(x) (p \cdot u(x)) \\ &= (2\pi)^{-3} 4\pi m^3 e^{\mu(x)/T(x)} (m/T(x))^{-1} K_2(m/T(x)), \end{aligned} \quad (2.2.19)$$

$$e(x) \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}}(x) (p \cdot u(x))^2 = m n(x) \left[\frac{K_3(m/T(x))}{K_2(m/T(x))} - (m/T(x))^{-1} \right], \quad (2.2.20)$$

$$p(x) \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}}(x) (-p^\mu p^\nu \Delta_{\mu\nu}(x)/3) = n(x) T(x), \quad (2.2.21)$$

where we have introduced the second- and third-order modified Bessel functions $K_2(z)$ and $K_3(z)$ defined in Appendix A.2. It is remarkable that $N^\mu(x)$ and $T^{\mu\nu}(x)$ in Eqs. (2.2.16) and (2.2.17) are identical to those in the relativistic Euler equation, which describes the fluid dynamics with no dissipative effects, and $n(x)$, $e(x)$, and $p(x)$ defined by Eqs. (2.2.19)-(2.2.21) are the equations of state of the dilute gas. Since the entropy-conserving distribution function $f_p^{\text{eq}}(x)$ reproduces the relativistic Euler equation, we find that the dissipative effect is attributable to the deviation of $f_p(x)$ from $f_p^{\text{eq}}(x)$.

2.3 Chapman-Enskog expansion and Maxwell - Grad moment methods

To take into account the deviation of $f_p(x)$ from $f_p^{\text{eq}}(x)$, the Chapman-Enskog expansion method [47, 48, 75] and Maxwell-Grad moment method [47, 48, 50, 76] have

2.3. CHAPMAN-ENSKOG EXPANSION AND MAXWELL -GRAD MOMENT METHODS

been developed independently and often used. In this section, we shall first clarify the ad-hoc aspects of the ansatz made in the standard Chapman-Enskog expansion and Maxwell-Grad moment methods for the derivation of the relativistic dissipative hydrodynamic equations. Then, we shall present the microscopic representation of the transport coefficients, e.g., the shear viscosity, obtained by the Chapman-Enskog expansion and Maxwell-Grad moment methods, respectively, and point out that the representations are not in accord with each other.

2.3.1 Ad-hoc aspect in Chapman-Enskog expansion method

In the Chapman-Enskog expansion method [47, 48, 75], one starts from the following form of the relativistic Boltzmann equation,

$$p \cdot u(x) D f_p(x) + p \cdot \nabla f_p(x) = \frac{1}{\epsilon} C[f]_p(x), \quad (2.3.1)$$

where

$$D \equiv u^\mu(x) \partial_\mu, \quad (2.3.2)$$

$$\nabla^\mu \equiv \Delta^{\mu\nu}(x) \partial_\nu. \quad (2.3.3)$$

We should notice the parameter ϵ in front of the right-hand side of Eq. (2.3.1), which measures the relative strength of the gradient and is called the non-uniformity parameter. One makes the perturbative expansion with respect to ϵ as follows:

$$f_p(x) = f_p^{(0)}(x) + \epsilon f_p^{(1)}(x) + \epsilon^2 f_p^{(2)}(x) + \dots \equiv f_p^{(0)}(x) + \delta f_p(x), \quad (2.3.4)$$

Substituting the perturbative expansion (2.3.4) into Eq. (2.3.1), we can obtain series of the perturbative equation with respect to ϵ . In fact, it is found that the zeroth-order solution is given by the local equilibrium distribution function, i.e., the Jüttner function given by Eq. (2.2.13);

$$f_p^{(0)}(x) = f_p^{\text{eq}}(x). \quad (2.3.5)$$

We note that $\delta f_p(x)$ defined in Eq. (2.3.4) is identically the deviation of $f_p(x)$ from $f_p^{\text{eq}}(x)$, which produces the dissipative effects.

Then, we solve the perturbative equations order by order to construct the explicit form of $\delta f_p(x)$ up to an arbitrary order of ϵ : For example, the first-order equation reads

$$\sum_q \mathcal{L}_{pq}(x) (f_q^{\text{eq}}(x))^{-1} f_q^{(1)}(x) = (f_p^{\text{eq}}(x))^{-1} (p \cdot u(x) D + p \cdot \nabla) f_p^{\text{eq}}(x). \quad (2.3.6)$$

Here, $\mathcal{L}_{pq}(x)$ denotes the linearized collision operator given by

$$\mathcal{L}_{pq}(x) \equiv (f_p^{\text{eq}}(x))^{-1} \frac{\partial}{\partial f_q} C[f]_p \Big|_{f=f^{\text{eq}}} (x) f_q^{\text{eq}}(x), \quad (2.3.7)$$

whose eigenvectors belonging to the zero eigenvalue are the collision invariants $(1, p^\mu)$. In fact, by differentiating Eq. (2.2.15) with respect to the five independent variables $\mu(x)/T(x)$ and $u^\mu(x)/T(x)$, we can show that

$$\sum_q \mathcal{L}_{pq}(x) (1, q^\mu) = 0. \quad (2.3.8)$$

Owing to this property of $\mathcal{L}_{pq}(x)$, the solution to the linear inhomogeneous equation (2.3.6) is composed of a sum of the special solution and an arbitrary linear combination of the collision invariants. Thus, we need some appropriate constraints to determine uniquely the form of the solution to Eq. (2.3.6).

It is customary [47] to assume that the five constraints are given as the following forms: First, the particle-number density and internal energy in the non-equilibrium state is the same as those in the local equilibrium state, and set

$$n(x) \equiv u_\mu(x) \left[\sum_p \frac{1}{p^0} p^\mu f_p(x) \right] = u_\mu(x) \left[\sum_p \frac{1}{p^0} p^\mu f_p^{\text{eq}}(x) \right], \quad (2.3.9)$$

$$e(x) \equiv u_\mu(x) u_\nu(x) \left[\sum_p \frac{1}{p^0} p^\mu p^\nu f_p(x) \right] = u_\mu(x) u_\nu(x) \left[\sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}}(x) \right]. \quad (2.3.10)$$

For consistency, one also imposes the constraints to the higher-order terms

$$u_\mu(x) \left[\sum_p \frac{1}{p^0} p^\mu \delta f_p(x) \right] = 0, \quad (2.3.11)$$

$$u_\mu(x) u_\nu(x) \left[\sum_p \frac{1}{p^0} p^\mu p^\nu \delta f_p(x) \right] = 0. \quad (2.3.12)$$

Then, to obtain the hydrodynamic equation in the particle (Eckart) frame, another constraint is imposed

$$\Delta_{\mu\nu}(x) \left[\sum_p \frac{1}{p^0} p^\nu \delta f_p(x) \right] = 0. \quad (2.3.13)$$

Instead, if one wants to obtain the hydrodynamic equation in the energy (Landau-Lifshitz) frame, one imposes the constraint

$$\Delta_{\mu\nu}(x) u_\rho(x) \left[\sum_p \frac{1}{p^0} p^\nu p^\rho \delta f_p(x) \right] = 0. \quad (2.3.14)$$

These conditions imposed to the distribution function in the higher orders are called the *conditions of fit*; the zeroth-order constraint is not a constraint but an identity.

It is noteworthy that although a foundation for them has never been given, such conditions of fit (2.3.9)-(2.3.14) are also imposed in an ad-hoc way even when the Maxwell-Grad moment method is adopted [47]. We stress here that these constraints

are actually equivalent with a strong physical assumption that there are no particle-number density nor internal energy of the dissipative origin, although the distribution function in the non-equilibrium state is quite different from that in the local equilibrium state. It is, therefore, an urgent but yet unsolved problem to verify or elaborate these ad-hoc constraints somehow, say, from a microscopic theory, or by experiment, if possible.

In the RG method which we adopt, one needs no such conditions of fit for derivation, and rather the correct forms of them are obtained as a property of the derived equation: We will see that the conditions of fit in the energy frame is compatible with the underlying relativistic Boltzmann equation and physical, but those in the particle frame is not.

2.3.2 Ad-hoc aspect in Maxwell-Grad moment method

In the Maxwell-Grad moment method [47, 48, 75], the distribution function $f_p(x)$ around $f_p^{\text{eq}}(x)$ is expanded as

$$f_p(x) = f_p^{\text{eq}}(x) \left(1 + \Phi_p(x) \right), \quad (2.3.15)$$

where it is assumed that the deviation $\Phi_p(x)$ is small and the second and higher order can be neglected. Then, the ansatz on $\Phi_p(x)$ is imposed as

$$\Phi_p(x) = a(x) + p^\mu b_\mu(x) + p^\mu p^\nu c_{\mu\nu}(x) \equiv \Phi_p^{\text{IS}}(x). \quad (2.3.16)$$

Here, $a(x)$, $b^\mu(x)$, and $c^{\mu\nu}(x)$ are expansion coefficients whose x -dependence is not yet determined nor $T(x)$, $\mu(x)$, and $u^\mu(x)$ in $f_p^{\text{eq}}(x)$. Furthermore, we can impose that

$$c^{\mu\nu}(x) = c^{\nu\mu}(x), \quad (2.3.17)$$

$$c^\mu_\mu(x) = 0, \quad (2.3.18)$$

without loss of generality due to the following properties: $p^\mu p^\nu = p^\nu p^\mu$ and $p^\mu p_\mu = m^2$. It is noted that $T(x)$, $\mu(x)$, $u^\mu(x)$, $a(x)$, $b^\mu(x)$, and $c^{\mu\nu}(x)$ are not independent quantities because the conditions of fit (2.3.11)-(2.3.14) are imposed to the distribution function (2.3.15). Thus, the total number of independent components of $T(x)$, $\mu(x)$, $u^\mu(x)$, $a(x)$, $b^\mu(x)$, and $c^{\mu\nu}(x)$ is fourteen. In the following, we shall suppress (x) when no misunderstanding is expected.

To determine the x -dependence of the fourteen coefficients T , μ , u^μ , a , b^μ , and $c^{\mu\nu}$, we utilize the fourteen-moment equations which are derived by multiplying the relativistic Boltzmann equation (2.2.1) by appropriate fourteen quantities dependent on the momentum p , and integrating them with respect to p . In Israel-Stewart's theory, the five collision invariants $(1, p^\mu)$ and the second moment $p^\mu p^\nu$ are adopted as the fourteen quantities: In fact, the number of the independent components of

$$1, p^\mu, p^\mu p^\nu, \quad (2.3.19)$$

is fourteen because of $p^\mu p_\mu = m^2$. The fourteen-moment equations consist of the five continuity equations

$$\sum_p \frac{1}{p^0} (1, p^\mu) p^\lambda \partial_\lambda (f_p^{\text{eq}} (1 + \Phi_p)) = 0, \quad (2.3.20)$$

and the nine relaxation equations

$$\sum_p \frac{1}{p^0} p^\mu p^\nu p^\lambda \partial_\lambda (f_p^{\text{eq}} (1 + \Phi_p)) = \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}} \sum_q \mathcal{L}_{pq} \Phi_q. \quad (2.3.21)$$

In the derivation of the relaxation equations, we have used

$$C[f]_p = C[f^{\text{eq}}]_p + \sum_q \frac{\partial}{\partial f_q} C[f]_p \bigg|_{f=f^{\text{eq}}} f_q^{\text{eq}} \Phi_q + O(\Phi^2) = f_p^{\text{eq}} \sum_q \mathcal{L}_{pq} \Phi_q, \quad (2.3.22)$$

where we have utilized Eqs. (2.2.15) and (2.3.7) and neglected $O(\Phi^2)$. Here, we remark that the relaxation equation (2.3.21) is a unique equation in the moment method, which is not seen in the Chapman-Enskog expansion method, and new coefficients appearing in the the relaxation equation are called relaxation times.

It is important to note that the derivation of Israel and Stewart contains one additional approximation besides the expansion (2.3.16) and the conditions of fit (2.3.11)-(2.3.14): As mentioned above, they utilized the five collision invariants $(1, p^\mu)$ and the second moment $p^\mu p^\nu$ to extract the equations of motion for the fourteen independent variables. However, this choice to extract the equations of motion is ambiguous, because any moment of the relativistic Boltzmann equation will lead to a closed set of equations, once the expansion and the conditions of fit are applied. The transport coefficients appearing in the final equations depend on the choice of the moment. In fact, Denicol et al. [67] proposed the moment method based on an alternative expansion

$$\Phi_p = \frac{1}{p \cdot u} (a + p^\mu b_\mu + p^\mu p^\nu c_{\mu\nu}) \equiv \Phi_p^{\text{D}}. \quad (2.3.23)$$

and the choice of the fourteen independent quantities as

$$1, p^\mu, p^\mu/(p \cdot u), p^\mu p^\nu/(p \cdot u). \quad (2.3.24)$$

This moment method leads us to an equation, whose form is the same as the Israel-Stewart equation but the coefficients take different values. In the case that the interactions between each particles can be treated as a hard sphere (a constant differential cross section), they showed that for the one-dimensional scaling expansion, their equation is in better agreement with the numerical solution obtained from the relativistic Boltzmann equation.

2.3.3 Microscopic representations of transport coefficients and relaxation time

Although there exist ad-hoc aspects in the Chapman-Enskog expansion and Maxwell -Grad moment methods shown in the previous subsections, the microscopic representation of the transport coefficients can be obtained as the coefficients in the resultant equation [47, 48, 67]. Here, we discuss the shear viscosity and corresponding relaxation time for example.

We give here the shear viscosity obtained by Maxwell-Grad moment method with the ansatz by Israel-Stewart and Denicol et al., i.e., Φ_p^{IS} and Φ_p^{D} , respectively,

$$\eta^{\text{IS}} = -\frac{1}{10T} \frac{\left(\sum_p \frac{1}{p^0} \pi_p^{\mu\nu} \pi_{p\mu\nu}\right)^2}{\sum_{pq} \frac{1}{p^0} \pi_p^{\rho\sigma} \mathcal{L}_{pq} \pi_{q\rho\sigma}}, \quad (2.3.25)$$

$$\eta^{\text{D}} = -\frac{1}{10T} \frac{\left(\sum_p \frac{1}{p^0} \frac{1}{p \cdot u} \pi_p^{\mu\nu} \pi_{p\mu\nu}\right)^2}{\sum_{pq} \frac{1}{p^0} \frac{1}{p \cdot u} \pi_p^{\rho\sigma} \mathcal{L}_{pq} \pi_{q\rho\sigma} \frac{1}{q \cdot u}}, \quad (2.3.26)$$

when

$$\pi_p^{\mu\nu} \equiv \left(\Delta^{\mu\rho} \Delta^{\nu\sigma} - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma}\right) p_\rho p_\sigma. \quad (2.3.27)$$

It is apparent that η^{IS} differs from η^{D} , and both of η^{IS} and η^{D} are not in agreement with η^{CE} , i.e., the shear viscosity obtained by Chapman-Enskog expansion method which is given by

$$\eta^{\text{CE}} = -\frac{1}{10T} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \pi_p^{\mu\nu} \mathcal{L}_{pq}^{-1} \pi_{q\mu\nu}. \quad (2.3.28)$$

Here, we have introduced $\mathcal{L}_{pq}^{-1}$ as the inverse matrix of the linearized collision operator $\mathcal{L}_{pq}$.

Next, we present the microscopic representation of the relaxation time corresponding to the shear viscosity by Maxwell-Grad moment method as

$$\tau_\pi^{\text{IS}} = -\frac{\sum_p \frac{1}{p^0} (p \cdot u) \pi_p^{\mu\nu} \pi_{p\mu\nu}}{\sum_{pq} \frac{1}{p^0} \pi_p^{\rho\sigma} \mathcal{L}_{pq} \pi_{p\rho\sigma}}, \quad (2.3.29)$$

$$\tau_\pi^{\text{D}} = -\frac{\sum_p \frac{1}{p^0} \frac{1}{p \cdot u} \pi_p^{\mu\nu} \pi_{p\mu\nu}}{\sum_{pq} \frac{1}{p^0} \frac{1}{p \cdot u} \pi_p^{\rho\sigma} \mathcal{L}_{pq} \pi_{q\rho\sigma} \frac{1}{q \cdot u}}. \quad (2.3.30)$$

We note that τ^{IS} is also manifestly different from τ^{D} , as well as the shear viscosity. Furthermore, we note that this discrepancy can be also seen in the other transport coefficients, i.e., the bulk viscosity and heat conductivity, and corresponding relaxation times [47, 48, 67].

Chapter 3

First-order relativistic dissipative hydrodynamics

In this chapter, we try to obtain the first-order relativistic dissipative hydrodynamics by constructing the distribution function in the asymptotic regime as the invariant manifold of the dynamical system, and thereby analyze the ambiguity of the definition of the frame and the unphysical instability of the equilibrium state i.e., the problems (a) and (b) introduced in Chap. 1. In Sec. 3.1, we introduce a time-like four vector which is called the macroscopic frame vector defining a Lorentz-covariant coordinate system suitable for describing the slow dynamics, prove that the macroscopic frame vector is nothing but the fluid velocity if the macroscopic frame vector is independent of the momentum p^μ , and derive the relativistic Boltzmann equation defined in the covariant coordinate system. In Sec. 3.2, the perturbative expansion of the distribution function with respect to the spatial derivative is performed with the zeroth-order being the local equilibrium distribution function; the dissipative effect is taken into account as a deformation of the distribution function made by the spatial inhomogeneity as the perturbation. In Sec. 3.3, we examine some properties of the resultant equation, concerning the frame, the stability of the equilibrium state, and the transport coefficients.

3.1 Macroscopic-frame vector

In this section, we introduce a macroscopic frame vector to specify a local rest frame on which the macroscopic dynamics is described: We shall see that the introduction of the macroscopic frame vector enables us to have a coarse-grained and covariant equation. As in the standard method like the Chapman-Enskog expansion method and others [47], we take the spatial inhomogeneity as the origin of the dissipation.

3.1.1 Macroscopic-frame vector as time-like Lorentz vector

Since we are interested in the hydrodynamic regime where the time and space dependence of the physical quantities are small, we try to solve Eq. (2.2.1) in the situation where the space-time variation of $f_p(x)$ is small and the space-time scales are coarse-grained from those in the kinetic regime. To make a coarse graining with the Lorentz covariance being retained, we introduce a time-like Lorentz vector denoted by

$$\mathbf{a}^\mu, \quad (3.1.1)$$

with $\mathbf{a}^2 > 0$ and $\mathbf{a}^0 > 0$ [92, 93], which may depend on x^μ ; $\mathbf{a}^\mu = \mathbf{a}^\mu(x)$. Thus, $\mathbf{a}^\mu$ specifies the covariant but macroscopic coordinate system where the local rest frame of the flow velocity and/or the flow velocity itself are defined: Since such a coordinate system is called *frame*, we call $\mathbf{a}^\mu$ the *macroscopic frame vector*. In fact, with the use of $\mathbf{a}^\mu$, we define the covariant and macroscopic coordinate system (τ, σ^μ) from the space-time coordinate x^μ as

$$d\tau \equiv \mathbf{a}^\mu dx_\mu, \quad (3.1.2)$$

$$d\sigma^\mu \equiv \left(g^{\mu\nu} - \frac{\mathbf{a}^\mu \mathbf{a}^\nu}{\mathbf{a}^2} \right) dx_\nu \equiv \Delta^{\mu\nu} dx_\nu. \quad (3.1.3)$$

It is noted that the definitions (3.1.2) and (3.1.3) lead to derivatives given by

$$\frac{\partial}{\partial \tau} = \frac{1}{\mathbf{a}^2} \mathbf{a}^\mu \partial_\mu, \quad (3.1.4)$$

$$\frac{\partial}{\partial \sigma_\mu} = \Delta^{\mu\nu} \partial_\nu. \quad (3.1.5)$$

Although the flow velocity u^μ is simply adopted as $\mathbf{a}^\mu$ to define the frame in the existing literature, our point is that this choice is mere customary and lacks in a physical foundation. Thus, we now prove that $\mathbf{a}^\mu$ must be proportional to the flow velocity u^μ , provided that $\mathbf{a}^\mu$ should be independent of the momentum p^μ and time-like vector with the Lorentz covariance.

Noting that u^μ and ∂^μ are the only available Lorentz vectors at hand, we see that the generic form of a Lorentz-covariant vector reads

$$\mathbf{a}^\mu = A_1 u^\mu + A_2 \partial^\mu T + A_3 \partial^\mu \mu + A_4 u^\nu \partial_\nu u^\mu, \quad (3.1.6)$$

where A_i with $i = 1, 2, 3, 4$ are arbitrary Lorentz-scalar functions of the temperature and the chemical potential; $A_i = A_i(T, \mu)$. Now the derivative ∂^μ can be decomposed into the time-like and space-like components as $\partial^\mu = u^\mu u^\nu \partial_\nu + \Delta^{\mu\nu} \partial_\nu = u^\mu D + \nabla^\mu$, where $\Delta^{\mu\nu}$, D , and ∇^μ have been defined in Eqs. (2.3.2)-(2.2.18), respectively. With the use of this decomposition, we write Eq. (3.1.6) as

$$\begin{aligned} \mathbf{a}^\mu &= (A_1 + A_2 DT + A_3 D\mu) u^\mu + A_2 \nabla^\mu T + A_3 \nabla^\mu \mu + A_4 Du^\mu \\ &\equiv C_t(T, \mu) u^\mu + \delta w^\mu, \end{aligned} \quad (3.1.7)$$

3.1. MACROSCOPIC-FRAME VECTOR

with $C_t(T, \mu) = A_1 + A_2 DT + A_3 D\mu$ and $\delta w^\mu = A_2 \nabla^\mu T + A_3 \nabla^\mu \mu + A_4 Du^\mu$. The relative magnitudes of $C_t(T, \mu)$ and δw^μ are only constrained by the inequality $\mathbf{a}^2 > 0$ in the present stage.

However, it should be emphasized that the space-like terms with the coefficients A_2 , A_3 , and A_4 in δw^μ are all derivative terms, which are supposed to be small in the hydrodynamic regime even in the dissipative regime if the dynamics is governed by the hydrodynamics at all. In fact, the space-like terms with the coefficients A_2 and A_3 in δw^μ are of higher order with respect to the dissipative effect and should be ignored in the perturbative approach which we adopt, where dissipative effects are treated as a perturbation. Furthermore, since we start with a stationary solution with vanishing time-dependence in the RG approach, the term with A_4 in δw^μ should be also ignored in the outset. As will be shown in Sec. 3.3.4, even if we include δw^μ , the leading term of δw^μ in the resultant hydrodynamic equation is of third order with respect to temporal and spatial gradients of the hydrodynamic variables, which is of higher order than the dissipative terms in the usual hydrodynamics and negligible. Thus, we have

$$\mathbf{a}^\mu = C_t(T, \mu) u^\mu, \quad (3.1.8)$$

with $C_t(T, \mu) > 0$, and accordingly

$$\Delta^{\mu\nu} = \Delta^{\mu\nu}. \quad (3.1.9)$$

With the use of this macroscopic frame vector, the covariant and macroscopic coordinate system (τ, σ^μ) reads $d\tau = C_t(T, \mu) u^\mu dx_\mu$ and $d\sigma^\mu = \Delta^{\mu\nu} dx_\nu$. Here, the “normalization” factor $C_t(T, \mu)$ is a redundant degree of freedom for the dynamics because $C_t(T, \mu)$ can be always made unity by converting τ into the new temporal coordinate τ' as $d\tau' \equiv C_t(T, \mu)^{-1} d\tau = u^\mu dx_\mu$. Thus, we arrive at

$$\mathbf{a}^\mu = u^\mu. \quad (3.1.10)$$

This completes the foundation for setting that $\mathbf{a}^\mu = u^\mu$, which has been taken for granted without a foundation in the literatures so far.

It is remarkable that this natural choice uniquely leads to the hydrodynamic equation in the energy (Landau-Lifshitz) frame, as will be shown and discussed later [92, 94]. Conversely, a choice of $C_t(T, \mu)$ different from unity with a momentum dependence could lead to various hydrodynamic equations other than that of Landau and Lifshitz, including the one in the particle frame for viscous fluids as was shown by the present authors [92, 94]. However, it is worth emphasizing that the particle frame can be only realized when $C_t(T, \mu)$ has a peculiar momentum dependence such as $C_t(T, \mu) = m/(p \cdot u)$ ($\mathbf{a}^\mu = (m/(p \cdot u)) u^\mu$) [92, 94]. In retrospect, however, the possible momentum dependence of $C_t(T, \mu)$ can not be legitimate for $\mathbf{a}^\mu$ to play a macroscopic frame vector, because it means that the covariant and macroscopic space-time in the particle frame is defined for a respective particle state with a definite energy-momentum, which is certainly unnatural and lead to a trouble in a

physical interpretation. Thus, we naturally require that $C_t(T, \mu)$ is independent of the momentum p^μ and hence $\mathbf{a}^\mu = u^\mu$.

Then, the relativistic Boltzmann equation (2.2.1) in the new coordinate system (τ, σ^μ) is written as

$$p \cdot u(\tau, \sigma) \frac{\partial}{\partial \tau} f_p(\tau, \sigma) + p \cdot \nabla f_p(\tau, \sigma) = C[f]_p(\tau, \sigma), \quad (3.1.11)$$

where $u^\mu(\tau, \sigma) \equiv u^\mu(x)$ and $f_p(\tau, \sigma) \equiv f_p(x)$. We remark the prefactor of the time derivative is a Lorentz scalar and positive definite;

$$p \cdot u(\tau, \sigma) > 0, \quad (3.1.12)$$

which is easily verified by taking the rest frame of p^0 .

Since we are interested in a hydrodynamic solution to Eq. (3.1.11) as mentioned above, we suppose that the time variation of $u^\mu(\tau, \sigma)$ is much smaller than that of the microscopic processes and hence $u^\mu(\tau, \sigma)$ has no τ dependence, i.e.,

$$u^\mu(\tau, \sigma) = u^\mu(\sigma). \quad (3.1.13)$$

Then, with the use of Eq. (3.1.13), we shall convert Eq. (3.1.11) into

$$\frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot u(\sigma)} C[f]_p(\tau, \sigma) - \epsilon \frac{1}{p \cdot u(\sigma)} p \cdot \nabla f_p(\tau, \sigma). \quad (3.1.14)$$

Here, the parameter ϵ , which will be set back to unity eventually, represents a measure of the non-uniformity of the fluid, which may be identified with the ratio of the average particle distance over the mean free path, i.e., the Knudsen number. Since ϵ appears in front of the second term of the right-hand side of Eq. (3.1.14), the relativistic Boltzmann equation has a form to which the perturbative expansion is applicable.

In the present analysis based on the RG method, the perturbative expansion of the distribution function with respect to ϵ is first performed with the zeroth-order being the local equilibrium one; the dissipative effect is taken into account as a deformation of the distribution function made by the spatial inhomogeneity as the perturbation. Thus, the above rewrite of the equation with ϵ reflects a physical assumption that only the spatial inhomogeneity is the origin of the dissipation. It is noteworthy that our RG method applied to the non-relativistic Boltzmann equation with the corresponding assumption successfully leads to the Navier-Stokes equation [87, 88], which means that the present approach [92, 93] is simply a relativistic generalization of the non-relativistic case.

3.2 Reduction of relativistic Boltzmann equation to hydrodynamics with RG method

In this section, starting from Eq. (3.1.14) we shall derive the relativistic dissipative hydrodynamic equation as the infrared asymptotic dynamics of the classical relativistic Boltzmann equation on the basis of the RG method: The five hydrodynamic

3.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO HYDRODYNAMICS WITH RG METHOD

variables, i.e., the flow velocity, local temperature, and particle-number density (or chemical potential), naturally correspond to the zero modes of the linearized collision operator. Then, without recourse to any ansatz such as the conditions of fit given by Eqs. (2.3.9)-(2.3.14), the excited modes which are to modify the local equilibrium distribution function are naturally defined in the sense that they are precisely orthogonal to the zero modes with a properly defined inner product for the distribution functions. It will be shown on the basis of the inner product that the so-called Burnett term does not affect the hydrodynamic equation owing to the fact that the hydrodynamic modes are the zero modes of the linearized collision operator.

3.2.1 Construction of approximate solution around arbitrary initial time

In accordance with the general formulation of the RG method [81, 85, 86], we first try to obtain the perturbative solution $\tilde{f}_p$ to Eq. (3.1.14) around the arbitrary initial time $\tau = \tau_0$ with the initial value $f_p(\tau_0, \sigma)$;

$$\tilde{f}_p(\tau = \tau_0, \sigma; \tau_0) = f_p(\tau_0, \sigma), \quad (3.2.1)$$

where we have made explicit that the solution has the τ_0 dependence. The initial value is not yet specified, we suppose that the initial value is on an exact solution. The initial value as well as the solution is expanded with respect to ϵ as follows;

$$\tilde{f}_p(\tau, \sigma; \tau_0) = \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) + \epsilon \tilde{f}_p^{(1)}(\tau, \sigma; \tau_0) + \epsilon^2 \tilde{f}_p^{(2)}(\tau, \sigma; \tau_0) + \dots, \quad (3.2.2)$$

and

$$f_p(\tau_0, \sigma) = f_p^{(0)}(\tau_0, \sigma) + \epsilon f_p^{(1)}(\tau_0, \sigma) + \epsilon^2 f_p^{(2)}(\tau_0, \sigma) + \dots. \quad (3.2.3)$$

The respective initial conditions at $\tau = \tau_0$ are set up as

$$\tilde{f}_p^{(l)}(\tau_0, \sigma; \tau_0) = f_p^{(l)}(\tau_0, \sigma), \quad l = 0, 1, 2, \dots. \quad (3.2.4)$$

In the expansion, the zeroth-order value $\tilde{f}_p^{(0)}(\tau_0, \sigma; \tau_0) = f_p^{(0)}(\tau_0, \sigma)$ is supposed to be as close as possible to an exact solution.

Substituting the above expansions into Eq. (3.1.14) in the τ -independent but τ_0 -dependent coordinate system with

$$u^\mu(\sigma) = u^\mu(\sigma; \tau_0), \quad (3.2.5)$$

we obtain the series of the perturbative equations with respect to ϵ .

Now the zeroth-order equation reads

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{p \cdot u(\sigma; \tau_0)} C[\tilde{f}^{(0)}]_p(\tau, \sigma; \tau_0). \quad (3.2.6)$$

Since we are interested in the slow motion which would be realized asymptotically as $\tau \rightarrow \infty$, we should take the following stationary solution or the fixed point,

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = 0, \quad (3.2.7)$$

which is realized when

$$\frac{1}{p \cdot u(\sigma; \tau_0)} C[\tilde{f}^{(0)}]_p(\tau, \sigma; \tau_0) = 0, \quad (3.2.8)$$

for arbitrary σ . Thus, we see that $\ln \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)$ can be represented as a linear combination of the five collision invariants $(1, p^\mu)$ as mentioned in Sec. 2.2, and hence $\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0)$ is found to be a local equilibrium distribution function and thus given by the Jüttner function (2.2.13):

$$\tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) = \frac{1}{(2\pi)^3} \exp \left[\frac{\mu(\sigma; \tau_0) - p^\mu u_\mu(\sigma; \tau_0)}{T(\sigma; \tau_0)} \right] \equiv f_p^{\text{eq}}(\sigma; \tau_0). \quad (3.2.9)$$

with $u^\mu(\sigma; \tau_0) u_\mu(\sigma; \tau_0) = 1$, which implies that

$$f_p^{(0)}(\tau_0, \sigma) = \tilde{f}_p^{(0)}(\tau_0, \sigma; \tau_0) = f_p^{\text{eq}}(\sigma; \tau_0). \quad (3.2.10)$$

It should be noticed that the five would-be integral constants $T(\sigma; \tau_0)$, $\mu(\sigma; \tau_0)$, and $u_\mu(\sigma; \tau_0)$ are independent of τ but may depend on τ_0 as well as σ . In the following, we shall suppress the coordinate arguments $(\sigma; \tau_0)$ and the momentum subscript, e.g., p when no misunderstanding is expected.

3.2.2 Linearized collision operator and inner product

Now the first-order equation reads

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(1)}(\tau) = \sum_q A_{pq} \tilde{f}_q^{(1)}(\tau) + F_p, \quad (3.2.11)$$

where the linear evolution operator A and the inhomogeneous term F are defined by

$$\begin{aligned} A_{pq} &\equiv \left. \frac{1}{p \cdot u} \frac{\partial}{\partial f_q} C[f]_p \right|_{f=f^{\text{eq}}} \\ &= \frac{1}{p \cdot u} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) \\ &\quad \times (\delta_{p_2 q} f_{p_3}^{\text{eq}} + f_{p_2}^{\text{eq}} \delta_{p_3 q} - \delta_{p q} f_{p_1}^{\text{eq}} - f_p^{\text{eq}} \delta_{p_1 q}), \end{aligned} \quad (3.2.12)$$

$$F_p \equiv -\frac{1}{p \cdot u} p \cdot \nabla f_p^{\text{eq}}, \quad (3.2.13)$$

respectively.

3.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO HYDRODYNAMICS WITH RG METHOD

To obtain the solution which describes a slow motion, it is convenient to first analyze the spectral properties of A . For this purpose, we convert A to another linear operator

$$L_{pq} \equiv (f_p^{\text{eq}})^{-1} A_{pq} f_q^{\text{eq}} = [(f^{\text{eq}})^{-1} A f^{\text{eq}}]_{pq}, \quad (3.2.14)$$

with the diagonal matrix $f_{pq}^{\text{eq}} \equiv f_p^{\text{eq}} \delta_{pq}$; the explicit form of L is given by

$$L_{pq} = -\frac{1}{p \cdot u} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) f_{p_1}^{\text{eq}} (\delta_{pq} + \delta_{p_1 q} - \delta_{p_2 q} - \delta_{p_3 q}). \quad (3.2.15)$$

Here, we have used the identity

$$\omega(p, p_1 | p_2, p_3) f_{p_2}^{\text{eq}} f_{p_3}^{\text{eq}} = \omega(p, p_1 | p_2, p_3) f_p^{\text{eq}} f_{p_1}^{\text{eq}}, \quad (3.2.16)$$

which follows from Eqs. (2.2.4) and (3.2.9).

Let us define an inner product between arbitrary non-zero vectors φ and ψ by

$$\langle \varphi, \psi \rangle_{\text{eq}} \equiv \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} \varphi_p \psi_p. \quad (3.2.17)$$

We note that the norm defined through this inner product is positive definite

$$\langle \varphi, \varphi \rangle_{\text{eq}} = \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} (\varphi_p)^2 > 0, \quad \varphi_p \neq 0, \quad (3.2.18)$$

since

$$(p \cdot u) > 0, \quad (3.2.19)$$

in accord with Eq. (3.1.12). Notice that the other factors than $(p \cdot u)$ in Eq. (3.2.18) are all positive definite. We shall see that this positive definiteness (3.2.18) of the inner product plays an essential role in making the resultant hydrodynamic equation assure the stability of the thermal equilibrium state, as it should be, in contrast to some phenomenological equations.

With the inner product, it is found that L is self-adjoint

$$\begin{aligned} \langle \varphi, L \psi \rangle_{\text{eq}} &= -\frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) f_p^{\text{eq}} f_{p_1}^{\text{eq}} \\ &\quad \times (\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}) (\psi_p + \psi_{p_1} - \psi_{p_2} - \psi_{p_3}) \\ &= \langle L \varphi, \psi \rangle_{\text{eq}}, \end{aligned} \quad (3.2.20)$$

and semi-negative definite

$$\begin{aligned} \langle \varphi, L \varphi \rangle_{\text{eq}} &= -\frac{1}{2!} \frac{1}{4} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) f_p^{\text{eq}} f_{p_1}^{\text{eq}} \\ &\quad \times (\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3})^2 \\ &\leq 0, \end{aligned} \quad (3.2.21)$$

which means that the eigenvalues of L are zero or negative. In the derivation of Eqs. (3.2.20) and (3.2.21), we have used Eqs. (2.2.4) and (3.2.16). We note that the inner product defined by Eq. (3.2.17) is a unique one which ensures the self-adjoint nature of L .

The eigenvectors belonging to the zero eigenvalue are found to be

$$\varphi_{0p}^\alpha \equiv \begin{cases} p^\mu, & \alpha = \mu, \\ 1 \times m, & \alpha = 4, \end{cases} \quad (3.2.22)$$

which span the kernel of L and satisfy

$$[L \varphi_0^\alpha]_p = 0. \quad (3.2.23)$$

We call φ_{0p}^α the zero modes. It is noted that these zero modes described by the five vectors are collision invariants shown in Eqs. (2.2.7) and (2.2.8), and the factor m in φ_{0p}^4 is introduced merely for convenience so that our method can be applied to the case of massless particles.

Following Ref. [86], we define the projection operator P onto the kernel of L which is called the P space and the projection operator Q onto the Q space complement to the P space:

$$[P \psi]_p \equiv \varphi_{0p}^\alpha \eta_{0\alpha\beta}^{-1} \langle \varphi_0^\beta, \psi \rangle_{\text{eq}}, \quad (3.2.24)$$

$$Q \equiv 1 - P, \quad (3.2.25)$$

where $\eta_{0\alpha\beta}^{-1}$ is the inverse matrix of the the P-space metric matrix $\eta_0^{\alpha\beta}$ defined by

$$\eta_0^{\alpha\beta} \equiv \langle \varphi_0^\alpha, \varphi_0^\beta \rangle_{\text{eq}}. \quad (3.2.26)$$

3.2.3 First-order solution

The solution to Eq. (3.2.11) with the initial condition $\tilde{f}^{(1)}(\tau = \tau_0) = f^{(1)}$, i.e., $\tilde{f}_p^{(1)}(\tau = \tau_0, \sigma; \tau_0) = f_p^{(1)}(\sigma; \tau_0)$ is expressed as

$$\tilde{f}^{(1)}(\tau) = e^{(\tau - \tau_0)A} \left\{ f^{(1)} + A^{-1} \bar{Q} F \right\} + (\tau - \tau_0) \bar{P} F - A^{-1} \bar{Q} F, \quad (3.2.27)$$

where we have introduced the modified projection operators

$$\bar{P} \equiv f^{\text{eq}} P f^{\text{eq}-1}, \quad (3.2.28)$$

$$\bar{Q} \equiv f^{\text{eq}} Q f^{\text{eq}-1}. \quad (3.2.29)$$

We remark that the first term in Eq. (3.2.27) would be a fast motion coming from the Q space, which can be simply eliminated by choosing the initial value $f^{(1)}$, which has not yet been specified, as

$$f^{(1)} = \tilde{f}^{(1)}(\tau_0) = -A^{-1} \bar{Q} F. \quad (3.2.30)$$

3.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO HYDRODYNAMICS WITH RG METHOD

Thus, we have the first-order solution

$$\tilde{f}^{(1)}(\tau) = (\tau - \tau_0) \bar{P} F - A^{-1} \bar{Q} F, \quad (3.2.31)$$

with the initial value (3.2.30). We notice the appearance of the secular term proportional to $\tau - \tau_0$, which apparently invalidates the perturbative solution when $|\tau - \tau_0|$ becomes large. It is worth mentioning that the standard Chapman-Enskog expansion method includes a set of conditions for making secular terms disappear; the conditions are the solubility conditions of the balance equations (2.2.9) and (2.2.10) [47]. In applying the solubility conditions, one needs apply the ad-hoc constraints on the distribution function for defining the flow, e.g., Eckart flow or Landau-Lifshitz flow, as well as the ad-hoc constraints on the particle-number density and internal energy [47], as given by Eqs. (2.3.9)-(2.3.14). In the present RG method, secular terms are allowed to appear and no constraints are imposed on the distribution function; rather the secular terms will be utilized to obtain the slow dynamics.

We remark here that we could apply the RG method here to Eq. (3.2.31), which will give the relativistic Euler equation without dissipation effects. In fact, by applying the RG equation to $\tilde{f}(\tau) = \tilde{f}^{(0)}(\tau) + \epsilon \tilde{f}^{(1)}(\tau)$, we have $\partial_\mu T^{(0)\mu\nu} = \partial_\mu N^{(0)\mu} = 0$ with

$$T^{(0)\mu\nu} = e u^\mu u^\nu - p \Delta^{\mu\nu}, \quad (3.2.32)$$

$$N^{(0)\mu} = n u^\mu, \quad (3.2.33)$$

where e , p , and n denote the internal energy, pressure, particle-number density defined in Eqs. (3.3.2)-(3.3.4), respectively. We note that $T^{(0)\mu\nu}$ and $N^{(0)\mu}$ coincide with the energy-momentum tensor and particle current in Eqs. (2.2.16) and (2.2.17), respectively. A detailed derivation will be given in the case of the second order in Sec. 3.2.5.

To get a dissipative hydrodynamic equation, we need to proceed to the second order in our method.

3.2.4 Second-order solution

The second-order equation is written as

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(2)}(\tau) = \sum_q A_{pq} \tilde{f}_q^{(2)}(\tau) + (\tau - \tau_0)^2 G_p + (\tau - \tau_0) H_p + I_p, \quad (3.2.34)$$

where

$$G_p \equiv \sum_q \sum_r \frac{1}{2} B_{pqr} [\bar{P} F]_q [\bar{P} F]_r, \quad (3.2.35)$$

$$\begin{aligned} H_p \equiv & - \sum_q \sum_r \frac{1}{2} B_{pqr} \left([\bar{P} F]_q [A^{-1} \bar{Q} F]_r + [A^{-1} \bar{Q} F]_q [\bar{P} F]_r \right) \\ & - \frac{1}{p \cdot u} p \cdot \nabla [\bar{P} F]_p, \end{aligned} \quad (3.2.36)$$

$$I_p \equiv \sum_q \sum_r \frac{1}{2} B_{pqr} [A^{-1} \bar{Q} F]_q [A^{-1} \bar{Q} F]_r + \frac{1}{p \cdot u} p \cdot \nabla [A^{-1} \bar{Q} F]_p, \quad (3.2.37)$$

with

$$\begin{aligned}
 B_{pqr} &\equiv \frac{1}{p \cdot u} \frac{\partial^2}{\partial f_q \partial f_r} C[f]_p \Big|_{f=f^{\text{eq}}} \\
 &= \frac{1}{p \cdot u} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p, p_1 | p_2, p_3) \\
 &\quad \times (\delta_{p_2q} \delta_{p_3r} + \delta_{p_2r} \delta_{p_3q} - \delta_{pq} \delta_{p_1r} - \delta_{pr} \delta_{p_1q}).
 \end{aligned} \tag{3.2.38}$$

The solution to Eq. (3.2.34) is found to be

$$\begin{aligned}
 \tilde{f}^{(2)}(\tau) &= e^{(\tau-\tau_0)A} \left\{ f^{(2)} + 2A^{-3} \bar{Q} G + A^{-2} \bar{Q} H + A^{-1} \bar{Q} I \right\} \\
 &\quad + \frac{1}{3} (\tau - \tau_0)^3 \bar{P} G + \frac{1}{2} (\tau - \tau_0)^2 \left\{ \bar{P} H - 2A^{-1} \bar{Q} G \right\} \\
 &\quad + (\tau - \tau_0) \left\{ \bar{P} I - 2A^{-2} \bar{Q} G - A^{-1} \bar{Q} H \right\} \\
 &\quad + \left\{ -2A^{-3} \bar{Q} G - A^{-2} \bar{Q} H - A^{-1} \bar{Q} I \right\}.
 \end{aligned} \tag{3.2.39}$$

Again the would-be fast motion can be eliminated by a choice of the initial value $f^{(2)}$ so that

$$f^{(2)} = \tilde{f}^{(2)}(\tau_0) = -2A^{-3} \bar{Q} G - A^{-2} \bar{Q} H - A^{-1} \bar{Q} I. \tag{3.2.40}$$

Then, we have a second-order solution

$$\begin{aligned}
 \tilde{f}^{(2)}(\tau) &= \frac{1}{3} (\tau - \tau_0)^3 \bar{P} G + \frac{1}{2} (\tau - \tau_0)^2 \left\{ \bar{P} H - 2A^{-1} \bar{Q} G \right\} \\
 &\quad + (\tau - \tau_0) \left\{ \bar{P} I - 2A^{-2} \bar{Q} G - A^{-1} \bar{Q} H \right\} \\
 &\quad + \left\{ -2A^{-3} \bar{Q} G - A^{-2} \bar{Q} H - A^{-1} \bar{Q} I \right\},
 \end{aligned} \tag{3.2.41}$$

We notice again the appearance of secular terms and that no constraints on the solution are imposed for defining the flow, in contrast to the standard Chapman-Enskog expansion method [47].

Summing up the perturbative solutions up to the second order, we have an approximate solution around $\tau \simeq \tau_0$ to this order;

$$\tilde{f}_p(\tau, \sigma; \tau_0) = \tilde{f}_p^{(0)}(\tau, \sigma; \tau_0) + \epsilon \tilde{f}_p^{(1)}(\tau, \sigma; \tau_0) + \epsilon^2 \tilde{f}_p^{(2)}(\tau, \sigma; \tau_0) + O(\epsilon^3). \tag{3.2.42}$$

We emphasize that this solution contains the secular terms which apparently invalidates the perturbative expansion for τ away from the initial time τ_0 .

3.2.5 RG improvement of perturbative expansion

The point of the RG method lies in the fact that we can utilize the secular terms to obtain an asymptotic solution valid in a global domain. Following the general

3.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO HYDRODYNAMICS WITH RG METHOD

argument of the RG method presented in Sec. 2.1 we can see that we have a family of curves $\tilde{f}_p(\tau, \sigma; \tau_0)$ parameterized with τ_0 : They are all on the exact solution $f_p(\sigma; \tau)$ at $\tau = \tau_0$ up to $O(\epsilon^3)$, but only valid locally for τ near τ_0 . So it is conceivable that the envelope of the family of curves which contacts with each local solution at $\tau = \tau_0$ will give a global solution in our asymptotic situation. The envelope which contact with any curve in the family at $\tau = \tau_0$ is obtained by

$$\left. \frac{\partial}{\partial \tau_0} \tilde{f}_p(\tau, \sigma; \tau_0) \right|_{\tau_0=\tau} = 0, \quad (3.2.43)$$

or explicitly

$$\frac{\partial}{\partial \tau} \{ f^{\text{eq}} - \epsilon A^{-1} \bar{Q} F \} - \epsilon \bar{P} F - \epsilon^2 \{ \bar{P} I - 2 A^{-2} \bar{Q} G - A^{-1} \bar{Q} H \} + O(\epsilon^3) = 0, \quad (3.2.44)$$

which gives the equation of motion governing the dynamics of the five slow variables $T(\sigma; \tau)$, $\mu(\sigma; \tau)$, and $u^\mu(\sigma; \tau)$ in $f_p^{\text{eq}}(\sigma; \tau)$. The global solution in the asymptotic region is given as an envelope function,

$$\begin{aligned} f_{\text{Ep}}(\tau, \sigma) &\equiv \tilde{f}_p(\tau, \sigma; \tau_0 = \tau) \\ &= f_p^{\text{eq}}(\sigma; \tau) - \epsilon [A^{-1} \bar{Q} F]_p(\sigma; \tau) \\ &\quad - \epsilon^2 \{ 2 [A^{-3} \bar{Q} G]_p(\sigma; \tau) + [A^{-2} \bar{Q} H]_p(\sigma; \tau) \\ &\quad + [A^{-1} \bar{Q} I]_p(\sigma; \tau) \} + O(\epsilon^3), \end{aligned} \quad (3.2.45)$$

where the exact solution of Eq. (3.2.44) is inserted. As proved in Sec. 2.1, the envelope function $f_{\text{Ep}}(\tau, \sigma)$ satisfies Eq. (3.1.14) in a global domain up to $O(\epsilon^3)$ owing to the condition (3.2.43), although $\tilde{f}_p(\tau, \sigma; \tau_0)$ itself was constructed as a local solution around $\tau \sim \tau_0$. Thus, one sees that $f_{\text{Ep}}(\tau, \sigma)$ now describes a coarse-grained evolution of the one-particle distribution function in Eq. (3.1.14), because the time-derivatives of the quantities in $f_{\text{Ep}}(\tau, \sigma)$ are all in the order of ϵ or higher. We emphasize that we have derived the slow-motion equation of Eq. (3.1.14) in the form of the pair of Eqs. (3.2.44) and (3.2.45).

3.2.6 Reduction of RG equation to simpler form

Now let us see that the RG/envelope equation (3.2.44) is actually the hydrodynamic equation governing the five slow variables, $T(\sigma; \tau)$, $\mu(\sigma; \tau)$, and $u^\mu(\sigma; \tau)$. To show this explicitly, we apply $\bar{P}$ defined in Eq. (3.2.28) from the left and then take the inner product with the five zero modes φ_0^α . In this procedure, we first note that the direct use of the definitions of Eqs. (3.2.24), (3.2.26), and (3.2.28) leads to the following identity;

$$\sum_p \frac{1}{p^0} (p \cdot u) \varphi_{0p}^\alpha [\bar{P} \psi]_p = \sum_p \frac{1}{p^0} (p \cdot u) \varphi_{0p}^\alpha \psi_p. \quad (3.2.46)$$

Furthermore, noting that φ_{0p}^α are the collision invariants as shown in Eqs. (2.2.7) and (2.2.8), we have

$$\begin{aligned} & \sum_p \frac{1}{p^0} (p \cdot u) \varphi_{0p}^\alpha \sum_q \sum_r \frac{1}{2} B_{pqr} [A^{-1} \bar{Q} F]_q [A^{-1} \bar{Q} F]_r \\ &= \sum_p \frac{1}{p^0} \varphi_{0p}^\alpha C[A^{-1} \bar{Q} F]_p = 0, \end{aligned} \quad (3.2.47)$$

where the definitions of Eqs. (2.2.2) and (3.2.38) have been used.

Thus, we have

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[(p \cdot u) \frac{\partial}{\partial \tau} + \epsilon p \cdot \nabla \right] \left\{ f_p^{\text{eq}} - \epsilon [A^{-1} \bar{Q} F]_p \right\} + O(\epsilon^3) = 0, \quad (3.2.48)$$

Putting back $\epsilon = 1$, we arrive at

$$\partial_\mu J_{1\text{st}}^{\mu\alpha} = 0, \quad (3.2.49)$$

with

$$J_{1\text{st}}^{\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha \left\{ f_p^{\text{eq}} - [A^{-1} \bar{Q} F]_p \right\}. \quad (3.2.50)$$

Here, we have used the identity

$$(p \cdot u) \frac{\partial}{\partial \tau} + p \cdot \nabla = p^\mu \partial_\mu, \quad (3.2.51)$$

which follows from Eqs. (3.1.4) and (3.1.5) together with $\mathbf{a}^\mu = u^\mu$. It is noted that $J_{1\text{st}}^{\mu\alpha}$ perfectly agrees with the one obtained by inserting the solution $f_{\text{Ep}}(\tau, \sigma)$ in Eq. (3.2.45) into N^μ and $T^{\mu\nu}$ in Eqs. (2.2.9) and (2.2.10):

$$N^\mu = m^{-1} J_{1\text{st}}^{\mu 4} \equiv N_{1\text{st}}^\mu, \quad (3.2.52)$$

$$T^{\mu\nu} = J_{1\text{st}}^{\mu\nu} \equiv T_{1\text{st}}^{\mu\nu}. \quad (3.2.53)$$

Therefore, we conclude that Eq. (3.2.49) is identically the relativistic dissipative hydrodynamic equation: We note that the subscript of “1st” in $J_{1\text{st}}^{\mu\alpha}$, $N_{1\text{st}}^\mu$, and $T_{1\text{st}}^{\mu\nu}$ means that the obtained equation is in a class of the so-called first-order relativistic dissipative hydrodynamics. As is well known [47], the first-order equations can be derived phenomenologically with use of the entropy current which includes dissipative effects up to the first order.

It is worth mentioning that the problematic Burnett term is absent in Eq. (3.2.48) thanks to Eq. (3.2.47): The Burnett terms are given as a product of spatial gradients of T , μ , and u^μ , such as

$$\nabla^\mu \frac{\mu}{T} \nabla_\mu \frac{\mu}{T}. \quad (3.2.54)$$

3.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO HYDRODYNAMICS WITH RG METHOD

If the Burnett term were to remain, the particle-number and energy-momentum conservation laws are lost; moreover, boundary conditions might have to be taken care of simultaneously because its magnitude is comparable to the Burnett term [103]. In fact, the presence of the Burnett term is known to be inevitable when the Chapman-Enskog expansion method [103] is applied to derive the Navier-Stokes equation from the non-relativistic Boltzmann equation.

We decompose $J_{1st}^{\mu\alpha}$ into two parts as $J_{1st}^{\mu\alpha} = J_{1st}^{(0)\mu\alpha} + \delta J_{1st}^{\mu\alpha}$, where

$$J_{1st}^{(0)\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha f_p^{\text{eq}}, \quad (3.2.55)$$

$$\delta J_{1st}^{\mu\alpha} \equiv - \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha [A^{-1} \bar{Q} F]_p = - \langle \tilde{\varphi}_1^{\mu\alpha}, L^{-1} Q f^{\text{eq}-1} F \rangle_{\text{eq}}, \quad (3.2.56)$$

with

$$\tilde{\varphi}_{1p}^{\mu\alpha} \equiv p^\mu \varphi_{0p}^\alpha \frac{1}{p \cdot u}. \quad (3.2.57)$$

Needless to say, $J_{1st}^{(0)\mu\alpha}$ and $\delta J_{1st}^{\mu\alpha}$ represent the currents in the perfect-fluid and dissipative part, respectively. Corresponding to $J_{1st}^{(0)\mu\alpha}$ and $\delta J_{1st}^{\mu\alpha}$, N_{1st}^μ and $T_{1st}^{\mu\nu}$ in Eqs. (3.2.52) and (3.2.53) are decomposed as

$$N_{1st}^\mu = N_{1st}^{(0)\mu} + \delta N_{1st}^\mu, \quad (3.2.58)$$

$$T_{1st}^{\mu\nu} = T_{1st}^{(0)\mu\nu} + \delta T_{1st}^{\mu\nu}, \quad (3.2.59)$$

where

$$N_{1st}^{(0)\mu} \equiv m^{-1} J_{1st}^{(0)\mu 4}, \quad (3.2.60)$$

$$\delta N_{1st}^\mu \equiv m^{-1} \delta J_{1st}^{\mu 4}, \quad (3.2.61)$$

$$T_{1st}^{(0)\mu\nu} \equiv J_{1st}^{(0)\mu\nu}, \quad (3.2.62)$$

$$\delta T_{1st}^{\mu\nu} \equiv \delta J_{1st}^{\mu\nu}. \quad (3.2.63)$$

For later convenience, we present a simpler form of $\delta J_{1st}^{\mu\alpha}$. First, we note that F in Eq. (3.2.13) is expressed as

$$F_p = -f_p^{\text{eq}} \left(\tilde{\varphi}_{1p}^{\mu 4} m^{-1} \nabla_\mu \frac{\mu}{T} - \tilde{\varphi}_{1p}^{\mu\nu} \nabla_\mu \frac{u_\nu}{T} \right) = -f_p^{\text{eq}} \tilde{\varphi}_{1p}^{\mu\alpha} \bar{X}_{\mu\alpha}, \quad (3.2.64)$$

where

$$\bar{X}_{\mu\alpha} \equiv \nabla_\mu X_\alpha, \quad (3.2.65)$$

with

$$X_\alpha \equiv \begin{cases} -u_\nu/T, & \alpha = \nu, \\ m^{-1} \mu/T, & \alpha = 4. \end{cases} \quad (3.2.66)$$

Using the above representation of F and the identity

$$\langle \varphi, L^{-1} Q \psi \rangle_{\text{eq}} = \langle Q \varphi, L^{-1} Q \psi \rangle_{\text{eq}}, \quad (3.2.67)$$

we can reduce the dissipative part $\delta J_{1\text{st}}^{\mu\alpha}$ to the following form:

$$\delta J_{1\text{st}}^{\mu\alpha} = \eta_1^{\mu\alpha\nu\beta} \bar{X}_{\nu\beta}, \quad (3.2.68)$$

where

$$\eta_1^{\mu\alpha\nu\beta} \equiv \langle \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}}. \quad (3.2.69)$$

Here, we have introduced an important new vector defined by

$$\varphi_{1p}^{\mu\alpha} \equiv [Q \tilde{\varphi}_1^{\mu\alpha}]_p. \quad (3.2.70)$$

We call $\varphi_{1p}^{\mu\alpha}$ the *first-excited modes*. It is noteworthy that $\delta J_{1\text{st}}^{\mu\alpha}$ is represented as a product of $\eta_1^{\mu\alpha\nu\beta}$ and $\bar{X}_{\nu\beta}$: $\eta_1^{\mu\alpha\nu\beta}$ has some information about the transport coefficients, while $\bar{X}_{\nu\beta}$ is identical to the corresponding thermodynamic forces.

Finally, we write down the relativistic dissipative hydrodynamic equation defined in the covariant coordinate system $(\sigma; \tau)$:

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[(p \cdot u) \frac{\partial}{\partial \tau} + p \cdot \nabla \right] \left[f_p^{\text{eq}} \left(1 + [L^{-1} \varphi_1^{\nu\beta}]_p \bar{X}_{\nu\beta} \right) \right] = 0, \quad (3.2.71)$$

which can be derived straightforwardly from Eqs. (3.2.48) and (3.2.64). This form of the relativistic dissipative hydrodynamic equation will be found to play an essential role in the stability analysis to be presented in Sec. 3.3.3.

3.3 Relativistic dissipative hydrodynamic equation

In this section, we give explicit forms of the relativistic dissipative hydrodynamic equations by integrating out the right-hand side of Eqs. (3.2.55) and (3.2.56) with respect to the momentum p^μ . Then, we examine some properties of the resultant equation, concerning the frame, the stability of the equilibrium state, and the transport coefficients.

3.3.1 Uniqueness of Landau-Lifshitz energy frame

After the integration, the currents of perfect-fluid part $J_{1\text{st}}^{(0)\mu\alpha}$ reads

$$J_{1\text{st}}^{(0)\mu\alpha} = \begin{cases} e u^\mu u^\nu - p \Delta^{\mu\nu} = T_{1\text{st}}^{(0)\mu\nu}, & \alpha = \nu, \\ m n u^\mu = m N_{1\text{st}}^{(0)\mu}, & \alpha = 4, \end{cases} \quad (3.3.1)$$

3.3. RELATIVISTIC DISSIPATIVE HYDRODYNAMIC EQUATION

where e , p , and n denote the internal energy, pressure, and particle-number density, respectively. These quantities are defined by

$$n \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}}(p \cdot u) = (2\pi)^{-3} 4\pi m^3 e^{\frac{\mu}{T}} z^{-1} K_2(z), \quad (3.3.2)$$

$$e \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}}(p \cdot u)^2 = m n \left[\frac{K_3(z)}{K_2(z)} - z^{-1} \right], \quad (3.3.3)$$

$$p \equiv \sum_p \frac{1}{p^0} f_p^{\text{eq}}(-p^\mu p^\nu \Delta_{\mu\nu}/3) = n T, \quad (3.3.4)$$

where we have introduced the second- and third-order modified Bessel functions $K_2(z)$ and $K_3(z)$ with z being the dimensionless variable defined by

$$z \equiv \frac{m}{T}. \quad (3.3.5)$$

The explicit form of the modified Bessel functions is presented in Eq. (A.2.6) in Appendix A.2.

Then, we shall now give the explicit representation of $\delta J_{1\text{st}}^{\mu\alpha} = \eta_1^{\mu\alpha\nu\beta} \bar{X}_{\nu\beta}$. For this purpose, we derive an explicit form of $\eta_1^{\mu\alpha\nu\beta}$ in Eq. (3.2.68), which task is tantamount to obtaining an explicit representation of $\varphi_{1p}^{\mu\alpha}$. After a straightforward calculation presented in Appendix A.2, one can find that $\varphi_{1p}^{\mu\alpha}$ are written as

$$\varphi_{1p}^{\mu\alpha} = \begin{cases} -\Delta^{\mu\nu} \hat{\Pi}_p + \hat{\pi}_p^{\mu\nu}, & \alpha = \nu, \\ z \hat{h}^{-1} \hat{J}_p^\mu, & \alpha = 4. \end{cases} \quad (3.3.6)$$

Here, we have introduced $\hat{\Pi}_p$, $\hat{J}_p^\mu$, and $\hat{\pi}_p^{\mu\nu}$ defined by

$$(\hat{\Pi}_p, \hat{J}_p^\mu, \hat{\pi}_p^{\mu\nu}) \equiv (\Pi_p, J_p^\mu, \pi_p^{\mu\nu}) \frac{1}{p \cdot u}, \quad (3.3.7)$$

$$\Pi_p \equiv \left(\frac{4}{3} - \gamma \right) (p \cdot u)^2 + \left((\gamma - 1) T \hat{h} - \gamma T \right) (p \cdot u) - \frac{1}{3} m^2, \quad (3.3.8)$$

$$J_p^\mu \equiv - \left((p \cdot u) - T \hat{h} \right) \Delta^{\mu\nu} p_\nu, \quad (3.3.9)$$

$$\pi_p^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma} p_\rho p_\sigma. \quad (3.3.10)$$

It is noted that $\hat{\Pi}_p$, $\hat{J}_p^\mu$, and $\hat{\pi}_p^{\mu\nu}$ belong to the Q space, and Π_p , J_p^μ , and $\pi_p^{\mu\nu}$ are identically the microscopic representations of dissipative currents in the literature [47]. The definitions of $\hat{h}$, γ , and $\Delta^{\mu\nu\rho\sigma}$ used in Eqs. (3.3.6)-(3.3.10) are

$$\hat{h} \equiv \frac{e + p}{n T}, \quad (3.3.11)$$

$$\gamma \equiv 1 + (z^2 - \hat{h}^2 + 5 \hat{h} - 1)^{-1}, \quad (3.3.12)$$

$$\Delta^{\mu\nu\rho\sigma} \equiv \frac{1}{2} \left(\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \right), \quad (3.3.13)$$

where $\hat{h}$ and γ denote the reduced enthalpy per particle and the ratio of the heat capacities, respectively.

With use of $\varphi_{1p}^{\mu\alpha}$ in Eq. (3.3.6), we can write down $\eta_1^{\mu\alpha\nu\beta} = \langle \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}}$ as

$$\eta_1^{\mu\rho\nu\sigma} = -T \zeta \Delta^{\mu\rho} \Delta^{\nu\sigma} - 2T \eta \Delta^{\mu\rho\nu\sigma}, \quad (3.3.14)$$

$$\eta_1^{\mu\rho\nu 4} = \eta_1^{\nu 4\mu\rho} = 0, \quad (3.3.15)$$

$$\eta_1^{\mu 4\nu 4} = T^2 \lambda z^2 \hat{h}^{-2} \Delta^{\mu\nu}. \quad (3.3.16)$$

Here, we have introduced the transport coefficients, i.e., the bulk viscosity ζ , the heat conductivity λ , and the shear viscosity η given by

$$\zeta \equiv -\frac{1}{T} \langle \hat{\Pi}, L^{-1} \hat{\Pi} \rangle_{\text{eq}}, \quad (3.3.17)$$

$$\lambda \equiv \frac{1}{3T^2} \langle \hat{J}^\mu, L^{-1} \hat{J}_\mu \rangle_{\text{eq}}, \quad (3.3.18)$$

$$\eta \equiv -\frac{1}{10T} \langle \hat{\pi}^{\mu\nu}, L^{-1} \hat{\pi}_{\mu\nu} \rangle_{\text{eq}}, \quad (3.3.19)$$

respectively. In the derivations of Eqs. (3.3.14)-(3.3.16), we have used the following identities

$$\langle \hat{J}^\mu, L^{-1} \hat{J}^\nu \rangle_{\text{eq}} = \frac{1}{3} \Delta^{\mu\nu} \langle \hat{J}^a, L^{-1} \hat{J}_a \rangle_{\text{eq}}, \quad (3.3.20)$$

$$\langle \hat{\pi}^{\mu\nu}, L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} = \frac{1}{5} \Delta^{\mu\nu\rho\sigma} \langle \hat{\pi}^{ab}, L^{-1} \hat{\pi}_{ab} \rangle_{\text{eq}}. \quad (3.3.21)$$

Finally, the dissipative currents $\delta J_{1\text{st}}^{\mu\alpha}$ are obtained from $\eta_1^{\mu\alpha\nu\beta}$ in Eqs. (3.3.14)-(3.3.16) and $\bar{X}_{\nu\beta}$ in Eq. (3.2.65), as follows:

$$\delta J_{1\text{st}}^{\mu\alpha} = \begin{cases} -\zeta \Delta^{\mu\nu} X_\Pi + 2\eta X_\pi^{\mu\nu} = \delta T_{1\text{st}}^{\mu\nu}, & \alpha = \nu, \\ -T \lambda z \hat{h}^{-1} X_J^\mu = m \delta N_{1\text{st}}^\mu, & \alpha = 4, \end{cases} \quad (3.3.22)$$

where the new thermodynamic forces X_Π , X_J^μ , and $X_\pi^{\mu\nu}$ are defined by

$$X_\Pi \equiv T \Delta^{\mu\nu} \bar{X}_{\mu\nu} = -\nabla \cdot u, \quad (3.3.23)$$

$$X_J^\mu \equiv -T \Delta^{\mu\nu} z \hat{h}^{-1} \bar{X}_{\nu 4} = -\hat{h}^{-1} \nabla^\mu (\mu/T), \quad (3.3.24)$$

$$X_\pi^{\mu\nu} \equiv -T \Delta^{\mu\nu\rho\sigma} \bar{X}_{\rho\sigma} = \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma. \quad (3.3.25)$$

Here, the following relations have been used: $u^\mu \bar{X}_{\mu\alpha} = 0$ and $u^\nu \nabla_\mu u_\nu = 0$.

Thus, combining Eqs. (3.3.1) and (3.3.22), we arrive at the explicit form of the energy-momentum tensor and particle-number current

$$T_{1\text{st}}^{(0)\mu\nu} = e u^\mu u^\nu - p \Delta^{\mu\nu}, \quad (3.3.26)$$

$$\delta T_{1\text{st}}^{\mu\nu} = \zeta \Delta^{\mu\nu} \nabla \cdot u + 2\eta \Delta^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma, \quad (3.3.27)$$

$$N_{1\text{st}}^{(0)\mu} = n u^\mu, \quad (3.3.28)$$

$$\delta N_{1\text{st}}^\mu = \lambda \hat{h}^{-2} \nabla^\mu \frac{\mu}{T}, \quad (3.3.29)$$

3.3. RELATIVISTIC DISSIPATIVE HYDRODYNAMIC EQUATION

respectively.

A remark is in order here: We have calculated and presented the thermodynamic quantities and transport coefficients using our model equation, i.e., the relativistic Boltzmann equation, for completeness. Then, the explicit forms of them are inherently for the relativistic rarefied gas. We would like to remind the reader, however, that the main purpose of the present work is to determine the form of the relativistic hydrodynamic equations for a viscous fluid, and expect that the forms of the macroscopic hydrodynamic equations which contain the thermodynamic quantities and transport coefficients only parametrically, and hence the forms are independent of the microscopic expressions of these quantities.

In the rest of this subsection, we discuss the properties of $\delta T_{1st}^{\mu\nu}$ and δN_{1st}^μ in Eqs. (3.3.27) and (3.3.29).

First, it is clear that these formulae completely agree with those proposed by Landau and Lifshitz [41]. Indeed, the respective dissipative parts $\delta T_{1st}^{\mu\nu}$ and δN_{1st}^μ in Eqs. (3.3.27) and (3.3.29) meet Landau and Lifshitz's ansatz

$$u_\mu \delta T_{1st}^{\mu\nu} u_\nu = 0, \quad (3.3.30)$$

$$u_\mu \delta N_{1st}^\mu = 0, \quad (3.3.31)$$

$$\Delta_{\mu\nu} \delta T_{1st}^{\nu\rho} u_\rho = 0. \quad (3.3.32)$$

which are nothing but the constraints imposed in a heuristic way by Landau and Lifshitz in their phenomenological derivation [41, 47].

Next, we shall discuss the underlying meaning of Eqs. (3.3.30)-(3.3.32) in the level of the kinetic equation using the distribution function on the basis of the previous results [92, 93]. As was mentioned above, these equations are usually just imposed [47] to the higher-order terms of the distribution function as the conditions of fit without any foundation to select the hydrodynamic equation in the energy frame. We shall clarify that these conditions are equivalent to the orthogonality condition for the excited modes expressed in terms of the inner product [92, 93] and hence an inevitable consequence for the relativistic hydrodynamics in our analysis which is free from any ansatz.

A manipulation shows [92, 93] that Eqs. (3.2.63) and (3.2.61) can be rewritten as

$$\delta T_{1st}^{\mu\nu} = \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}} \bar{\phi}_p, \quad (3.3.33)$$

$$\delta N_{1st}^\mu = \sum_p \frac{1}{p^0} p^\mu f_p^{\text{eq}} \bar{\phi}_p, \quad (3.3.34)$$

with

$$\bar{\phi}_p \equiv [L^{-1} \varphi_1^{\mu\alpha}]_p \bar{X}_{\mu\alpha}. \quad (3.3.35)$$

It should be emphasized that $\bar{\phi}_p$ belongs to the Q space and thus orthogonal to the zero modes,

$$\langle \varphi_0^\alpha, \bar{\phi} \rangle_{\text{eq}} = 0, \quad \alpha = 0, 1, 2, 3, 4. \quad (3.3.36)$$

Recalling the definition Eq. (3.2.17) of the inner product, we see that Eq. (3.3.36) with $\alpha = \mu = 0, 1, 2, 3$ can be recast into the following form,

$$0 = \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} p^\mu \bar{\phi}_p = u_\nu \left[\sum_p \frac{1}{p^0} p^\nu p^\mu f_p^{\text{eq}} \bar{\phi}_p \right] = u_\nu \delta T_{1\text{st}}^{\mu\nu}, \quad (3.3.37)$$

which readily leads to Eqs. (3.3.30) and (3.3.32). Quite similarly, Eq. (3.3.36) with $\alpha = 4$ is reduced to

$$0 = \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} \bar{\phi}_p = u_\nu \left[\sum_p \frac{1}{p^0} p^\nu f_p^{\text{eq}} \bar{\phi}_p \right] = u_\nu \delta N_{1\text{st}}^\nu, \quad (3.3.38)$$

which is nothing but Eq. (3.3.31).

We now make three remarks here: (1) Our proof of the uniqueness of the energy frame can be traced back to the natural identification of the time-like vector u^μ with the flow velocity and the physical assumption that the dissipative effects can be solely attributed to the spatial inhomogeneity, apart from the Gibbs-Duhem relation in the local equilibrium which leads to the Jüttner function (3.2.9). Conversely, if one of these conditions were not satisfied, the uniqueness of the energy frame could be violated. (2) Recently an interesting paper by Minami and Hidaka [104] appeared, which showed, on the basis of Mori's projection operator method [105], that only the energy frame is natural one for the relativistic hydrodynamics, at least in the linear regime, if the hydrodynamics is an effective dynamics described solely by the genuine slow variables of microscopic Hamiltonian dynamics at all. This is a complementary work to ours which is valid even in the nonlinear regime although based on the Boltzmann equation that is applicable to a dilute gas. (3) Although it is certainly preferable to give an intuitive explanation why the Landau-Lifshitz energy frame should be uniquely chosen, we have, unfortunately, not succeeded in it. We hope that we are able to give an intuitive explanation for the uniqueness near future. It is clear that further studies are needed for establishing the uniqueness of the energy frame in the relativistic hydrodynamics for viscous fluids.

3.3.2 Microscopic representations of transport coefficients

The transport coefficients ζ , λ , and η given by Eqs. (3.3.17)-(3.3.19) can be converted into

$$\zeta = -\frac{1}{T} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \Pi_p \mathcal{L}_{pq}^{-1} \Pi_q, \quad (3.3.39)$$

$$\lambda = \frac{1}{3T^2} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} J_p^\mu \mathcal{L}_{pq}^{-1} J_{q\mu}, \quad (3.3.40)$$

$$\eta = -\frac{1}{10T} \sum_{pq} \frac{1}{p^0} f_p^{\text{eq}} \pi_p^{\mu\nu} \mathcal{L}_{pq}^{-1} \pi_{q\mu\nu}. \quad (3.3.41)$$

Here, Π_p , J_p^μ , and $\pi_p^{\mu\nu}$ are the microscopic representations of dissipative currents given by Eqs. (3.3.8)-(3.3.10), and $\mathcal{L}_{pq}^{-1} = L_{pq}^{-1} (q \cdot u)^{-1}$ denotes the inverse matrix

3.3. RELATIVISTIC DISSIPATIVE HYDRODYNAMIC EQUATION

of $\mathcal{L}_{pq} = (p \cdot u) L_{pq}$, which is derived from Eqs. (2.3.7) and (3.2.15). It turns out that these expressions are in agreement with those obtained by the Chapman-Enskog expansion method [47]; see η^{CE} given by Eq. (2.3.28).

Now, let us rewrite the expressions of ζ , λ , and η in a more familiar form, i.e., the Green-Kubo formula [106, 107, 108] in the linear response theory [69, 70, 71, 72, 109]. With the use of the identity given by

$$\sum_q [L^{-1}]_{pq} (\hat{\Pi}_q, \hat{J}_q^\mu, \hat{\pi}_q^{\mu\nu}) = - \int_0^\infty ds \sum_q [e^{sL}]_{pq} (\hat{\Pi}_q, \hat{J}_q^\mu, \hat{\pi}_q^{\mu\nu}), \quad (3.3.42)$$

due to $\hat{\Pi}_p$, $\hat{J}_p^\mu$, and $\hat{\pi}_p^{\mu\nu}$ belonging to the Q space, we can rewrite Eqs. (3.3.17)-(3.3.19) as

$$\zeta = \int_0^\infty ds R_\Pi(s), \quad (3.3.43)$$

$$\lambda = \int_0^\infty ds R_J(s), \quad (3.3.44)$$

$$\eta = \int_0^\infty ds R_\pi(s), \quad (3.3.45)$$

where

$$R_\Pi(s) \equiv \frac{1}{T} \langle \hat{\Pi}(0), \hat{\Pi}(s) \rangle_{\text{eq}}, \quad (3.3.46)$$

$$R_J(s) \equiv -\frac{1}{3T^2} \langle \hat{J}^\mu(0), \hat{J}_\mu(s) \rangle_{\text{eq}}, \quad (3.3.47)$$

$$R_\pi(s) \equiv \frac{1}{10T} \langle \hat{\pi}^{\mu\nu}(0), \hat{\pi}_{\mu\nu}(s) \rangle_{\text{eq}}, \quad (3.3.48)$$

with the “time-evolved” vectors defined by

$$(\hat{\Pi}_p(s), \hat{J}_p^\mu(s), \hat{\pi}_p^{\mu\nu}(s)) \equiv \sum_q [e^{sL}]_{pq} (\hat{\Pi}_q, \hat{J}_q^\mu, \hat{\pi}_q^{\mu\nu}). \quad (3.3.49)$$

It is noted that $R_\Pi(s)$, $R_J(s)$, and $R_\pi(s)$ in Eqs. (3.3.46)-(3.3.48) are called the relaxation function in the linear response theory [69, 70, 71, 72, 109].

3.3.3 Generic stability of relativistic hydrodynamic equation in energy frame

In this subsection, we shall provide a proof [93, 95] that generic constant solutions of the relativistic dissipative hydrodynamic equation in the energy frame is stable against a small perturbation [46], on account of the positive definiteness of the inner product as shown in Eq. (3.2.18).

Now, a generic constant solution means that it describes a system having a finite homogeneous flow with a constant temperature and a constant chemical potential, as

follows:

$$T(\sigma; \tau) = T_0, \quad (3.3.50)$$

$$\mu(\sigma; \tau) = \mu_0, \quad (3.3.51)$$

$$u_\mu(\sigma; \tau) = u_{0\mu}, \quad (3.3.52)$$

where T_0 , μ_0 , and $u_{0\mu}$ are constant. We note that these states include the thermal equilibrium state as a special case.

We shall show the linear stability of the constant solution of the relativistic dissipative hydrodynamic equation in the energy frame. We represent T , μ , and u_μ around the constant solution as follows:

$$T(\sigma; \tau) = T_0 + \delta T(\sigma; \tau), \quad (3.3.53)$$

$$\mu(\sigma; \tau) = \mu_0 + \delta \mu(\sigma; \tau), \quad (3.3.54)$$

$$u_\mu(\sigma; \tau) = u_{0\mu} + \delta u_\mu(\sigma; \tau), \quad (3.3.55)$$

where the deviations δT , $\delta \mu$, and δu_μ are assumed to so small that terms in the second or higher orders of them can be neglected. Instead of these six variables which are not independent of each other because $\delta u_\mu u_0^\mu = 0$, we use the following five independent variables in accordance with Eq. (3.2.66),

$$\delta X_\alpha \equiv \begin{cases} -\delta(u_\mu/T) = -\delta u_\mu/T_0 + \delta T u_{0\mu}/T_0^2, & \alpha = \mu, \\ m^{-1} \delta(\mu/T) = m^{-1} (\delta \mu/T_0 - \delta T \mu_0/T_0^2), & \alpha = 4. \end{cases} \quad (3.3.56)$$

Substituting Eq. (3.3.56) into Eq. (3.2.71) and with some manipulation, we obtain the linearized equation governing δX_α as

$$\begin{aligned} & \left(\langle \varphi_0^\alpha, \varphi_0^\beta \rangle_{\text{eq}} + \langle \varphi_0^\alpha, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \nabla_\nu \right) \frac{\partial}{\partial \tau} \delta X_\beta \\ & + \left(\langle \tilde{\varphi}_1^{\mu\alpha}, \varphi_0^\beta \rangle_{\text{eq}} \nabla_\mu + \langle \tilde{\varphi}_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \nabla_\mu \nabla_\nu \right) \delta X_\beta = 0. \end{aligned} \quad (3.3.57)$$

Here, we have used the following simple relation

$$\delta(f_p^{\text{eq}}) = f_p^{\text{eq}} \varphi_{0p}^\alpha \delta X_\alpha. \quad (3.3.58)$$

We note that all the coefficients in Eq. (3.3.57) take constant values because they are solely given by the constant solution $(T, \mu, u_\mu) = (T_0, \mu_0, u_{0\mu})$. Owing to the orthogonality between the P and Q spaces, Eq. (3.3.57) is reduced to

$$\eta_0^{\alpha\beta} \frac{\partial}{\partial \tau} \delta X_\beta + D^{\alpha\beta} \delta X_\beta = 0. \quad (3.3.59)$$

Here $\eta_0^{\alpha\beta}$ is the metric tensor defined in (3.2.26) and $D^{\alpha\beta}$ is defined by

$$D^{\alpha\beta} \equiv \langle \tilde{\varphi}_1^{\mu\alpha}, \varphi_0^\beta \rangle_{\text{eq}} \nabla_\mu + \eta_1^{\mu\alpha\nu\beta} \nabla_\mu \nabla_\nu, \quad (3.3.60)$$

3.3. RELATIVISTIC DISSIPATIVE HYDRODYNAMIC EQUATION

with $\eta_1^{\mu\alpha\nu\beta} = \langle \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}}$ given by Eqs. (3.3.14)-(3.3.16). Both of $\eta_0^{\alpha\beta}$ and $D^{\alpha\beta}$ are symmetric tensors.

With the ansatz

$$\delta X_\alpha(\sigma; \tau) = \delta \tilde{X}_\alpha(k; \Lambda) e^{ik \cdot \sigma - \Lambda \tau}, \quad (3.3.61)$$

Eq. (3.3.59) leads to the following algebraic equation,

$$(\Lambda \eta_0^{\alpha\beta} - \tilde{D}^{\alpha\beta}) \delta \tilde{X}_\beta = 0, \quad (3.3.62)$$

with

$$\tilde{D}^{\alpha\beta} \equiv i \langle \tilde{\varphi}_1^{\mu\alpha}, \varphi_0^\beta \rangle_{\text{eq}} k_\mu - \eta_1^{\mu\alpha\nu\beta} k_\mu k_\nu. \quad (3.3.63)$$

Thus, we have the eigenvalue equation as follows,

$$\det(\Lambda \eta_0 - \tilde{D}) = 0, \quad (3.3.64)$$

which leads to the dispersion relation $\Lambda = \Lambda(k)$. The stability of the generic constant solution (3.3.50) against a small perturbation is assured when the real part of $\Lambda(k)$ is nonnegative for any k^μ , which is shown to be the case as follows.

Now, we note that the matrix η_0 is a real symmetric and positive-definite matrix:

$$\begin{aligned} w_\alpha \eta_0^{\alpha\beta} w_\beta &= \langle w_\alpha \varphi_0^\alpha, w_\beta \varphi_0^\beta \rangle_{\text{eq}} \\ &= \langle \varphi, \varphi \rangle_{\text{eq}} > 0, \quad w_\alpha \neq 0, \end{aligned} \quad (3.3.65)$$

with $\varphi_p \equiv w_\alpha \varphi_0^\alpha$. Here, we have used the positive definiteness of the inner product (3.2.18). Equation (3.3.65) means that the inverse matrix η_0^{-1} exists, and η_0^{-1} is also a real symmetric positive-definite matrix. Thus, using the Cholesky decomposition, we can represent η_0^{-1} as

$$\eta_0^{-1} = {}^t U U, \quad (3.3.66)$$

where U denotes a real matrix and ${}^t U$ a transposed matrix of U . Then, Eq. (3.3.64) is converted to

$$\det(\Lambda I - U \tilde{D} {}^t U) = 0, \quad (3.3.67)$$

where I denotes the unit matrix. Equation (3.3.67) tells us that $\Lambda(k)$ is an eigenvalue of $U \tilde{D} {}^t U$.

There is a following theorem: The real part of the eigenvalue of a complex matrix C is nonnegative when the hermitian matrix $\text{Re}(C) \equiv (C + C^\dagger)/2$ is semi-positive definite. Here, we shall present a proof of this theorem. We define the eigenvalue and eigenvector of C as λ and χ , respectively;

$$C \chi = \lambda \chi, \quad (3.3.68)$$

with $\chi^\dagger \chi = 1$. If $\text{Re}(C)$ is semi-positive definite, we can suppose that

$$\psi^\dagger \text{Re}(C) \psi \geq 0, \quad \forall \psi. \quad (3.3.69)$$

By setting $\psi = \chi$ in Eq. (3.3.69) and using Eq. (3.3.68) and its hermitian conjugate, we have

$$(\lambda + \lambda^*)/2 \geq 0, \quad (3.3.70)$$

which means that the real part of the eigenvalue of C is nonnegative. This completes the proof.

Applying this theorem to the present case, we find that the real part of $\Lambda(k)$ becomes nonnegative for any k^μ when $\text{Re}(U \tilde{D}^t U)$ is a semi-positive definite matrix, which is shown to be the case, as follows;

$$\begin{aligned} w_\alpha [\text{Re}(U \tilde{D}^t U)]^{\alpha\beta} w_\beta &= w_\alpha [U \text{Re}(\tilde{D})^t U]^{\alpha\beta} w_\beta \\ &= [wU]_\alpha [\text{Re}(\tilde{D})]^{\alpha\beta} [wU]_\beta \\ &= -[wU]_\alpha \eta_1^{\mu\alpha\nu\beta} k_\mu k_\nu [wU]_\beta \\ &= -\langle k_\mu [wU]_\alpha \varphi_1^{\mu\alpha}, L^{-1} k_\nu [wU]_\beta \varphi_1^{\nu\beta} \rangle_{\text{eq}} \\ &= -\langle \psi, L^{-1} \psi \rangle_{\text{eq}} \geq 0, \quad w_\alpha \neq 0, \end{aligned} \quad (3.3.71)$$

with $\psi_p \equiv k_\mu [wU]_\alpha \varphi_1^{\mu\alpha}$. This completes the proof that the generic constant solution in Eq. (3.3.50) is stable against a small perturbation.

3.3.4 Higher-order correction to relativistic hydrodynamic equation caused by derivative terms in macroscopic frame vector

In this subsection, we show that although we start with the generic form of the macroscopic frame vector $\mathbf{a}^\mu = C_t(T, \mu) u^\mu + \delta w^\mu$ given in Eq. (3.1.7), the leading term of the hydrodynamic equation with respect to δw^μ is a third-order derivative term, which is of higher order than that in the Landau-Lifshitz energy frame derived by setting $\mathbf{a}^\mu = u^\mu$ from the outset.

Here, we set $C_t(T, \mu) = 1$ without loss of generality as discussed in Sec. 3.1.

By replacing u^μ by $\mathbf{a}^\mu$ in all of the equations except for that in f_p^{eq} in Sec. 3.2, we can derive straightforwardly the hydrodynamic equation with $\mathbf{a}^\mu$ not being specified. In fact, we have $\partial_\mu N^\mu(\mathbf{a}) = \partial_\mu T^{\mu\nu}(\mathbf{a}) = 0$ with

$$N^\mu(\mathbf{a}) = N^{(0)\mu}(\mathbf{a}) + \delta N^\mu(\mathbf{a}), \quad (3.3.72)$$

$$T^{\mu\nu}(\mathbf{a}) = T^{(0)\mu\nu}(\mathbf{a}) + \delta T^{\mu\nu}(\mathbf{a}), \quad (3.3.73)$$

where

$$N^{(0)\mu}(\mathbf{a}) = \sum_p \frac{1}{p^0} p^\mu f_p^{\text{eq}}, \quad (3.3.74)$$

3.3. RELATIVISTIC DISSIPATIVE HYDRODYNAMIC EQUATION

$$\delta N^\mu(\mathbf{a}) = - \sum_p \frac{1}{p^0} p^\mu f_p^{\text{eq}} [L^{-1}(\mathbf{a}) \varphi_1^{\nu\beta}(\mathbf{a})]_p \Delta_{\nu\rho}(\mathbf{a}) \partial^\rho X_\beta, \quad (3.3.75)$$

$$T^{(0)\mu\nu}(\mathbf{a}) = \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}}, \quad (3.3.76)$$

$$\delta T^{\mu\nu}(\mathbf{a}) = - \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{eq}} [L^{-1}(\mathbf{a}) \varphi_1^{\nu\beta}(\mathbf{a})]_p \Delta_{\nu\rho}(\mathbf{a}) \partial^\rho X_\beta. \quad (3.3.77)$$

Here, we have made it explicit that L^{-1} , $\varphi_1^{\mu\alpha}$, and $\Delta_{\mu\nu}$ depend on the macroscopic frame vector $\mathbf{a}^\mu$. We note that the zeroth terms $N^{(0)\mu}(\mathbf{a})$ and $T^{(0)\mu\nu}(\mathbf{a})$ are independent of $\mathbf{a}^\mu$, and agree with the particle current and energy-momentum tensor in the relativistic Euler equation as

$$N^{(0)\mu}(\mathbf{a}) = n u^\mu = N_{\text{1st}}^{(0)\mu}, \quad (3.3.78)$$

$$T^{(0)\mu\nu}(\mathbf{a}) = e u^\mu u^\nu - p \Delta^{\mu\nu} = T_{\text{1st}}^{(0)\mu\nu}, \quad (3.3.79)$$

respectively. It is apparent that $N^{(0)\mu}(\mathbf{a})$ and $T^{(0)\mu\nu}(\mathbf{a})$ contain no differential operator ∂_μ . On the other hand, $\delta N^\mu(\mathbf{a})$ and $\delta T^{\mu\nu}(\mathbf{a})$ have a dependence on $\mathbf{a}^\mu$, and manifestly contain terms of the first order of ∂_μ .

Here, we shall count the number of ∂_μ in $\delta N^\mu(\mathbf{a})$ and $\delta T^{\mu\nu}(\mathbf{a})$. For this purpose, we first expand $\delta N^\mu(\mathbf{a})$ and $\delta T^{\mu\nu}(\mathbf{a})$ with respect to δw^μ as

$$\delta N^\mu(\mathbf{a}) = \delta N^\mu(u) + \left. \frac{\partial}{\partial \mathbf{a}^\nu} \delta N^\mu(\mathbf{a}) \right|_{\mathbf{a}=u} \delta w^\nu + \dots, \quad (3.3.80)$$

$$\delta T^{\mu\nu}(\mathbf{a}) = \delta T^{\mu\nu}(u) + \left. \frac{\partial}{\partial \mathbf{a}^\rho} \delta T^{\mu\nu}(\mathbf{a}) \right|_{\mathbf{a}=u} \delta w^\rho + \dots, \quad (3.3.81)$$

where $\delta N^\mu(u)$ and $\delta T^{\mu\nu}(u)$ are nothing but those in the energy frame, i.e., $\delta N^\mu(u) = \delta N_{\text{1st}}^\mu$ and $\delta T^{\mu\nu}(u) = \delta T_{\text{1st}}^{\mu\nu}$.

Then, let us count the number of ∂_μ in each terms of $N^\mu(\mathbf{a})$. We find that

$$N_{\text{1st}}^{(0)\mu} \sim O(\partial^0), \quad (3.3.82)$$

$$\delta N^\mu(u) \sim O(\partial^1), \quad (3.3.83)$$

$$\left. \frac{\partial}{\partial \mathbf{a}^\nu} \delta N^\mu(\mathbf{a}) \right|_{\mathbf{a}=u} \sim O(\partial^1), \quad (3.3.84)$$

$$\delta w^\nu \sim O(\partial^1), \quad (3.3.85)$$

where we note that the counting in Eq. (3.3.84) has been derived from the fact that the partial derivative with respect to $\mathbf{a}^\mu$ does not change the number of ∂_μ that manifestly exists in $\delta N^\mu(\mathbf{a})$. Thus, we have

$$\left. \frac{\partial}{\partial \mathbf{a}^\nu} \delta N^\mu(\mathbf{a}) \right|_{\mathbf{a}=u} \delta w^\nu \sim O(\partial^2). \quad (3.3.86)$$

Finally, by combining Eqs. (3.3.82), (3.3.83), and (3.3.86), we have

$$N^\mu(\mathbf{a}) = N^\mu(u) + O(\partial^2). \quad (3.3.87)$$

The same argument can be given for $T^{\mu\nu}(\mathbf{a})$, and we obtain

$$T^{\mu\nu}(\mathbf{a}) = T^{\mu\nu}(u) + O(\partial^2). \quad (3.3.88)$$

Thus, the hydrodynamic equation with $\mathbf{a}^\mu = u^\mu + \delta w^\mu$ reads

$$0 = \partial_\mu N^\mu(\mathbf{a}) = \partial_\mu N^\mu(u) + O(\partial^3), \quad (3.3.89)$$

$$0 = \partial_\mu T^{\mu\nu}(\mathbf{a}) = \partial_\mu T^{\mu\nu}(u) + O(\partial^3). \quad (3.3.90)$$

The above equations tell us that even if δw^μ is left, the resultant hydrodynamic equation is consistent with one in the energy frame in the hydrodynamic regime.

Chapter 4

Second-order relativistic dissipative hydrodynamics

In this chapter, we report on our attempt to construct the second-order relativistic dissipative hydrodynamics as the mesoscopic dynamics of the relativistic Boltzmann equation, and thereby discuss the problem of the lack of causality, i.e., the problem (c) introduced in Chap. 1. In Sec. 4.1, we develop a method based on the RG method to extract the mesoscopic dynamics from a generic evolution equation by constructing the invariant/attractive manifold incorporating some excited modes as well as the zero modes. It turns out that the number and form of the excited modes that should be included in the invariant/attractive manifold are uniquely determined based on the general principles of the reduction theory. In Sec. 4.2, we apply the method to extract the mesoscopic dynamics of the relativistic Boltzmann equation. Then, we examine some properties of the resultant equation, concerning the causality and the relaxation times as well as the frame, the stability of the equilibrium state, and the transport coefficients.

4.1 Mesoscopic dynamics derived from generic evolution equation with RG method

In this section, we shall develop a method on the basis of the RG method to extract the mesoscopic dynamics from a generic evolution equation by constructing the invariant/attractive manifold incorporating some appropriate excited modes as well as the zero modes of its linearized evolution operator based on the following consistency condition and general principle of the reduction theory of the dynamics [78]: (A) the resultant dynamics should be consistent with the slow dynamics obtained by employing only the zero modes in the asymptotic regime; (B) the resultant dynamics should be as simple as possible because we are interested to reduce the dynamics to a simpler one. As mentioned in Sec. 1.2, we use the principle (B) to derive an equation describing the mesoscopic dynamics, where the number of dynamical variables and terms are reduced as few as possible. We will see that these condition and principle

uniquely determine the number and form of the excited modes that should be included in the invariant/attractive manifold on which the mesoscopic dynamics of the evolution equation is defined.

4.1.1 Evolution equation

As a generic evolution equation, we shall treat a system of differential equations with two non-linear terms, which represent the relaxation to a static solution and weak perturbation, respectively. The equation reads

$$\frac{\partial}{\partial t} X_i = G_i(X_1, \dots, X_N) + \epsilon F_i(X_1, \dots, X_N), \quad i = 1, \dots, N, \quad (4.1.1)$$

which is also rewritten in a more convenient vector form

$$\frac{\partial}{\partial t} \mathbf{X} = \mathbf{G}(\mathbf{X}) + \epsilon \mathbf{F}(\mathbf{X}). \quad (4.1.2)$$

In Eq. (4.1.2), t denotes time, $\mathbf{X}$ dynamical variables represented as a vector whose dimension is N ($1 < N \leq \infty$), $\mathbf{G}(\mathbf{X})$ and $\mathbf{F}(\mathbf{X})$ non-linear functions with respect to $\mathbf{X}$, and ϵ an indicator of the smallness of $\mathbf{F}(\mathbf{X})$ that is finally set equal to 1; the vector $\mathbf{X}(t)$ governed by Eq. (4.1.2) without $\mathbf{F}(\mathbf{X})$ relaxes to the static solution $\mathbf{X}_{\text{st}}$ under time evolution as

$$\mathbf{X}(t \rightarrow \infty) \rightarrow \mathbf{X}_{\text{st}}, \quad (4.1.3)$$

which is given as a solution to

$$\mathbf{G}(\mathbf{X}_{\text{st}}) = 0. \quad (4.1.4)$$

Here, we suppose that the static solution $\mathbf{X}_{\text{st}}$ forms a well-defined M_0 -dimensional invariant manifold with M_0 being smaller than or equal to N . This means that $\mathbf{X}_{\text{st}}$ is parametrized by M_0 integral constants $C_{(\alpha)}$ with $\alpha = 1, \dots, M_0$;

$$\mathbf{X}_{\text{st}} = \mathbf{X}_{\text{st}}(C_{(1)}, \dots, C_{(M_0)}). \quad (4.1.5)$$

We first define the linearized evolution operator A by

$$A_{ij} \equiv \left. \frac{\partial}{\partial X_j} G_i(X_1, \dots, X_N) \right|_{\mathbf{X}=\mathbf{X}_{\text{st}}}. \quad (4.1.6)$$

We note that in accordance with Eq. (4.1.5), A has the M_0 eigenvectors belonging to the zero eigenvalue, i.e., zero modes, and the dimension of the kernel of A is M_0 . In fact, by differentiating Eq. (4.1.4) with respect to the M_0 integral constants $C_{(\alpha)}$, we have

$$A \partial \mathbf{X}_{\text{st}} / \partial C_{(\alpha)} = 0, \quad (4.1.7)$$

4.1. MESOSCOPIC DYNAMICS DERIVED FROM GENERIC EVOLUTION EQUATION WITH RG METHOD

which means that $\varphi_0^{(\alpha)}$ defined by

$$\varphi_0^{(\alpha)} = \partial \mathbf{X}_{\text{st}} / \partial C_{(\alpha)}, \quad (4.1.8)$$

are the M_0 zero modes. The invariant manifold is spanned by $\varphi_0^{(\alpha)}$ with $\alpha = 1, \dots, M_0$.

We shall define the projection operator P_0 onto the kernel of A , which is called the P_0 space, and the projection operator Q_0 onto the Q_0 space as the complement to the P_0 space: With the use of an inner product which satisfies the positive definiteness of the norm as

$$\langle \psi, \psi \rangle > 0, \quad \psi \neq 0, \quad (4.1.9)$$

we define

$$P_0 \psi \equiv \sum_{\alpha=1}^{M_0} \sum_{\beta=1}^{M_0} \varphi_0^{(\alpha)} \eta_{0(\alpha)(\beta)}^{-1} \langle \varphi_0^{(\beta)}, \psi \rangle, \quad (4.1.10)$$

$$Q_0 \equiv 1 - P_0, \quad (4.1.11)$$

where $\eta_{0(\alpha)(\beta)}^{-1}$ is the inverse matrix of the P_0 -space metric matrix $\eta_0^{(\alpha)(\beta)}$ defined by

$$\eta_0^{(\alpha)(\beta)} \equiv \langle \varphi_0^{(\alpha)}, \varphi_0^{(\beta)} \rangle. \quad (4.1.12)$$

Here, we shall treat a specific case of the relaxation, i.e., exponential damping without oscillation, which is equivalent to the case that the other eigenvalues of A are real negative. Thus, we can suppose that with the inner product A is self-adjoint,

$$\langle A \psi, \chi \rangle = \langle \psi, A \chi \rangle, \quad (4.1.13)$$

with ψ and χ being arbitrary vectors. We will see that this self-adjoint nature of A plays an essential role in making the form of the resultant equation simpler.

Furthermore, we shall focus on the case that A has no Jordan cell.

4.1.2 Construction of approximate solution around arbitrary initial time

In accordance with the general formulation of the RG method [81, 82, 83, 85, 86], we first try to obtain the perturbative solution $\tilde{\mathbf{X}}$ to Eq. (4.1.2) around an arbitrary initial time $t = t_0$ with the initial value $\mathbf{X}(t_0)$,

$$\tilde{\mathbf{X}}(t = t_0; t_0) = \mathbf{X}(t_0), \quad (4.1.14)$$

where we have made explicit that the solution has a t_0 dependence. It is noted that in the RG method the initial value $\mathbf{X}(t_0)$ and the RG equation applied to the perturbative solution $\tilde{\mathbf{X}}(t; t_0)$ provide the invariant/attractive manifold and the

reduced dynamics defined on it, respectively. We suppose that the initial value is on a yet unknown exact solution. The initial value as well as the perturbative solution is expanded with respect to ϵ as follows:

$$\tilde{\mathbf{X}}(t; t_0) = \tilde{\mathbf{X}}_0(t; t_0) + \epsilon \tilde{\mathbf{X}}_1(t; t_0) + \epsilon^2 \tilde{\mathbf{X}}_2(t; t_0) + \cdots, \quad (4.1.15)$$

$$\mathbf{X}(t_0) = \mathbf{X}_0(t_0) + \epsilon \mathbf{X}_1(t_0) + \epsilon^2 \mathbf{X}_2(t_0) + \cdots. \quad (4.1.16)$$

The respective initial conditions at $t = t_0$ are set up as

$$\tilde{\mathbf{X}}_i(t = t_0; t_0) = \mathbf{X}_i(t_0), \quad i = 0, 1, 2, \cdots. \quad (4.1.17)$$

In the expansion, the zeroth-order initial value $\tilde{\mathbf{X}}_0(t_0; t_0) = \mathbf{X}_0(t_0)$ is supposed to be as close as possible to an exact solution.

Substituting the above expansions into Eq. (4.1.2), we obtain the series of the perturbative equations with respect to ϵ . Here, we shall carry out the perturbative analysis up to the second order.

The zeroth-order equation reads

$$\frac{\partial}{\partial t} \tilde{\mathbf{X}}_0(t; t_0) = \mathbf{G}(\tilde{\mathbf{X}}_0(t; t_0)). \quad (4.1.18)$$

As mentioned above, we shall construct the invariant/attractive manifold by incorporating appropriate excited modes, as additional components to the invariant manifold spanned by the M_0 zero modes. To achieve the construction in a systematic manner, we suppose that the initial value $\mathbf{X}_0(t_0)$ can be expanded around the stationary solution $\mathbf{X}_{\text{st}}(t_0)$ as

$$\mathbf{X}_0(t_0) = \mathbf{X}_{\text{st}}(t_0) + \boldsymbol{\phi}(t_0), \quad (4.1.19)$$

where $\boldsymbol{\phi}(t_0)$ denotes additional components of the excited modes. Without loss of generality, we can impose the boundary condition

$$\tilde{\mathbf{X}}_0(t \rightarrow \infty; t_0) \rightarrow \mathbf{X}_{\text{st}}(t_0), \quad (4.1.20)$$

whose geometrical image is shown in Fig. 4.1.

In most cases, it is difficult to obtain an exact solution to Eq. (4.1.18) with the initial value $\mathbf{X}_0(t_0)$ as given in Eq. (4.1.19) due to the non-linearity of $\mathbf{G}(\mathbf{X})$. Here, we shall treat $\boldsymbol{\phi}(t_0)$ as a small quantity, whose amplitude of $\boldsymbol{\phi}(t_0)$ is the same as that of the perturbation term $\mathbf{F}(\mathbf{X})$ in the evolution equation (4.1.2), i.e.,

$$\boldsymbol{\phi}(t_0) \sim O(\epsilon). \quad (4.1.21)$$

For the viewpoint of the consistency of the perturbative analysis with respect to ϵ , we should identify $\boldsymbol{\phi}(t_0)$ as the first-order initial value $\mathbf{X}_1(t_0)$:

$$\mathbf{X}_1(t_0) = \tilde{\mathbf{X}}_1(t = t_0; t_0) = \boldsymbol{\phi}(t_0), \quad (4.1.22)$$

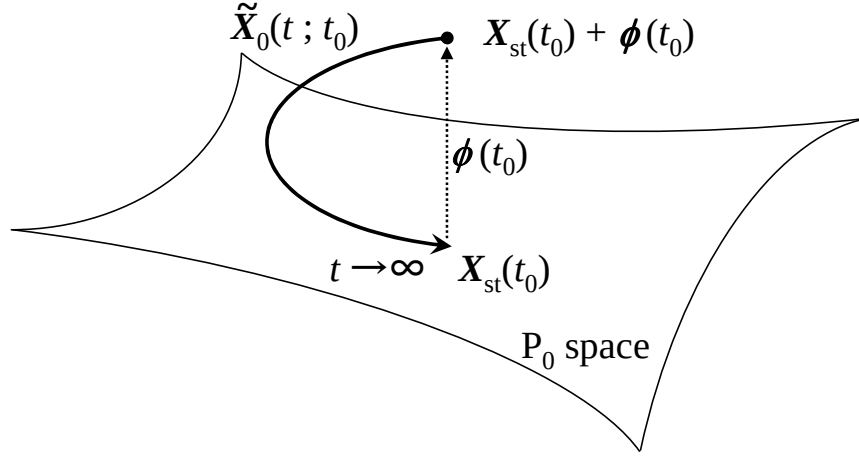


Figure 4.1: The geometrical image of the initial value and solution in the zeroth-order analysis. The solution $\tilde{\mathbf{X}}_0(t; t_0)$ represents the relaxation from the initial value $\mathbf{X}_{\text{st}}(t_0) + \boldsymbol{\phi}(t_0)$ to the stationary solution $\mathbf{X}_{\text{st}}(t_0)$ defined on the invariant manifold belonging to the P_0 space under the time evolution governed by Eq. (4.1.18).

which is constructed to be consistent with Eq. (4.1.20).

We obtain the zeroth-order solution and initial value as

$$\tilde{\mathbf{X}}_0(t; t_0) = \mathbf{X}_{\text{st}}(t_0), \quad (4.1.23)$$

and

$$\mathbf{X}_0(t_0) = \tilde{\mathbf{X}}_0(t = t_0; t_0) = \mathbf{X}_{\text{st}}(t_0), \quad (4.1.24)$$

respectively. We note that $\mathbf{X}_{\text{st}}(t_0)$ depends on t_0 through the would-be M_0 integral constants $C_{(\alpha)}(t_0)$ with $\alpha = 1, \dots, M_0$ defined in Eq. (4.1.5);

$$\mathbf{X}_{\text{st}}(t_0) = \mathbf{X}_{\text{st}}(C_{(1)}(t_0), \dots, C_{(M_0)}(t_0)). \quad (4.1.25)$$

In the following, we shall suppress the initial time t_0 when no misunderstanding is expected.

4.1.3 Doublet scheme

As announced, we determine the first-order initial value $\boldsymbol{\phi}$ on the basis of the consistency condition (A) and general principle of the reduction theory of the dynamics (B). Here, we present an explicit form of $\boldsymbol{\phi}$. For that, let us utilize the *doublet scheme*, which will be explained below.

First, we introduce the M_1 vectors $\tilde{\boldsymbol{\varphi}}_1^{(\mu)}$ with $\mu = 1, \dots, M_1$, which are obtained from the perturbation term $\mathbf{F}(\mathbf{X})$ with $\mathbf{X} = \mathbf{X}_{\text{st}}$ as follows:

$$\mathbf{F}_0 \equiv \mathbf{F}(\mathbf{X}_{\text{st}}) = \sum_{\mu=1}^{M_1} \tilde{\boldsymbol{\varphi}}_1^{(\mu)} \delta \bar{X}_{(\mu)}, \quad (4.1.26)$$

where $\delta\bar{X}_{(\mu)}$ denote expansion coefficients corresponding to $\tilde{\varphi}_1^{(\mu)}$ and depend on $\mathbf{X}_{\text{st}}$. We emphasize that the number and form of $\tilde{\varphi}_1^{(\mu)}$ can be uniquely determined since the above expansion can be carried out without mathematical ambiguity.

Then, with the use of $\tilde{\varphi}_1^{(\mu)}$, we shall define the P_1 space that ϕ belongs to, and introduce the Q_1 space which is the complement to the P_0 and P_1 spaces. We introduce the vectors spanning the P_1 space as

$$\Phi_1^{(n,\mu)} \equiv A^{-n} \varphi_1^{(\mu)}, \quad n = 0, 1, \quad (4.1.27)$$

with

$$\varphi_1^{(\mu)} \equiv Q_0 \tilde{\varphi}_1^{(\mu)}. \quad (4.1.28)$$

We note that the existence of Q_0 in front of $\tilde{\varphi}_1^{(\mu)}$ ensures that the zero modes $\varphi_0^{(\alpha)}$ are not included in $\varphi_1^{(\mu)}$, and hence A^{-1} can be applied to $\varphi_1^{(\mu)}$ as shown in Eq. (4.1.27) without mathematical ambiguity. Since $\Phi_1^{(1,\mu)} = A^{-1} \varphi_1^{(\mu)}$ do not contain the zero modes either, the following identity is satisfied:

$$Q_0 \Phi_1^{(n,\mu)} = \Phi_1^{(n,\mu)}. \quad (4.1.29)$$

We call $\Phi_1^{(n,\mu)}$ with $n = 0, 1$ the *doublet mode*. It is noted that the doublet mode cannot be the eigenvector of A , and is generically written as a linear combination of the eigenvectors of different finite eigenvalues of A . The projection operators onto the P_1 and Q_1 spaces are given by

$$P_1 \psi \equiv \sum_{n=0,1} \sum_{\mu=1}^{M_1} \sum_{m=0,1} \sum_{\nu=1}^{M_1} \Phi_1^{(n,\mu)} \eta_{1(n,\mu)(m,\nu)}^{-1} \langle \Phi_1^{(m,\nu)}, \psi \rangle, \quad (4.1.30)$$

$$Q_1 \equiv Q_0 - P_1, \quad (4.1.31)$$

respectively, where $\eta_{1(n,\mu)(m,\nu)}^{-1}$ has been introduced as the inverse matrix of the P_1 -space metric matrix given by

$$\eta_1^{(n,\mu)(m,\nu)} \equiv \langle \Phi_1^{(n,\mu)}, \Phi_1^{(m,\nu)} \rangle. \quad (4.1.32)$$

We note that P_0 , P_1 , and Q_1 satisfy the following properties:

$$1 = P_0 + P_1 + Q_1, \quad (4.1.33)$$

$$P_0^2 = P_0, \quad (4.1.34)$$

$$P_1^2 = P_1, \quad (4.1.35)$$

$$Q_1^2 = Q_1, \quad (4.1.36)$$

$$P_0 P_1 = P_1 P_0 = P_0 Q_1 = 0, \quad (4.1.37)$$

$$Q_1 P_0 = P_1 Q_1 = Q_1 P_1 = 0. \quad (4.1.38)$$

It is noted that we utilize the projection operators P_0 , P_1 , and Q_1 to construct the first- and second-order solutions, where only the motion caused by the P_0 and P_1 spaces appears.

4.1. MESOSCOPIC DYNAMICS DERIVED FROM GENERIC EVOLUTION EQUATION WITH RG METHOD

Table 4.1: The doublet scheme in the RG method and the mechanism to produce a simple reduced dynamics which can reproduce the slow dynamics as described with only the zero modes in the asymptotic regime are summarized. Owing to Eqs. (4.1.42), (4.1.60), and (4.1.61) derived from the explicit form of the modes belonging to the P_0 and P_1 spaces and the would-be integral constants correspondent to the modes, we can obtain Eqs. (4.1.58) and (4.1.59) as an equation governing the dynamics of $\delta X_{(\mu)}$ and $C_{(\alpha)}$, which is consistent with the evolution equation (4.1.2) in the mesoscopic regime. We have defined $P_A = P_1 (A - \partial/\partial\tau) P_1 (A - \partial/\partial\tau)^{-1} Q_0$ and $\mathbf{R}(\tau) = \mathbf{F}_0$ or $\mathbf{K}(\tau)$.

Subspace	Mode	Integration constant	
P_0	$\varphi_0^{(\alpha)}$	$\leftrightarrow$	$C_{(\alpha)}$
P_1	$\Phi_1^{(1,\mu)} \equiv A^{-1} \varphi_1^{(\mu)}$	$\leftrightarrow$	$\delta X_{(\mu)}$
Q_0	$\Phi_1^{(0,\mu)} \equiv \varphi_1^{(\mu)}$	$\rightarrow$	$Q_1 A^{-1} Q_0 \mathbf{F}_0 = 0$ in Eq. (4.1.42)
Q_1		$\rightarrow$	$\langle \Phi_1^{(1,\mu)}, P_A \mathbf{R}(\tau) \rangle$ $= \langle \Phi_1^{(1,\mu)}, \mathbf{R}(\tau) \rangle$ in Eqs. (4.1.60) and (4.1.61)

Finally, with the use of one of the doublet mode, i.e., $\Phi_1^{(1,\mu)}$, we define ϕ as

$$\phi = \sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta X_{(\mu)}, \quad (4.1.39)$$

where $\Phi_1^{(1,\mu)}$ with $\mu = 1, \dots, M_1$ are the excited modes that should be considered as additional components to the invariant manifold spanned by the zero modes $\varphi_0^{(\alpha)}$ with $\alpha = 1, \dots, M_0$, and $\delta X_{(\mu)}$ with $\mu = 1, \dots, M_1$ denote the would-be integral constants that acquire the time dependence by the RG equation as well as $C_{(\alpha)}$ with $\alpha = 1, \dots, M_0$ in $\mathbf{X}_{\text{st}}$.

We call the scheme of the construction of the mesoscopic dynamics based on the choice of the P_1 -space vectors in Eq. (4.1.27) and the form of ϕ in Eq. (4.1.39) the *doublet scheme* in the RG method. We will see that the doublet scheme makes the resultant equation governing the dynamics of $C_{(\alpha)}$ and $\delta X_{(\mu)}$ satisfy the consistency condition (A) and general principle of the reduction theory of the dynamics (B) by the mechanism shown in Table 4.1.

4.1.4 First-order solution

The first-order equation reads

$$\frac{\partial}{\partial t} \tilde{\mathbf{X}}_1(t) = A \tilde{\mathbf{X}}_1(t) + \mathbf{F}_0. \quad (4.1.40)$$

By solving Eq. (4.1.40) with the initial value $\tilde{\mathbf{X}}_1(t_0) = \boldsymbol{\phi}$ shown in Eq. (4.1.22), we have the first-order solution given by

$$\begin{aligned}
\tilde{\mathbf{X}}_1(t; t_0) &= e^{A(t-t_0)} \tilde{\mathbf{X}}_1(t_0) + \int_{t_0}^t ds e^{A(t-s)} \mathbf{F}_0 \\
&= e^{A(t-t_0)} \boldsymbol{\phi} + (t - t_0) P_0 \mathbf{F}_0 + (e^{A(t-t_0)} - 1) A^{-1} Q_0 \mathbf{F}_0 \\
&= e^{A(t-t_0)} \left[\boldsymbol{\phi} + Q_1 A^{-1} Q_0 \mathbf{F}_0 \right] \\
&\quad + (t - t_0) P_0 \mathbf{F}_0 + (e^{A(t-t_0)} - 1) P_1 A^{-1} Q_0 \mathbf{F}_0 - Q_1 A^{-1} Q_0 \mathbf{F}_0 \\
&= e^{A(t-t_0)} \boldsymbol{\phi} + (t - t_0) P_0 \mathbf{F}_0 + (e^{A(t-t_0)} - 1) P_1 A^{-1} Q_0 \mathbf{F}_0. \quad (4.1.41)
\end{aligned}$$

We should stress that in the first-order solution the motion coming from the P_0 and P_1 spaces appears and the fast motion caused by the Q_1 space is absent due to

$$Q_1 A^{-1} Q_0 \mathbf{F}_0 = Q_1 \sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta \bar{X}_{(\mu)} = 0, \quad (4.1.42)$$

because $\Phi_1^{(1,\mu)}$ belongs to the P_1 space as shown in Eq. (4.1.27).

We note the appearance of the secular term proportional to $t - t_0$ in Eq. (4.1.41), which apparently invalidate the perturbative solution when $|t - t_0|$ becomes large.

4.1.5 Second-order solution

The second-order equation is written as

$$\frac{\partial}{\partial t} \tilde{\mathbf{X}}_2(t) = A \tilde{\mathbf{X}}_2(t) + \mathbf{K}(t - t_0), \quad (4.1.43)$$

with the time-dependent inhomogeneous term given by

$$\begin{aligned}
\mathbf{K}(\tau) &\equiv \frac{1}{2} B \left[e^{A\tau} \boldsymbol{\phi} + \tau P_0 \mathbf{F}_0 + (e^{A\tau} - 1) P_1 A^{-1} Q_0 \mathbf{F}_0 \right]^2 \\
&\quad + F_1 \left[e^{A\tau} \boldsymbol{\phi} + \tau P_0 \mathbf{F}_0 + (e^{A\tau} - 1) P_1 A^{-1} Q_0 \mathbf{F}_0 \right]. \quad (4.1.44)
\end{aligned}$$

Here, we have introduced B and F_1 whose components are given by

$$B_{ijk} \equiv \frac{\partial^2}{\partial X_j \partial X_k} G_i(X) \Big|_{\mathbf{X}=\mathbf{X}_{\text{st}}}, \quad (4.1.45)$$

$$F_{1ij} \equiv \frac{\partial}{\partial X_j} F_i(X) \Big|_{\mathbf{X}=\mathbf{X}_{\text{st}}}. \quad (4.1.46)$$

To obtain appropriate initial values and solutions with the motion coming from the P_0 and P_1 spaces to Eq. (4.1.43), we utilize the formulae (A.3.5) and (A.3.6) given in Appendix A.3: By setting $\mathbf{R}(t - t_0) = \mathbf{K}(t - t_0)$ in Eqs. (A.3.5) and (A.3.6), we find that the initial value and solution to Eq. (4.1.43) read

$$\tilde{\mathbf{X}}_2(t_0) = -Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0}, \quad (4.1.47)$$

and

$$\begin{aligned}\tilde{\mathbf{X}}_2(t; t_0) = & (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{K}(\tau) \Big|_{\tau=0} \\ & + (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial\tau}) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \\ & - e^{(t-t_0)\partial/\partial\tau} Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0},\end{aligned}\quad (4.1.48)$$

respectively. We notice again the appearance of secular terms in Eq. (4.1.48).

Summing up the perturbative solutions up to the second order with respect to ϵ , we have the full expression of the initial value and the approximate solution around $t \sim t_0$ to the second order:

$$\mathbf{X}(t_0) = \mathbf{X}_{\text{st}} + \epsilon \phi - \epsilon^2 Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} + O(\epsilon^3), \quad (4.1.49)$$

and

$$\begin{aligned}\tilde{\mathbf{X}}(t; t_0) = & \mathbf{X}_{\text{st}} + \epsilon \left[e^{A(t-t_0)} \phi + (t - t_0) P_0 \mathbf{F}_0 + (e^{A(t-t_0)} - 1) P_1 A^{-1} Q_0 \mathbf{F}_0 \right] \\ & + \epsilon^2 \left[(1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{K}(\tau) \Big|_{\tau=0} \right. \\ & + (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial\tau}) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \\ & \left. - e^{(t-t_0)\partial/\partial\tau} Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \right] + O(\epsilon^3).\end{aligned}\quad (4.1.50)$$

We note that in Eq. (4.1.50) the fast motion caused by the Q_1 space has been eliminated by adopting the appropriate initial value (4.1.49).

4.1.6 RG improvement of perturbative expansion

We emphasize that the solution (4.1.50) contains the secular terms that apparently invalidate the perturbative expansion for t away from the initial time t_0 . The point of the RG method lies in the fact that we can utilize the secular terms to obtain a solution valid in a global domain as discussed in the previous chapters and in Refs. [81, 82, 83, 84, 85, 86, 87, 88]. By applying the RG equation to the local solution (4.1.50), we can improve the local approximate solution to the global one:

$$\frac{\partial}{\partial t_0} \tilde{\mathbf{X}}(t; t_0) \Big|_{t_0=t} = 0, \quad (4.1.51)$$

which is reduced to

$$\begin{aligned}& \frac{\partial}{\partial t} \mathbf{X}_{\text{st}} + \epsilon \left[-A \phi + \frac{\partial}{\partial t} \phi - P_0 \mathbf{F}_0 - A P_1 A^{-1} Q_0 \mathbf{F}_0 \right] \\ & + \epsilon^2 \left[-P_0 \mathbf{K}(0) - (A - \partial/\partial\tau) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \right. \\ & \left. - (-\partial/\partial\tau) Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \right] + O(\epsilon^3) = 0.\end{aligned}\quad (4.1.52)$$

It is noted that the RG equation (4.1.52) gives the equation of motion governing the dynamics of the would-be integral constant $C_{(\alpha)}$ in $\mathbf{X}_{\text{st}}$ and $\delta X_{(\mu)}$ in ϕ . The global solution can be obtained as the initial value (4.1.49)

$$\mathbf{X}_{\text{E}}(t) \equiv \mathbf{X}(t_0 = t) = \mathbf{X}_{\text{st}} + \epsilon \phi - \epsilon^2 Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \Big|_{t_0=t} + O(\epsilon^3), \quad (4.1.53)$$

where the exact solution to Eq. (4.1.52) is inserted. It is noteworthy that we have derived the mesoscopic dynamics of Eq. (4.1.2) in the form of the pair of Eqs. (4.1.52) and (4.1.53): Equation (4.1.53) is nothing but the invariant/attractive manifold of Eq. (4.1.2), and Eq. (4.1.52) describes the mesoscopic dynamics defined on it.

4.1.7 Reduction of RG equation to simpler form

Here, we shall convert Eq. (4.1.52) into a more convenient form with the use of ϕ and P_1 defined in Eqs. (4.1.39) and (4.1.30), respectively.

By applying P_0 and P_1 from the left-hand side of Eq. (4.1.52), we have

$$P_0 \frac{\partial}{\partial t} \mathbf{X}_{\text{st}} + \epsilon \left[P_0 \frac{\partial}{\partial t} \phi - P_0 \mathbf{F}_0 \right] - \epsilon^2 P_0 \mathbf{K}(0) + O(\epsilon^3) = 0, \quad (4.1.54)$$

and

$$P_1 \frac{\partial}{\partial t} \mathbf{X}_{\text{st}} + \epsilon \left[-P_1 A \phi + P_1 \frac{\partial}{\partial t} \phi - P_1 A P_1 A^{-1} Q_0 \mathbf{F}_0 \right] - \epsilon^2 P_1 (A - \partial/\partial\tau) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} + O(\epsilon^3) = 0, \quad (4.1.55)$$

respectively.

Then, we take the inner product with the zero modes $\varphi_0^{(\alpha)}$ and the excited modes $\Phi_1^{(1,\mu)}$ used in the definition of ϕ , respectively. In this procedure, we note that the direct use of the definitions (4.1.10), (4.1.12), (4.1.30), and (4.1.32) leads to the followings:

$$\langle \varphi_0^{(\alpha)}, P_0 \psi \rangle = \langle \varphi_0^{(\alpha)}, \psi \rangle, \quad (4.1.56)$$

$$\langle \Phi_1^{(n,\mu)}, P_1 \psi \rangle = \langle \Phi_1^{(n,\mu)}, \psi \rangle, \quad (4.1.57)$$

with ψ being an arbitrary vector. Thus, we arrive at

$$\begin{aligned} & \langle \varphi_0^{(\alpha)}, \frac{\partial}{\partial t} (\mathbf{X}_{\text{st}} + \epsilon \phi) \rangle - \epsilon \langle \varphi_0^{(\alpha)}, \mathbf{F}_0 + \epsilon F_1 \phi \rangle \\ &= \epsilon^2 \frac{1}{2} \langle \varphi_0^{(\alpha)}, B \phi^2 \rangle + O(\epsilon^3), \end{aligned} \quad (4.1.58)$$

$$\begin{aligned} & \langle \Phi_1^{(1,\mu)}, \frac{\partial}{\partial t} (\mathbf{X}_{\text{st}} + \epsilon \phi) \rangle - \epsilon \langle \Phi_1^{(1,\mu)}, \mathbf{F}_0 + \epsilon F_1 \phi \rangle \\ &= \epsilon \langle \Phi_1^{(1,\mu)}, A \phi \rangle + \epsilon^2 \frac{1}{2} \langle \Phi_1^{(1,\mu)}, B \phi^2 \rangle + O(\epsilon^3). \end{aligned} \quad (4.1.59)$$

4.1. MESOSCOPIC DYNAMICS DERIVED FROM GENERIC EVOLUTION EQUATION WITH RG METHOD

In the derivation of Eqs. (4.1.58) and (4.1.59), we have used the following identities:

$$\begin{aligned}\langle \Phi_1^{(1,\mu)}, A P_1 A^{-1} Q_0 \mathbf{F}_0 \rangle &= \langle \Phi_1^{(0,\mu)}, P_1 A^{-1} Q_0 \mathbf{F}_0 \rangle \\ &= \langle \Phi_1^{(0,\mu)}, A^{-1} Q_0 \mathbf{F}_0 \rangle \\ &= \langle \Phi_1^{(1,\mu)}, \mathbf{F}_0 \rangle,\end{aligned}\tag{4.1.60}$$

and

$$\begin{aligned}&\langle \Phi_1^{(1,\mu)}, (A - \partial/\partial\tau) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &= \langle (A - \partial/\partial\tau) \Phi_1^{(1,\mu)}, P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &= \langle (\Phi_1^{(0,\mu)} - \Phi_1^{(1,\mu)} \partial/\partial\tau), P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &= \langle (\Phi_1^{(0,\mu)} - \Phi_1^{(1,\mu)} \partial/\partial\tau), (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &= \langle (A - \partial/\partial\tau) \Phi_1^{(1,\mu)}, (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &= \langle \Phi_1^{(1,\mu)}, \mathbf{K}(0) \rangle \\ &= \frac{1}{2} \langle \Phi_1^{(1,\mu)}, B \phi^2 \rangle + \langle \Phi_1^{(1,\mu)}, F_1 \phi \rangle,\end{aligned}\tag{4.1.61}$$

where we have used the self-adjoint nature of A shown in Eq. (4.1.13), the identities (4.1.56) and (4.1.57), and the relation derived from Eq. (4.1.44);

$$\mathbf{K}(0) = \frac{1}{2} B \phi^2 + F_1 \phi.\tag{4.1.62}$$

We note that the pair of Eqs. (4.1.58) and (4.1.59) is also the equation of motion governing $C_{(\alpha)}$ in $\mathbf{X}_{\text{st}}$ and $\delta X_{(\mu)}$ in ϕ , which is much simpler than Eq. (4.1.52).

With the use of the explicit forms of ϕ , $\mathbf{F}_0$, and $\Phi_1^{(1,\mu)}$ in Eqs. (4.1.39), (4.1.26), and (4.1.27), respectively, we obtain an alternative form of Eqs. (4.1.58) and (4.1.59):

$$\begin{aligned}&\langle \varphi_0^{(\alpha)}, \frac{\partial}{\partial t} \mathbf{X}_{\text{st}} \rangle + \epsilon \sum_{\mu=1}^{M_1} \langle \varphi_0^{(\alpha)}, \tilde{\varphi}_1^{(\mu)} \rangle \delta \bar{X}_{(\mu)} \\ &= -\epsilon \sum_{\mu=1}^{M_1} \langle \varphi_0^{(\alpha)}, \left[\frac{\partial}{\partial t} - \epsilon F_1 \right] A^{-1} \varphi_1^{(\mu)} \rangle \delta X_{(\mu)} \\ &\quad + \epsilon^2 \frac{1}{2} \sum_{\mu=1}^{M_1} \sum_{\nu=1}^{M_1} \langle \varphi_0^{(\alpha)}, B [A^{-1} \varphi_1^{(\mu)}] [A^{-1} \varphi_1^{(\nu)}] \rangle \delta X_{(\mu)} \delta X_{(\nu)} + O(\epsilon^3),\end{aligned}\tag{4.1.63}$$

and

$$\epsilon \sum_{\nu=1}^{M_1} \langle A^{-1} \varphi_1^{(\mu)}, \left[\frac{\partial}{\partial t} - \epsilon F_1 \right] A^{-1} \varphi_1^{(\nu)} \rangle \delta X_{(\nu)}$$

$$\begin{aligned}
&= \epsilon \sum_{\nu=1}^{M_1} \langle \varphi_1^{(\mu)}, A^{-1} \varphi_1^{(\nu)} \rangle (\delta X_{(\nu)} + \delta \bar{X}_{(\nu)}) \\
&\quad + \epsilon^2 \frac{1}{2} \sum_{\nu=1}^{M_1} \sum_{\rho=1}^{M_1} \langle A^{-1} \varphi_1^{(\mu)}, B [A^{-1} \varphi_1^{(\nu)}] [A^{-1} \varphi_1^{(\rho)}] \rangle \delta X_{(\nu)} \delta X_{(\rho)} + O(\epsilon^3).
\end{aligned} \tag{4.1.64}$$

Here, we have used the fact that $\partial \mathbf{X}_{\text{st}}/\partial t$ is the zero modes, which can be shown by the following identity:

$$\frac{\partial}{\partial t} \mathbf{X}_{\text{st}} = \sum_{\alpha=1}^{M_0} \frac{\partial}{\partial C_{(\alpha)}} \mathbf{X}_{\text{st}} \frac{\partial}{\partial t} C_{(\alpha)} = \sum_{\alpha=1}^{M_0} \varphi_0^{(\alpha)} \frac{\partial}{\partial t} C_{(\alpha)}, \tag{4.1.65}$$

where we have used Eqs. (4.1.8) and (4.1.25). Furthermore, the initial value (4.1.53) is written as

$$\begin{aligned}
\mathbf{X}_{\text{E}}(t) &= \mathbf{X}_{\text{st}} + \epsilon \sum_{\mu=1}^{M_1} A^{-1} \varphi_1^{(\mu)} \delta X_{(\mu)} \\
&\quad - \epsilon^2 Q_1 (A - \partial/\partial \tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \Big|_{t_0=t} + O(\epsilon^3),
\end{aligned} \tag{4.1.66}$$

where

$$\begin{aligned}
\mathbf{K}(\tau) &= \frac{1}{2} \sum_{\mu=1}^{M_1} \sum_{\nu=1}^{M_1} B \left[e^{A\tau} A^{-1} \varphi_1^{(\mu)} \delta X_{(\mu)} + \left(\tau P_0 \tilde{\varphi}_1^{(\mu)} + (e^{A\tau} - 1) A^{-1} \varphi_1^{(\mu)} \right) \delta \bar{X}_{(\mu)} \right]^2 \\
&\quad + \sum_{\mu=1}^{M_1} F_1 \left[e^{A\tau} A^{-1} \varphi_1^{(\mu)} \delta X_{(\mu)} + \left(\tau P_0 \tilde{\varphi}_1^{(\mu)} + (e^{A\tau} - 1) A^{-1} \varphi_1^{(\mu)} \right) \delta \bar{X}_{(\mu)} \right].
\end{aligned} \tag{4.1.67}$$

We note that the set of Eqs. (4.1.58)/(4.1.63), (4.1.59)/(4.1.64), and (4.1.66) is the main result in this section, and the global solution to Eq. (4.1.2) in the mesoscopic regime can be obtained by substituting the exact solution to Eqs. (4.1.58)/(4.1.63) and (4.1.59)/(4.1.64) into Eq. (4.1.66).

In Appendix B, as an easy example, we investigate the mesoscopic dynamics of a simple two-dimensional evolution equation with the use of Eqs. (4.1.63), (4.1.64), and (4.1.66). Furthermore, in Appendix C, we apply this method to the derivation of the causal non-relativistic dissipative hydrodynamics as the mesoscopic dynamics of the non-relativistic Boltzmann equation.

4.1.8 Consistency of mesoscopic dynamics with slow dynamics as described with zero modes in asymptotic regime

We will show that an asymptotic form of the mesoscopic dynamics obtained above is consistent with the slow dynamics obtained by employing only the zero modes from

4.1. MESOSCOPIC DYNAMICS DERIVED FROM GENERIC EVOLUTION EQUATION WITH RG METHOD

the outset, which is given by the following equation:

$$\begin{aligned} & \langle \boldsymbol{\varphi}_0^{(\alpha)}, \frac{\partial}{\partial t}(\mathbf{X}_{\text{st}} - \epsilon A^{-1} Q_0 \mathbf{F}_0) \rangle - \epsilon \langle \boldsymbol{\varphi}_0^{(\alpha)}, \mathbf{F}_0 - \epsilon F_1 A^{-1} Q_0 \mathbf{F}_0 \rangle \\ &= \epsilon^2 \frac{1}{2} \langle \boldsymbol{\varphi}_0^{(\alpha)}, B [A^{-1} Q_0 \mathbf{F}_0]^2 \rangle + O(\epsilon^3), \end{aligned} \quad (4.1.68)$$

of which derivation is shown in Appendix A.4.

It is important to note that Eq. (4.1.58) can be reduced to Eq. (4.1.68) when the equality

$$\boldsymbol{\phi} = -A^{-1} Q_0 \mathbf{F}_0 + O(\epsilon), \quad (4.1.69)$$

is satisfied in the asymptotic regime. In fact, $\boldsymbol{\phi}$ and $-A^{-1} Q_0 \mathbf{F}_0$ are composed of the same vectors, i.e., $\boldsymbol{\Phi}_1^{(1,\mu)}$, because

$$A^{-1} Q_0 \mathbf{F}_0 = \sum_{\mu=1}^{M_1} \boldsymbol{\Phi}_1^{(1,\mu)} \delta \bar{X}_{(\mu)}, \quad (4.1.70)$$

which can be derived from Eqs. (4.1.26) and (4.1.28). We emphasize that the definition of $\boldsymbol{\phi}$ based on $\boldsymbol{\Phi}_1^{(1,\mu)}$ adopted in the doublet scheme is just a necessary condition for the consistency with the slow dynamics, and our task is to show that the amplitudes of $\boldsymbol{\Phi}_1^{(1,\mu)}$ in $\boldsymbol{\phi}$ agree with those in $-A^{-1} Q_0 \mathbf{F}_0$ in the asymptotic regime.

Here, we should notice the *time-scale separation* between the fast motion of $\delta X_{(\mu)}$ caused by the P_1 space and the slow motion of $C_{(\alpha)}$ by the P_0 space. Owing to this separation which becomes significant in the asymptotic regime, we can obtain the closed equations with respect to $C_{(\alpha)}$ by eliminating $\delta X_{(\mu)}$ adiabatically from Eqs. (4.1.58)/(4.1.63) and (4.1.59)/(4.1.64). First, we obtain

$$\delta X_{(\mu)} = -\delta \bar{X}_{(\mu)} + O(\epsilon), \quad (4.1.71)$$

which has been derived from Eq. (4.1.64) governing the fast motion of $\delta X_{(\mu)}$ as a stationary solution that is realized asymptotically with $C_{(\alpha)}$ being treated as a constant. We note that Eq. (4.1.71) is equivalent to

$$\boldsymbol{\phi} = -\sum_{\mu=1}^{M_1} \boldsymbol{\Phi}_1^{(1,\mu)} \delta \bar{X}_{(\mu)} + O(\epsilon). \quad (4.1.72)$$

Then, combining Eq. (4.1.72) with (4.1.70), we obtain Eq. (4.1.69), and find that the slow dynamics of $C_{(\alpha)}$ is the same as Eq. (4.1.68). Thus, we complete the proof that Eqs. (4.1.58)/(4.1.63) and (4.1.59)/(4.1.64) can reproduce Eq. (4.1.68) asymptotically.

4.1.9 Discussion

We shall discuss whether or not there exist other schemes which can produce a mesoscopic dynamics that is simpler than that of the doublet scheme shown in Table 4.1

and respects the consistency with the slow dynamics given in terms of the zero modes in the asymptotic regime. Here, we should notice that the identities (4.1.42), (4.1.60), and (4.1.61) have played an essential role in the derivation of the simple form shown in Eqs. (4.1.58) and (4.1.59).

First, let us see the significance of Eq. (4.1.42): If Eq. (4.1.42) is not satisfied, that is, $\Phi_1^{(1,\mu)}$ does not belong to the P_1 space, $\Phi_1^{(1,\mu)}$ cannot be utilized in the construction of ϕ , and hence an asymptotic form of the mesoscopic dynamics is not in accord with the slow dynamics as described with only the zero modes as we discussed in Sec. 4.1.8. In fact, the equality (4.1.69) is not realized in the asymptotic regime because ϕ and $-A^{-1}Q_0\mathbf{F}_0$ are no longer composed of the same vectors in general. We conclude that Eq. (4.1.42) is necessary to obtain a reduced dynamics which can reproduce the slow dynamics given in terms of the zero modes in the asymptotic regime and $\Phi_1^{(1,\mu)}$ should be incorporated into the vectors that are used to define ϕ .

Next, we shall focus Eqs. (4.1.60) and (4.1.61). To make the explanation clearer, we consider the case where ϕ is constructed with the use of not only $\Phi_1^{(1,\mu)}$ but also $\Phi_0^{(0,\mu)}$ as follows:

$$\phi = \sum_{\mu=1}^{M_1} \Phi_1^{(0,\mu)} \delta X_{(0,\mu)} + \sum_{\mu=1}^{M_1} \Phi_1^{(1,\mu)} \delta X_{(1,\mu)}, \quad (4.1.73)$$

where $\delta X_{(0,\mu)}$ and $\delta X_{(1,\mu)}$ denote the dynamical variables. In this case, it becomes necessary to take the inner product with not only $\varphi_0^{(\alpha)}$ and $\Phi_1^{(1,\mu)}$ but also $\Phi_1^{(0,\mu)}$ in Eq. (4.1.55). In contrast to the inner product with $\Phi_1^{(1,\mu)}$, however, the inner product with $\Phi_1^{(0,\mu)}$ does not have a simple form:

$$\begin{aligned} & \langle \Phi_1^{(0,\mu)}, (A - \partial/\partial\tau) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &= \langle (A - \partial/\partial\tau) \Phi_1^{(0,\mu)}, P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle \\ &\neq \langle (A - \partial/\partial\tau) \Phi_1^{(0,\mu)}, (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \rangle = \langle \Phi_1^{(0,\mu)}, \mathbf{K}(0) \rangle, \end{aligned} \quad (4.1.74)$$

because $A\Phi_1^{(0,\mu)} = A\varphi_1^{(\mu)}$ does not belong to the P_1 space. Owing to the inequality shown in the final line of Eq. (4.1.74), the reduced RG equation corresponding to the choice of ϕ defined in Eq. (4.1.73) is given by the set of Eqs. (4.1.58), (4.1.59), and

$$\begin{aligned} & \langle \Phi_1^{(0,\mu)}, \frac{\partial}{\partial t} (\mathbf{X}_{\text{st}} + \epsilon \phi) \rangle - \epsilon \langle \Phi_1^{(0,\mu)}, A P_1 A^{-1} Q_0 (\mathbf{F}_0 + \epsilon F_1 \phi) \rangle \\ &= \epsilon \langle \Phi_1^{(0,\mu)}, A \phi \rangle + \epsilon^2 \frac{1}{2} \langle \Phi_1^{(0,\mu)}, A P_1 A^{-1} Q_0 B \phi^2 \rangle \\ & \quad + \epsilon^2 \sum_{k=1}^{\infty} \langle \Phi_1^{(0,\mu)}, (A P_1 A^{-1} - P_1) A^{-k} Q_0 \frac{\partial^k}{\partial \tau^k} \mathbf{K}(\tau) \Big|_{\tau=0} \rangle + O(\epsilon^3), \end{aligned} \quad (4.1.75)$$

4.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

where we have used the identity

$$\begin{aligned} & (A - \partial/\partial\tau) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{K}(\tau) \Big|_{\tau=0} \\ &= A P_1 A^{-1} \mathbf{K}(0) + (A P_1 A^{-1} - P_1) \sum_{k=1}^{\infty} A^{-k} Q_0 \frac{\partial^k}{\partial\tau^k} \mathbf{K}(\tau) \Big|_{\tau=0}. \end{aligned} \quad (4.1.76)$$

Noticing that $\partial^k \mathbf{K}(\tau)/\partial\tau^k|_{\tau=0}$ does not vanish for any k since $\mathbf{K}(\tau)$ has an exponential τ dependence as shown in Eq. (4.1.67), we find that the last term in Eq.(4.1.75) gives an infinite number of terms. Although we can avoid this difficulty by making the P_1 space spanned by $\Phi_1^{(n,\mu)}$ with $n = -1, 0, 1$ instead of $\Phi_1^{(n,\mu)}$ with $n = 0, 1$ shown in Eq. (4.1.27), the resultant equation becomes an equation describing the dynamics of $C_{(\alpha)}$, $\delta X_{(0,\mu)}$, and $\delta X_{(1,\mu)}$, which is more complicated than Eqs. (4.1.58) and (4.1.59), because the number of the variables $C_{(\alpha)}$, $\delta X_{(0,\mu)}$, and $\delta X_{(1,\mu)}$ is obviously more than that of $C_{(\alpha)}$ and $\delta X_{(\mu)}$ governed by Eqs. (4.1.58) and (4.1.59). Thus, the P_1 space should be spanned by $\Phi_1^{(0,\mu)}$ together with $\Phi_1^{(1,\mu)}$ that is utilized to define ϕ .

From the above observations, it is found that any modification to the doublet scheme does not make the reduced dynamics simpler than the original one, when the consistency with the slow dynamics as described with only the zero modes in the asymptotic regime is respected. Thus, we suggest that the doublet scheme shown in Table 4.1 is one of the schemes to produce the simplest equation describing the mesoscopic dynamics whose asymptotic form is in accord with that of the slow dynamics.

A remark is in order here: The doublet scheme in the RG method itself has a universal nature and can be applied to derive a mesoscopic dynamics from a wide class of evolution equations, as far as the equation can be written as Eq. (4.1.2) and the linearized evolution operator A is self-adjoint as shown in Eq. 4.1.13 and has no Jordan cell. As will be seen in Sec. 4.2, the relativistic Boltzmann equation satisfies the above conditions, and its mesoscopic dynamics can be extracted by the doublet scheme in the RG method. An extension of the doublet scheme to a method applicable to a more generic case where A is not self-adjoint or has Jordan cell will be studied elsewhere.

4.2 Reduction of relativistic Boltzmann equation to mesoscopic dynamics with RG method

In this section, we apply the RG method developed in Sec. 4.1 to extract the mesoscopic dynamics from the relativistic Boltzmann equation (3.1.14). Then, we examine some properties of the resultant equation, concerning the causality and the relaxation times as well as the frame, the stability of the equilibrium state, and the transport coefficients.

4.2.1 Set up in doublet scheme

By comparing the relativistic Boltzmann equation (3.1.14) with the evolution equation (4.1.2) discussed in Sec. 4.1, we find the following correspondences:

$$\mathbf{X} = \left\{ f_p(\tau, \sigma) \right\}_p, \quad (4.2.1)$$

$$\mathbf{G}(\mathbf{X}) = \left\{ \frac{1}{p \cdot u(\sigma)} C[f]_p(\tau, \sigma) \right\}_p, \quad (4.2.2)$$

$$\mathbf{F}(\mathbf{X}) = \left\{ -\frac{1}{p \cdot u(\sigma)} p \cdot \nabla f_p(\tau, \sigma) \right\}_p. \quad (4.2.3)$$

It is noted that the momentum p is interpreted as an index of the vector, while we treat the Lorentz-covariant space coordinate σ^μ solely as a parameter, in accordance with the past works by Kuramoto [78] and Hatta and Kunihiro [87]. Furthermore, we note that the Lorentz-covariant time coordinate τ is treated as the time variables. From now on, we omit $\{\cdot\}_p$.

In the rest of this subsection, according to Eqs. (4.2.1)-(4.2.3), we present explicit forms of $\mathbf{X}_{\text{st}}$, A , $\varphi_0^{(\alpha)}$, P_0 , $\mathbf{F}_0$, $\tilde{\varphi}_1^{(\mu)}$, $\delta \bar{X}_{(\mu)}$, $\varphi_1^{(\mu)}$, P_1 , $\delta X_{(\mu)}$, B , and F_1 introduced in the generic case discussed in Sec. 4.1.

With the use of Eq. (4.1.4), we find that the static solution reads

$$\mathbf{X}_{\text{st}}(t_0) = f_p^{\text{eq}}(\sigma; \tau_0) = \frac{1}{(2\pi)^3} \exp \left[\frac{\mu(\sigma; \tau_0) - p^\mu u_\mu(\sigma; \tau_0)}{T(\sigma; \tau_0)} \right], \quad (4.2.4)$$

which is the Jüttner function (3.2.9). We note that the five would-be integral constants $T(\sigma; \tau_0)$, $\mu(\sigma; \tau_0)$, and $u^\mu(\sigma; \tau_0)$ with $u^\mu(\sigma; \tau_0) u_\mu(\sigma; \tau_0) = 1$ corresponding to $C_{(\alpha)}(t_0)$ in Sec. 4.1 are lifted to the dynamical variables by applying the RG equation. In the following, we shall suppress $(\sigma; \tau_0)$ when no misunderstanding is expected.

Using Eqs. (4.1.6) and (4.2.2), we have the linearized evolution operator A as

$$A = \frac{1}{p \cdot u} \frac{\partial}{\partial f_q} C[f]_p \bigg|_{f=f^{\text{eq}}} = A_{pq} = f_p^{\text{eq}} L_{pq} (f_q^{\text{eq}})^{-1}, \quad (4.2.5)$$

where A_{pq} and L_{pq} have been defined in Eqs. (3.2.12) and (3.2.15), respectively. Here, let us examine the property of A . We define another inner product by

$$\langle \psi, \chi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot u) (f_p^{\text{eq}})^{-1} \psi_p \chi_p = \langle (f^{\text{eq}})^{-1} \psi, (f^{\text{eq}})^{-1} \chi \rangle_{\text{eq}}, \quad (4.2.6)$$

with ψ_p and χ_p being arbitrary vectors and $(f^{\text{eq}})^{-1}_{pq} = (f_p^{\text{eq}})^{-1} \delta_{pq}$.

With the use of the positive definiteness of the inner product defined in Eq. (3.2.17), we can show that the norm through the new inner product is also positive definite,

$$\langle \psi, \psi \rangle = \langle (f^{\text{eq}})^{-1} \psi, (f^{\text{eq}})^{-1} \psi \rangle_{\text{eq}} = \langle \psi', \psi' \rangle_{\text{eq}} > 0, \quad \psi_p \neq 0, \quad (4.2.7)$$

4.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

with $\psi'_p \equiv [(f^{\text{eq}})^{-1} \psi]_p$. It is remarkable that the inner product (4.2.6) ensures the self-adjoint nature of A not but $L = (f^{\text{eq}})^{-1} A f^{\text{eq}}$ in construct to the inner product (3.2.17). In fact, with the inner product, A is self-adjoint,

$$\begin{aligned}
 \langle \psi, A \chi \rangle &= \langle (f^{\text{eq}})^{-1} \psi, (f^{\text{eq}})^{-1} A \chi \rangle_{\text{eq}} \\
 &= \langle (f^{\text{eq}})^{-1} \psi, L (f^{\text{eq}})^{-1} \chi \rangle_{\text{eq}} \\
 &= \langle L (f^{\text{eq}})^{-1} \psi, (f^{\text{eq}})^{-1} \chi \rangle_{\text{eq}} \\
 &= \langle (f^{\text{eq}})^{-1} A \psi, (f^{\text{eq}})^{-1} \chi \rangle_{\text{eq}} \\
 &= \langle A \psi, \chi \rangle.
 \end{aligned} \tag{4.2.8}$$

and real semi-negative definite,

$$\langle \psi, A \psi \rangle = \langle (f^{\text{eq}})^{-1} \psi, L (f^{\text{eq}})^{-1} \psi \rangle_{\text{eq}} = \langle \psi', L \psi' \rangle_{\text{eq}} \leq 0. \tag{4.2.9}$$

Here, we have utilized the self-adjoint nature and real semi-negative definiteness of L shown in Eqs. (3.2.20) and (3.2.21), which are based on the inner product defined in Eq. (3.2.17). Owing to the property of A , we can apply the method presented in Sec. 4.1 to extract the mesoscopic dynamics from the relativistic Boltzmann equation (3.1.14).

The zero modes of A are found to be

$$\varphi_0^{(\alpha)} = f_p^{\text{eq}} \varphi_{0p}^\alpha = [f^{\text{eq}} \varphi_0^\alpha]_p, \quad \alpha = 0, 1, 2, 3, 4, \tag{4.2.10}$$

where φ_{0p}^α with $\alpha = 0, \dots, 4$ are identically the zero modes defined in Eq. (3.2.22). It is noteworthy that the dimension of the kernel space of A is five, i.e., $M_0 = 5$.

With the use of the zero modes and the inner product defined in Eqs. (4.2.10) and (4.2.6), respectively, we have the P_0 -space metric matrix

$$\eta_0^{(\alpha)(\beta)} = \langle f^{\text{eq}} \varphi_0^\alpha, f^{\text{eq}} \varphi_0^\beta \rangle = \langle \varphi_0^\alpha, \varphi_0^\beta \rangle_{\text{eq}} = \eta_0^{\alpha\beta}, \tag{4.2.11}$$

where $\eta_0^{\alpha\beta}$ has been defined in Eq. (3.2.26).

Thus, we have the projection operators P_0 and Q_0 given as

$$\begin{aligned}
 [P_0 \psi]_p &= f_p^{\text{eq}} \varphi_{0p}^\alpha \eta_{0\alpha\beta}^{-1} \langle f^{\text{eq}} \varphi_0^\beta, \psi \rangle \\
 &= f_p^{\text{eq}} \varphi_{0p}^\alpha \eta_{0\alpha\beta}^{-1} \langle \varphi_0^\beta, (f^{\text{eq}})^{-1} \psi \rangle_{\text{eq}} = [\bar{P} \psi]_p,
 \end{aligned} \tag{4.2.12}$$

$$Q_0 = 1 - P_0 = \bar{Q}, \tag{4.2.13}$$

where $\bar{P}$ and $\bar{Q}$ have been defined in Eqs. (3.2.28) and (3.2.28), respectively.

The definition shown in Eq. (4.1.26) leads us to

$$\mathbf{F}_0 = -\frac{1}{p \cdot u} p \cdot \nabla f_p^{\text{eq}} = -f_p^{\text{eq}} \tilde{\varphi}_{1p}^{\mu\alpha} \bar{X}_{\mu\alpha}, \tag{4.2.14}$$

where $\tilde{\varphi}_{1p}^{\mu\alpha}$ and $\bar{X}_{\mu\alpha}$ have been defined in Eqs. (3.2.57) and (3.2.65), respectively. By comparing Eq. (4.2.14) with Eq. (4.1.26), we can read off $\tilde{\varphi}_1^{(\mu)}$ and $\delta\bar{X}_{(\mu)}$ as

$$\tilde{\varphi}_1^{(\mu)} = f_p^{\text{eq}} \tilde{\varphi}_{1p}^{\mu\alpha}, \quad (4.2.15)$$

$$\delta\bar{X}_{(\mu)} = -\bar{X}_{\mu\alpha}, \quad (4.2.16)$$

respectively.

The vectors belonging to the P_1 space reads

$$\varphi_1^{(\mu)} = [Q_0 f^{\text{eq}} \tilde{\varphi}_1^{\mu\alpha}]_p = [\bar{Q} f^{\text{eq}} \tilde{\varphi}_1^{\mu\alpha}]_p = f_p^{\text{eq}} [Q \tilde{\varphi}_1^{\mu\alpha}]_p = f_p^{\text{eq}} \varphi_{1p}^{\mu\alpha}. \quad (4.2.17)$$

Here, let us recall that the first-excited modes $\varphi_{1p}^{\mu\alpha}$ defined in Eq. (3.2.70) has the explicit form shown in Eq. (3.3.6). Thus, the number of independent components of $\varphi_{1p}^{\mu\alpha}$, i.e., $\hat{\Pi}_p^\mu$, $\hat{J}_p^\mu$, and $\hat{\pi}_p^{\mu\nu}$, is nine, i.e., $M_1 = 9$. We stress that the number and form of the excited modes $\varphi_1^{(\mu)}$ have been automatically determined in the construction of the mesoscopic dynamics from the relativistic Boltzmann equation based on the doublet scheme in the RG method.

Using the P_1 -space vectors

$$\Phi_1^{(n,\mu)} = f_p^{\text{eq}} [L^{-n} \varphi_1^{\mu\alpha}]_p, \quad n = 0, 1, \quad (4.2.18)$$

we have the P_1 -space metric matrix

$$\eta_1^{(n,\mu)(m,\nu)} = \langle L^{-n} \varphi_1^{\mu\alpha}, L^{-m} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \equiv \eta_1^{n\mu\alpha, m\nu\beta}, \quad (4.2.19)$$

which implies that the projection operators P_1 and Q_1 are given as

$$[P_1 \psi]_p = f_p^{\text{eq}} \sum_{n=0,1} \sum_{m=0,1} [L^{-n} \varphi_1^{\mu\alpha}]_p \eta_{1n\mu\alpha, m\nu\beta}^{-1} \langle L^{-m} \varphi_1^{\nu\beta}, (f^{\text{eq}})^{-1} \psi \rangle_{\text{eq}}, \quad (4.2.20)$$

and $Q_1 = Q_0 - P_1$, respectively.

We shall introduce the integral constants that represents the deviation from f_p^{eq} as follows:

$$\delta X_{(\mu)}(t_0) = X_{\mu\alpha}(\sigma; \tau_0). \quad (4.2.21)$$

According to the definitions presented in Eqs. (4.1.45) and (4.1.46), we identify

$$B = \frac{1}{p \cdot u} \frac{\partial^2}{\partial f_q \partial f_r} C[f]_p \bigg|_{f=f^{\text{eq}}} = B_{pqr}, \quad (4.2.22)$$

$$F_1 = -\frac{1}{p \cdot u} p \cdot \nabla \delta_{pq}, \quad (4.2.23)$$

where B_{pqr} has been defined in Eq. (3.2.38).

4.2.2 Reduced dynamics by RG method

Substituting A , B , $\mathbf{F}_0$, F_1 , $\delta\bar{X}_{(\mu)}$, $\delta X_{(\mu)}$, $\varphi_0^{(\alpha)}$, and $\varphi_1^{(\mu)}$ obtained above into Eqs. (4.1.58)/(4.1.63), (4.1.59)/(4.1.64), and (4.1.66), we can obtain the mesoscopic dynamics of the relativistic Boltzmann equation.

First, the RG equations (4.1.58) and (4.1.64) read

$$\begin{aligned} & \sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[(p \cdot u) \frac{\partial}{\partial \tau} + \epsilon p \cdot \nabla \right] \left[f_p^{\text{eq}} \left(1 + \epsilon [L^{-1} \varphi_1^{\nu\beta}]_p X_{\nu\beta} \right) \right] \\ &= \epsilon^2 \frac{1}{2} \langle \varphi_0^\alpha, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta}]^2 \rangle_{\text{eq}} + O(\epsilon^3), \end{aligned} \quad (4.2.24)$$

and

$$\begin{aligned} & \epsilon \sum_p \frac{1}{p^0} [L^{-1} \varphi_1^{\mu\alpha}]_p \left[(p \cdot u) \frac{\partial}{\partial \tau} + \epsilon p \cdot \nabla \right] \left[f_p^{\text{eq}} [L^{-1} \varphi_1^{\nu\beta}]_p X_{\nu\beta} \right] \\ &= \epsilon \eta_1^{\mu\alpha\nu\beta} (X_{\nu\beta} - \bar{X}_{\nu\beta}) + \epsilon^2 \frac{1}{2} \langle L^{-1} \varphi_1^{\mu\alpha}, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta}]^2 \rangle_{\text{eq}} + O(\epsilon^3), \end{aligned} \quad (4.2.25)$$

respectively, where $\eta_1^{\mu\alpha\nu\beta}$ has been given by Eq. (3.2.69). Here, we can reduce Eq. (4.2.24) to

$$\sum_p \frac{1}{p^0} \varphi_{0p}^\alpha \left[(p \cdot u) \frac{\partial}{\partial \tau} + \epsilon p \cdot \nabla \right] \left[f_p^{\text{eq}} \left(1 + \epsilon [L^{-1} \varphi_1^{\nu\beta}]_p X_{\nu\beta} \right) \right] + O(\epsilon^3) = 0, \quad (4.2.26)$$

because the following identity is satisfied:

$$\begin{aligned} & \frac{1}{2} \langle \varphi_0^\alpha, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta}]^2 \rangle_{\text{eq}} \\ &= \sum_p \frac{1}{p^0} (p \cdot u) \varphi_{0p}^\alpha \sum_q \sum_r \frac{1}{2} B_{pqr} [f^{\text{eq}} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta}]_q [f^{\text{eq}} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta}]_r \\ &= \sum_p \frac{1}{p^0} \varphi_{0p}^\alpha C [f^{\text{eq}} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta}]_p \\ &= 0, \end{aligned} \quad (4.2.27)$$

which can be derived from the fact that φ_{0p}^α are the collision invariants as shown in Eqs. (2.2.7) and (2.2.8). We will see that Eq. (4.2.26) can produce the continuity equations, while Eq. (4.2.25) the relaxation equations. Note that Eqs. (4.2.26) and (4.2.25) actually give the evolution equations for $T(\sigma; \tau)$, $\mu(\sigma; \tau)$, $u^\mu(\sigma; \tau)$, and $X_{\mu\alpha}(\sigma; \tau)$.

Then, the invariant/attractive manifold (4.1.66) reads

$$f_{Ep}(\tau, \sigma) = f_p^{\text{eq}} \left(1 + \epsilon [L^{-1} \varphi_1^{\nu\beta}]_p X_{\nu\beta} \right) (\sigma; \tau_0 = \tau) + \epsilon^2 \Delta f_p(\sigma; \tau_0 = \tau) + O(\epsilon^3). \quad (4.2.28)$$

Here, Δf_p denotes the contribution from the Q_1 space given as

$$\Delta f_p \equiv -[Q_1 f^{\text{eq}} (L - \partial/\partial s)^{-1} (f^{\text{eq}})^{-1} Q_0 K(s) \Big|_{s=0}]_p, \quad (4.2.29)$$

where

$$\begin{aligned} K_p(s) = & \sum_q \sum_r \frac{1}{2} B_{pqr} [f^{\text{eq}} e^{Ls} L^{-1} \varphi_1^{\mu\alpha} X_{\mu\alpha} - s P_0 f^{\text{eq}} \tilde{\varphi}_1^{\mu\alpha} \bar{X}_{\mu\alpha} \\ & - f^{\text{eq}} (e^{Ls} - 1) L^{-1} \varphi_1^{\mu\alpha} \bar{X}_{\mu\alpha}]_q [f^{\text{eq}} e^{Ls} L^{-1} \varphi_1^{\nu\beta} X_{\nu\beta} - s P_0 f^{\text{eq}} \tilde{\varphi}_1^{\nu\beta} \bar{X}_{\nu\beta} \\ & - f^{\text{eq}} (e^{Ls} - 1) L^{-1} \varphi_1^{\nu\beta} \bar{X}_{\nu\beta}]_r \\ & + \sum_q F_{1pq} [f^{\text{eq}} e^{Ls} L^{-1} \varphi_1^{\mu\alpha} X_{\mu\alpha} - s P_0 f^{\text{eq}} \tilde{\varphi}_1^{\mu\alpha} \bar{X}_{\mu\alpha} \\ & - f^{\text{eq}} (e^{Ls} - 1) L^{-1} \varphi_1^{\mu\alpha} \bar{X}_{\mu\alpha}]_q. \end{aligned} \quad (4.2.30)$$

We note that the global solution to the relativistic Boltzmann equation (3.1.14) in the mesoscopic regime can be obtained by substituting the exact solution to Eqs. (4.2.26) and (4.2.25), i.e., $T(\sigma; \tau)$, $\mu(\sigma; \tau)$, $u^\mu(\sigma; \tau)$, and $X_{\mu\alpha}(\sigma; \tau)$ into $f_{\text{Ep}}(\tau, \sigma)$ in Eq. (4.2.28).

By setting ϵ equal to 1, we reduce Eq (4.2.26) to a form that is easy to see a correspondence to the first-order hydrodynamic equation (3.2.49) as

$$\partial_\mu J_{2\text{nd}}^{\mu\alpha} = 0, \quad (4.2.31)$$

with

$$J_{2\text{nd}}^{\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha f_p^{\text{eq}} (1 + [L^{-1} \varphi_1^{\nu\beta}]_p X_{\nu\beta}), \quad (4.2.32)$$

where the identity (3.2.51) has been utilized. Here, as in the case of $J_{1\text{st}}^{\mu\alpha}$, we decompose $J_{2\text{nd}}^{\mu\alpha}$ into two parts as $J_{2\text{nd}}^{\mu\alpha} = J_{2\text{nd}}^{(0)\mu\alpha} + \delta J_{2\text{nd}}^{\mu\alpha}$, where

$$J_{2\text{nd}}^{(0)\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha f_p^{\text{eq}}, \quad (4.2.33)$$

$$\delta J_{2\text{nd}}^{\mu\alpha} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi_{0p}^\alpha f_p^{\text{eq}} [L^{-1} \varphi_1^{\nu\beta}]_p X_{\nu\beta} = \langle \tilde{\varphi}_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} X_{\nu\beta} = \eta_1^{\mu\alpha\nu\beta} X_{\nu\beta}, \quad (4.2.34)$$

with $\tilde{\varphi}_{1p}^{\mu\alpha} = p^\mu \varphi_{0p}^\alpha / (p \cdot u)$ given in Eq. (3.2.57). Needless to say, $J_{2\text{nd}}^{(0)\mu\alpha}$ and $\delta J_{2\text{nd}}^{\mu\alpha}$ denote the currents in the perfect-fluid and dissipative part, respectively. Here, one should notice that $J_{2\text{nd}}^{(0)\mu\alpha}$ agrees with $J_{1\text{st}}^{(0)\mu\alpha}$ defined in Eq. (3.2.55);

$$J_{2\text{nd}}^{(0)\mu\alpha} = J_{1\text{st}}^{(0)\mu\alpha} = \begin{cases} e u^\mu u^\nu - p \Delta^{\mu\nu}, & \alpha = \nu, \\ m n u^\mu, & \alpha = 4, \end{cases} \quad (4.2.35)$$

where e , p , and n denote the internal energy, pressure, and particle-number density for the dilute gas, respectively, given by Eqs. (3.3.2)-(3.3.4),

4.2.3 Fourteen-moment equation in energy frame and microscopic representations of transport coefficients and relaxation times

We shall reduce Eqs. (4.2.31) and (4.2.25) to the second-order relativistic hydrodynamic equation, i.e., the set of the continuity equations and the relaxation equations, where the physical meaning of the dynamical variables is clearer than that of $X_{\mu\alpha}$.

Since $X_{\mu\alpha}$ appears in Eqs. (4.2.31) and (4.2.25) as the form of $\varphi_{1p}^{\mu\alpha} X_{\mu\alpha}$ or $\eta_1^{\mu\alpha\nu\beta} X_{\nu\beta}$, we can use the new variables Π , J^μ , and $\pi^{\mu\nu}$ as the fundamental quantities instead of $X_{\mu\alpha}$;

$$\Pi \equiv -T \Delta^{\mu\nu} X_{\mu\nu}, \quad (4.2.36)$$

$$J^\mu \equiv z \hat{h}^{-1} \Delta^{\mu\nu} X_{\nu 4}, \quad (4.2.37)$$

$$\pi^{\mu\nu} \equiv T \Delta^{\mu\nu\rho\sigma} X_{\rho\sigma}. \quad (4.2.38)$$

Indeed, we have

$$\varphi_{1p}^{\mu\alpha} X_{\mu\alpha} = \frac{1}{T} (\hat{\Pi}_p \Pi + \hat{J}_p^\mu J_\mu + \hat{\pi}_p^{\mu\nu} \pi_{\mu\nu}), \quad (4.2.39)$$

$$\eta_1^{\mu\alpha\nu\beta} X_{\nu\beta} = \begin{cases} -\zeta \Delta^{\mu\nu} \Pi + 2\eta \pi^{\mu\nu}, & \alpha = \nu, \\ -T \lambda z \hat{h}^{-1} J^\mu, & \alpha = 4, \end{cases} \quad (4.2.40)$$

where $\hat{\Pi}_p$, $\hat{J}_p^\mu$, and $\hat{\pi}_p^{\mu\nu}$ have been defined in Eqs. (3.3.7)-(3.3.10). It is noted that ζ , λ , and η appearing in Eq. (4.2.40) denote the bulk viscosity, heat conductivity, and shear viscosity, respectively, whose definitions have been given by Eqs. (3.3.17)-(3.3.19), respectively. We should emphasize that ζ , λ , and η are in agreement with the transport coefficients obtained in the first-order relativistic dissipative hydrodynamics, i.e., those of the Chapman-Enskog expansion method, instead of those of the Maxwell-Grad moment method; see η^{IS} , η^{D} , and η^{CE} given by Eqs. (2.3.25), (2.3.26), and (2.3.28), respectively. We also stress that Π , J^μ , and $\pi^{\mu\nu}$ denote the bulk pressure, heat flux, and viscous pressure, respectively. We note that the definitions of J^μ and $\pi^{\mu\nu}$ ensure

$$u_\mu J^\mu = 0, \quad (4.2.41)$$

$$\pi^{\mu\nu} = \pi^{\nu\mu}, \quad (4.2.42)$$

$$u_\mu \pi^{\mu\nu} = 0, \quad (4.2.43)$$

$$\pi^\mu{}_\mu = 0. \quad (4.2.44)$$

Owing to the properties shown in Eqs. (4.2.41)-(4.2.44), the number of independent components of J^μ and $\pi^{\mu\nu}$ is eight, and the total number of the would-be integral constants T , μ , u^μ , Π , J^μ , and $\pi^{\mu\nu}$ are fourteen. Although this number is the same as that of the dynamical variables introduced in the fourteen-moment approximation proposed by Israel and Stewart, we emphasize that this number and form of the

dynamical variables have been automatically determined from the relativistic Boltzmann equation by the doublet scheme in the RG method developed in Sec. 4.1, which does not demand any ansatz at all in contrast to the traditional approaches.

Using Eq. (4.2.39) and (4.2.40) together with the thermodynamic forces X_Π , X_J^μ , and $X_\pi^{\mu\nu}$ defined in Eqs. (3.3.23)-(3.3.25), we first reduce the dissipative part (4.2.34) in Eq. (4.2.31) to

$$\delta J_{2\text{nd}}^{\mu\alpha} = \begin{cases} -\zeta \Delta^{\mu\nu} \Pi + 2\eta \pi^{\mu\nu} \equiv \delta T_{2\text{nd}}^{\mu\nu}, & \alpha = \nu, \\ -T \lambda z \hat{h}^{-1} J^\mu \equiv m \delta N_{2\text{nd}}^\mu, & \alpha = 4. \end{cases} \quad (4.2.45)$$

Here, we should emphasize that $\delta T_{2\text{nd}}^{\mu\nu}$ and $\delta N_{2\text{nd}}^\mu$ is identically the energy-momentum tensor and particle-number current in the Landau-Lifshitz energy frame. In fact, $\delta T_{2\text{nd}}^{\mu\nu}$ and $\delta N_{2\text{nd}}^\mu$ meet the conditions of fit that are imposed into the energy-frame equation, i.e., (i), (ii), and (iv);

$$u_\mu \delta T_{2\text{nd}}^{\mu\nu} u_\nu = 0, \quad (4.2.46)$$

$$\Delta_{\mu\nu} \delta T_{2\text{nd}}^{\nu\rho} u_\rho = 0, \quad (4.2.47)$$

$$u_\mu \delta N_{2\text{nd}}^\mu = 0. \quad (4.2.48)$$

Next, we discuss the relaxation equation (4.2.25). For the sake of the simplicity, we treat only the leading term with respect to ϵ of Eq. (4.2.25) in this subsection. In Appendix A.5, we show the detailed derivation and full expressions of the relaxation equations. The resultant equations read

$$\Pi = X_\Pi - \tau_\Pi \frac{\partial}{\partial \tau} \Pi, \quad (4.2.49)$$

$$J^\mu = X_J^\mu - \tau_J \Delta^{\mu\rho} \frac{\partial}{\partial \tau} J_\rho, \quad (4.2.50)$$

$$\pi^{\mu\nu} = X_\pi^{\mu\nu} - \tau_\pi \Delta^{\mu\nu\rho\sigma} \frac{\partial}{\partial \tau} \pi_{\rho\sigma}. \quad (4.2.51)$$

We note that the τ -derivative terms denote the relaxation terms with the new coefficients τ_Π , τ_J , and τ_π , which are called the relaxation times and have the following microscopic representations:

$$\tau_\Pi \equiv -\frac{\langle \hat{\Pi}, L^{-2} \hat{\Pi} \rangle_{\text{eq}}}{\langle \hat{\Pi}, L^{-1} \hat{\Pi} \rangle_{\text{eq}}}, \quad (4.2.52)$$

$$\tau_J \equiv -\frac{\langle \hat{J}^\mu, L^{-2} \hat{J}_\mu \rangle_{\text{eq}}}{\langle \hat{J}^\nu, L^{-1} \hat{J}_\nu \rangle_{\text{eq}}}, \quad (4.2.53)$$

$$\tau_\pi \equiv -\frac{\langle \hat{\pi}^{\mu\nu}, L^{-2} \hat{\pi}_{\mu\nu} \rangle_{\text{eq}}}{\langle \hat{\pi}^{\rho\sigma}, L^{-1} \hat{\pi}_{\rho\sigma} \rangle_{\text{eq}}}. \quad (4.2.54)$$

These quantities are time constants characterizing the corresponding relaxation process, where Π , J^μ , and $\pi^{\mu\nu}$ can be reduced to X_Π , X_J^μ , and $X_\pi^{\mu\nu}$ asymptotically,

4.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

respectively;

$$\Pi \rightarrow X_\Pi, \quad (4.2.55)$$

$$J^\mu \rightarrow X_J^\mu, \quad (4.2.56)$$

$$\pi^{\mu\nu} \rightarrow X_\pi^{\mu\nu}. \quad (4.2.57)$$

We can see that $\delta J_{2\text{nd}}^{\mu\alpha}$ with substitution $\Pi = X_\Pi$, $J^\mu = X_J^\mu$, and $\pi^{\mu\nu} = X_\pi^{\mu\nu}$ are exactly in agreement with $\delta J_{1\text{st}}^{\mu\alpha}$ shown in Eq. (3.3.22);

$$\delta J_{2\text{nd}}^{\mu\alpha} \rightarrow \delta J_{1\text{st}}^{\mu\alpha} = \begin{cases} -\zeta \Delta^{\mu\nu} X_\Pi + 2\eta X_\pi^{\mu\nu}, & \alpha = \nu, \\ -T\lambda z \hat{h}^{-1} X_J^\mu, & \alpha = 4. \end{cases} \quad (4.2.58)$$

It is noteworthy that the second-order relativistic hydrodynamics can reproduce the first-order relativistic hydrodynamics after τ becomes large up to the same order as τ_Π , τ_J , and τ_π from zero.

With the use of the relaxation functions $R_\Pi(s)$, $R_J(s)$, and $R_\pi(s)$ defined in Eqs. (3.3.46)-(3.3.48), the relaxation times (4.2.52)-(4.2.54) can be converted into the following compact forms, as in the case of the transport coefficients in Eqs. (3.3.43)-(3.3.45):

$$\tau_\Pi = \frac{\int_0^\infty ds s R_\Pi(s)}{\int_0^\infty ds R_\Pi(s)}, \quad (4.2.59)$$

$$\tau_J = \frac{\int_0^\infty ds s R_J(s)}{\int_0^\infty ds R_J(s)}, \quad (4.2.60)$$

$$\tau_\pi = \frac{\int_0^\infty ds s R_\pi(s)}{\int_0^\infty ds R_\pi(s)}. \quad (4.2.61)$$

Here, we shall prove the equality in Eqs. (4.2.59)-(4.2.61): The equality is based on the relation

$$\begin{aligned} \sum_q [L^{-2}]_{pq} (\hat{\Pi}_q, \hat{J}_q^\mu, \hat{\pi}_q^{\mu\nu}) &= \sum_q \int_0^\infty ds s [e^{sL}]_{pq} (\hat{\Pi}_q, \hat{J}_q^\mu, \hat{\pi}_q^{\mu\nu}) \\ &= \int_0^\infty ds s (\hat{\Pi}_p(s), \hat{J}_p^\mu(s), \hat{\pi}_p^{\mu\nu}(s)). \end{aligned} \quad (4.2.62)$$

A proof of the relation (4.2.62) is given as follows: First, let us start from the identity

$$[(L - a)^{-1} \varphi]_p = - \int_0^\infty ds [e^{s(L-a)} \varphi]_p, \quad (4.2.63)$$

which is satisfied when a is a non-negative parameter and φ an arbitrary Q_0 -space vector. Next, we differentiate Eq. (4.2.63) with respect to a . Finally, by putting $a = 0$, we have

$$[L^{-2} \varphi]_p = \int_0^\infty ds s [e^{sL} \varphi]_p. \quad (4.2.64)$$

The setting of $\varphi_p = (\hat{\Pi}_p, \hat{J}_p^\mu, \hat{\pi}_p^{\mu\nu})$ leads us to the relation (4.2.62).

It is noteworthy that we can represent the relaxation times τ_Π , τ_J , and τ_π with use of the relaxation functions $R_\Pi(s)$, $R_J(s)$, and $R_\pi(s)$ in Eqs. (3.3.46)-(3.3.48), respectively. To the best of our knowledge, it is for the first time that the above representations are written down in the context of the derivation of the second-order relativistic hydrodynamic equation from the relativistic Boltzmann equation. Equations (4.2.52)-(4.2.54) make a physical meaning of τ_Π , τ_J , and τ_π clearer: The relaxation time is interpreted as a correlation time in the relaxation function. The formulae of the relaxation time obtained by Israel et al. [50] or Denicol et al. [67], e.g., τ_π^{IS} and τ_π^{D} given by Eqs. (2.3.29) and (2.3.30), respectively, cannot be represented by the relaxation function. Thus, it may be not easy to give a clear physical meaning to their formulae.

With the use of $L_{pq}^{-1} = \mathcal{L}_{pq}^{-1}(q \cdot u)$, $\hat{\Pi}_p = \Pi_p/p \cdot u$, $\hat{J}_p^\mu = J_p^\mu/p \cdot u$, and $\hat{\pi}_p^{\mu\nu} = \pi_p^{\mu\nu}/p \cdot u$, which is derived from Eqs. (2.3.7), (3.2.15), (3.3.8)-(3.3.10), we can show that the microscopic formulae of the transport coefficients ζ , λ , and η and relaxation times τ_Π , τ_J , and τ_π given in this subsection coincide with those introduced in Eqs. (1.3.11)-(1.3.16).

4.2.4 Causality and generic stability of fourteen-moment equation in energy frame

In this subsection, we shall show that the fluctuation around the static solution of the second-order relativistic hydrodynamic equation, i.e., the pair of Eqs. (4.2.26) and (4.2.25), propagates with a finite speed, which is less than the speed of light, i.e., the unity.

For this purpose, we first show that the static solution is stable against a small perturbation. A strategy used in this subsection is the same as one in the first-order case shown in Sec. 3.3.3.

A generic constant solution reads

$$T(\sigma; \tau) = T_0, \quad (4.2.65)$$

$$\mu(\sigma; \tau) = \mu_0, \quad (4.2.66)$$

$$u_\mu(\sigma; \tau) = u_{0\mu}, \quad (4.2.67)$$

$$X_{\mu\alpha}(\sigma; \tau) = 0, \quad (4.2.68)$$

where T_0 , μ_0 , and $u_{0\mu}$ are constant. To show the stability of the constant solution, we apply the linear stability analysis to the second-order relativistic hydrodynamic equation (4.2.26) and (4.2.25).

We expand T , μ , u_μ , and $X_{\mu\alpha}$ around the constant solution as follows:

$$T(\sigma; \tau) = T_0 + \delta T(\sigma; \tau), \quad (4.2.69)$$

$$\mu(\sigma; \tau) = \mu_0 + \delta \mu(\sigma; \tau), \quad (4.2.70)$$

$$u_\mu(\sigma; \tau) = u_{0\mu} + \delta u_\mu(\sigma; \tau), \quad (4.2.71)$$

$$X_{\mu\alpha}(\sigma; \tau) = \delta X_{\mu\alpha}(\sigma; \tau), \quad (4.2.72)$$

4.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

We assume that the higher term than second order in terms of δT , $\delta\mu$, δu_μ , and $\delta X_{\mu\alpha}$ can be neglected since these quantities are small. Instead of these six variables which are not independent of each other because $\delta u_\mu u_0^\mu = 0$, we use the following five independent variables composed of δT , $\delta\mu$, and δu_μ defined as

$$\delta X_{4\alpha} \equiv \begin{cases} -\delta(u_\mu/T) = -\delta u_\mu/T_0 + \delta T u_{0\mu}/T_0^2, & \alpha = \mu, \\ m^{-1} \delta(\mu/T) = m^{-1} (\delta\mu/T_0 - \delta T \mu_0/T_0^2), & \alpha = 4. \end{cases} \quad (4.2.73)$$

Substituting Eqs. (4.2.69)-(4.2.72) into the second-order relativistic hydrodynamic equation (4.2.26) and (4.2.25). we obtain the linearized equation governing $\delta X_{\alpha\beta} = (\delta X_{\mu\beta}, \delta X_{4\beta})$ as

$$\begin{aligned} & \langle \varphi_0^\alpha, \varphi_0^\beta \rangle_{\text{eq}} \frac{\partial}{\partial \tau} \delta X_{4\beta} + \langle \varphi_0^\alpha, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \frac{\partial}{\partial \tau} \delta X_{\nu\beta} \\ & + \langle \varphi_0^\alpha, v^\rho \varphi_0^\beta \rangle_{\text{eq}} \nabla_\rho \delta X_{4\beta} + \langle \varphi_0^\alpha, v^\rho L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \nabla_\rho \delta X_{\nu\beta} = 0, \end{aligned} \quad (4.2.74)$$

$$\begin{aligned} & -\eta_1^{\mu\alpha\nu\beta} \delta X_{\nu\beta} + \eta_1^{\mu\alpha\nu\beta} \nabla_\nu \delta X_{4\beta} \\ & + \langle L^{-1} \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \frac{\partial}{\partial \tau} \delta X_{\nu\beta} + \langle L^{-1} \varphi_1^{\mu\alpha}, v^\rho L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}} \nabla_\rho \delta X_{\nu\beta} = 0, \end{aligned} \quad (4.2.75)$$

where $v_p^\mu \equiv v_p^\mu \delta_{pq}$ with

$$v_p^\mu \equiv \Delta^{\mu\nu} p_\nu \frac{1}{p \cdot u}. \quad (4.2.76)$$

In the derivation of Eqs. (4.2.74) and (4.2.75), we have used

$$\delta(f_p^{\text{eq}}) = f_p^{\text{eq}} \varphi_{0p}^\alpha \delta X_{4\alpha}, \quad (4.2.77)$$

$$\delta(\bar{X}_{\mu\alpha}) = \nabla_\mu \delta X_{4\alpha}. \quad (4.2.78)$$

Here, we note that all of the coefficients in Eqs. (4.2.74) and (4.2.75) take a value of the constant solution $(T, \mu, u_\mu, X_{\mu\alpha}) = (T_0, \mu_0, u_{0\mu}, 0)$. Because of the orthogonality between the P_0 space and the Q_0 space, we can reduce Eqs. (4.2.74) and (4.2.75) to

$$A^{\alpha\beta, \gamma\delta} \frac{\partial}{\partial \tau} \delta X_{\gamma\delta} + B^{\alpha\beta, \gamma\delta} \delta X_{\gamma\delta} = 0, \quad (4.2.79)$$

where $A^{\alpha\beta, \gamma\delta}$ and $B^{\alpha\beta, \gamma\delta}$ are defined as

$$A^{\alpha\beta, \gamma\delta} \equiv \begin{cases} \langle L^{-1} \varphi_1^{\mu\beta}, L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}}, & (\alpha, \gamma) = (\mu, \nu), \\ 0, & (\alpha, \gamma) = (\mu, 4), \\ 0, & (\alpha, \gamma) = (4, \nu), \\ \eta_0^{\beta\delta}, & (\alpha, \gamma) = (4, 4), \end{cases} \quad (4.2.80)$$

$$B^{\alpha\beta, \gamma\delta} \equiv \begin{cases} -\eta_1^{\mu\beta\nu\delta} + \langle L^{-1} \varphi_1^{\mu\beta}, v^\rho L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}} \nabla_\rho, & (\alpha, \gamma) = (\mu, \nu), \\ \eta_1^{\mu\beta\rho\delta} \nabla_\rho, & (\alpha, \gamma) = (\mu, 4), \\ \eta_1^{\rho\beta\nu\delta} \nabla_\rho, & (\alpha, \gamma) = (4, \nu), \\ \langle \varphi_0^\beta, v^\rho \varphi_0^\delta \rangle_{\text{eq}} \nabla_\rho, & (\alpha, \gamma) = (4, 4). \end{cases} \quad (4.2.81)$$

Here, we have used $\eta_0^{\alpha\beta} = \langle \varphi_0^\alpha, \varphi_0^\beta \rangle_{\text{eq}}$ and $\eta_1^{\mu\alpha\nu\beta} = \langle \varphi_1^{\mu\alpha}, L^{-1} \varphi_1^{\nu\beta} \rangle_{\text{eq}}$.

We convert Eq. (4.2.79) into the algebraic equation, using the Fourier and Laplace transformations with respect to the spatial variable σ^μ and the temporal variable τ , respectively. By substituting

$$\delta X_{\alpha\beta}(\sigma; \tau) = \delta \tilde{X}_{\alpha\beta}(k; \Lambda) e^{ik \cdot \sigma - \Lambda \tau}, \quad (4.2.82)$$

into Eq. (4.2.79), we have

$$(\Lambda A^{\alpha\beta, \gamma\delta} - \tilde{B}^{\alpha\beta, \gamma\delta}) \delta \tilde{X}_{\gamma\delta} = 0, \quad (4.2.83)$$

where $\tilde{B}^{\alpha\beta, \gamma\delta}$ is defined as

$$\tilde{B}^{\alpha\beta, \gamma\delta} \equiv \begin{cases} -\eta_1^{\mu\beta\nu\delta} + i \langle L^{-1} \varphi_1^{\mu\beta}, v^\rho L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}} k_\rho, & (\alpha, \gamma) = (\mu, \nu), \\ i \eta_1^{\mu\beta\rho\delta} k_\rho, & (\alpha, \gamma) = (\mu, 4), \\ i \eta_1^{\rho\beta\nu\delta} k_\rho, & (\alpha, \gamma) = (4, \nu), \\ i \langle \varphi_0^\beta, v^\rho \varphi_0^\delta \rangle k_\rho, & (\alpha, \gamma) = (4, 4). \end{cases} \quad (4.2.84)$$

We note that k^μ is a space-like vector satisfying $k^\mu = \Delta^{\mu\nu} k_\nu$. In the rest of this section, we use the matrix representation when no misunderstanding is expected.

Since we are interested in a solution other than $\delta \tilde{X} = 0$, we can impose

$$\det(\Lambda A - \tilde{B}) = 0. \quad (4.2.85)$$

It is noted that Eq. (4.2.85) leads to the dispersion relation

$$\Lambda = \Lambda(k). \quad (4.2.86)$$

The stability of the constant solution given by Eqs. (4.2.65)-(4.2.65) against a small perturbation is equivalent to that δX does not increase with time evolution. Therefore, our task is to show that the real part of $\Lambda(k)$ is non-negative for any k^μ .

We show that A is a real symmetric positive-definite matrix as follows:

$$\begin{aligned} w_{\alpha\beta} A^{\alpha\beta, \gamma\delta} w_{\gamma\delta} &= w_{\mu\beta} \langle L^{-1} \varphi_1^{\mu\beta}, L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}} w_{\nu\delta} + w_{4\beta} \eta_0^{\beta\delta} w_{4\delta} \\ &= \langle w_{\mu\beta} L^{-1} \varphi_1^{\mu\beta}, w_{\nu\delta} L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}} + \langle w_{4\beta} \varphi_0^\beta, w_{4\delta} \varphi_0^\delta \rangle_{\text{eq}} \\ &= \langle w_{\mu\beta} L^{-1} \varphi_1^{\mu\beta} + w_{4\beta} \varphi_0^\beta, w_{\nu\delta} L^{-1} \varphi_1^{\nu\delta} + w_{4\delta} \varphi_0^\delta \rangle_{\text{eq}} \\ &= \langle \chi, \chi \rangle_{\text{eq}} > 0, \quad w_{\alpha\beta} \neq 0, \end{aligned} \quad (4.2.87)$$

with $\chi_p \equiv w_{\mu\alpha} [L^{-1} \varphi_1^{\mu\alpha}]_p + w_{4\alpha} \varphi_0^\alpha$. In Eq. (4.2.87), we have used the orthogonality between φ_0^α and $L^{-1} \varphi_1^{\mu\alpha}$ and the positive-definite property of the inner product (3.2.17).

Equation (4.2.87) means that the inverse matrix A^{-1} exists, and A^{-1} is also a real symmetric positive-definite matrix. Thus, with the use of the Cholesky decomposition, we can represent A^{-1} as

$$A^{-1} = {}^t U U, \quad (4.2.88)$$

4.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

where U denotes a real matrix and tU a trans matrix of U . Substituting Eq. (4.2.88) into Eq. (4.2.85), we have

$$\det(\Lambda I - U \tilde{B} {}^tU) = 0, \quad (4.2.89)$$

where I denotes the unit matrix. It is noted that $\Lambda(k)$ is an eigenvalue of $U \tilde{B} {}^tU$.

We find that the real part of $\Lambda(k)$ becomes non-negative for any k^μ when $\text{Re}(U \tilde{B} {}^tU)$ is a semi-positive definite matrix. In fact, we can show that $\text{Re}(U \tilde{B} {}^tU)$ is semi-positive definite as follows:

$$\begin{aligned} w_{\alpha\beta} [\text{Re}(U \tilde{B} {}^tU)]^{\alpha\beta, \gamma\delta} w_{\gamma\delta} &= w_{\alpha\beta} [U \text{Re}(\tilde{B}) {}^tU]^{\alpha\beta, \gamma\delta} w_{\gamma\delta} \\ &= [w U]_{\alpha\beta} [\text{Re}(\tilde{B})]^{\alpha\beta, \gamma\delta} [w U]_{\gamma\delta} \\ &= -[w U]_{\mu\beta} \eta_1^{\mu\beta\nu\delta} [w U]_{\nu\delta} \\ &= -\langle [w U]_{\mu\beta} \varphi_1^{\mu\beta}, L^{-1} [w U]_{\nu\delta} \varphi_1^{\nu\delta} \rangle \\ &= -\langle \psi, L^{-1} \psi \rangle \geq 0, \quad w_{\alpha\beta} \neq 0, \end{aligned} \quad (4.2.90)$$

with $\psi_p \equiv [w U]_{\mu\alpha} \varphi_{1p}^{\mu\alpha}$. Therefore, we conclude that the constant solution given by Eqs. (4.2.65)-(4.2.65) is stable against a small perturbation.

Now, we are in a position, where we show that the propagation speed of the fluctuation $\delta X_{\alpha\beta}$ is not beyond the unity, i.e., the speed of light. Here, we suppose that the propagation speed of $\delta X_{\alpha\beta}$ is given by a character speed, whose Lorentz-invariant form may be given by

$$v_{\text{ch}} \equiv \sqrt{-\Delta_{\mu\nu} v_{\text{ch}}^\mu v_{\text{ch}}^\nu}. \quad (4.2.91)$$

Here, we have introduced the space-like vector v_{ch}^μ defined as

$$v_{\text{ch}}^\mu \equiv \lim_{-k^2 \rightarrow \infty} \left[-i \frac{\partial}{\partial k_\mu} \Lambda(k) \right]. \quad (4.2.92)$$

In the rest of this subsection, we shall prove that v_{ch} does not exceed 1.

By differentiating Eq. (4.2.89) with respect to $i k_\mu$, we find that v_{ch}^μ is an eigenvalue of $U C^\mu {}^tU$, i.e.,

$$\det [v_{\text{ch}}^\mu - U C^\mu {}^tU] = 0, \quad (4.2.93)$$

where

$$\begin{aligned} [C^\rho]^{\alpha\beta, \gamma\delta} &\equiv \lim_{-k^2 \rightarrow \infty} \left[-i \frac{\partial}{\partial k_\rho} \tilde{B}^{\alpha\beta, \gamma\delta} \right] \\ &= \begin{cases} \langle L^{-1} \varphi_1^{\mu\beta}, v^\rho L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}}, & (\alpha, \gamma) = (\mu, \nu), \\ \langle L^{-1} \varphi_1^{\mu\beta}, v^\rho \varphi_0^\delta \rangle_{\text{eq}}, & (\alpha, \gamma) = (\mu, 4), \\ \langle v^\rho \varphi_0^\beta, L^{-1} \varphi_1^{\nu\delta} \rangle_{\text{eq}}, & (\alpha, \gamma) = (4, \nu), \\ \langle \varphi_0^\beta, v^\rho \varphi_0^\delta \rangle_{\text{eq}}, & (\alpha, \gamma) = (4, 4). \end{cases} \end{aligned} \quad (4.2.94)$$

An expectation value of $U C^\mu t U$ with respect to an arbitrary vector $w' \equiv {}^t(U^{-1}) w$ can be written as

$$\begin{aligned} \frac{[w U^{-1}]_{\alpha\beta} [U C^\mu t U]^{\alpha\beta, \gamma\delta} [{}^t(U^{-1}) w]_{\gamma\delta}}{w_{\bar{\alpha}\bar{\beta}} [U^{-1} {}^t(U^{-1})]^{\bar{\alpha}\bar{\beta}, \bar{\gamma}\bar{\delta}} w_{\bar{\gamma}\bar{\delta}}} &= \frac{w_{\alpha\beta} [C^\mu]^{\alpha\beta, \gamma\delta} w_{\gamma\delta}}{w_{\bar{\alpha}\bar{\beta}} A^{\bar{\alpha}\bar{\beta}, \bar{\gamma}\bar{\delta}} w_{\bar{\gamma}\bar{\delta}}} \\ &= \frac{\langle \chi, v^\mu \chi \rangle_{\text{eq}}}{\langle \chi, \chi \rangle_{\text{eq}}} \\ &= \langle v^\mu \rangle_\chi, \end{aligned} \quad (4.2.95)$$

with $\chi_p = w_{\mu\alpha} [L^{-1} \varphi_1^{\mu\alpha}]_p + w_{4\alpha} \varphi_{0p}^\alpha$. Here, we have introduced

$$\langle O \rangle_\chi \equiv \frac{\langle \chi, O \chi \rangle_{\text{eq}}}{\langle \chi, \chi \rangle_{\text{eq}}}, \quad (4.2.96)$$

with O being an arbitrary operator.

It is important to note that if the inequality

$$\sqrt{-\Delta_{\mu\nu} \langle v^\mu \rangle_\chi \langle v^\nu \rangle_\chi} \leq 1, \quad (4.2.97)$$

are satisfied for any χ_p , we can conclude

$$v_{\text{ch}} = \sqrt{-\Delta_{\mu\nu} v_{\text{ch}}^\mu v_{\text{ch}}^\nu} \leq 1. \quad (4.2.98)$$

Indeed, we can show that the inequality (4.2.97) is satisfied in this case. The proof is given as follows: First, with the use of the identities

$$-\Delta_{\mu\nu} v_p^\mu v_p^\nu = \frac{(p \cdot u)^2 - m^2}{(p \cdot u)^2} \leq 1, \quad (4.2.99)$$

$$\langle 1 \rangle_\chi = 1, \quad (4.2.100)$$

we obtain

$$\langle -\Delta_{\mu\nu} v^\mu v^\nu \rangle_\chi \leq 1. \quad (4.2.101)$$

Then, we notice

$$\langle -\Delta_{\mu\nu} v^\mu v^\nu \rangle_\chi = -\Delta_{\mu\nu} \langle v^\mu \rangle_\chi \langle v^\nu \rangle_\chi + \langle -\Delta_{\mu\nu} \delta v^\mu \delta v^\nu \rangle_\chi \geq -\Delta_{\mu\nu} \langle v^\mu \rangle_\chi \langle v^\nu \rangle_\chi, \quad (4.2.102)$$

where $\delta v_{pq}^\mu \equiv \delta v_p^\mu \delta_{pq}$ with $\delta v_p^\mu \equiv v_p^\mu - \langle v^\mu \rangle_\chi$, because

$$-\Delta_{\mu\nu} \delta v_p^\mu \delta v_p^\nu \geq 0, \quad (4.2.103)$$

due to the fact that δv_p^μ is also a space-like vector. By combining Eq. (4.2.102) with Eq. (4.2.101), we complete the proof.

Thus, we emphasize that our fourteen-moment equation given by Eqs. (4.2.26) and (4.2.25) respects the causality in the linear analysis around the homogeneous steady state (4.2.65)-(4.2.69), in addition to the stability.

4.2.5 Novel ansatz in moment method consistent to mesoscopic dynamics of relativistic Boltzmann equation

Our second-order relativistic hydrodynamic equation consisting Eqs. (4.2.31) and (4.2.25) suggests a novel “ansatz” for the functional form of the distribution function in the Maxwell-Grad moment method, which should be adopted so as to be compatible with the relativistic Boltzmann equation in the mesoscopic regime. To show that this is the case, it is convenient to start with Eqs. (4.1.58) and (4.1.59) instead of Eqs. (4.2.31) and (4.2.25).

Substituting A , B , $\mathbf{F}_0$, F_1 , $\delta\bar{X}_{(\mu)}$, $\delta X_{(\mu)}$, $\varphi_0^{(\alpha)}$, and $\varphi_1^{(\mu)}$ obtained in Sec. 4.2.1 into Eqs. (4.1.58) and (4.1.59), we have

$$\sum_p \frac{1}{p^0} (1, p^\mu) p^\lambda \partial_\lambda \left[f_p^{\text{eq}} \left(1 + \frac{1}{T} ([L^{-1} \hat{\Pi}]_p \Pi + [L^{-1} \hat{J}^\rho]_p J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_p \pi_{\rho\sigma}) \right) \right] = 0, \quad (4.2.104)$$

and

$$\begin{aligned} & \sum_p \frac{1}{p^0} [L^{-1} (\hat{\Pi}, \hat{J}^\mu, \hat{\pi}^{\mu\nu})]_p p^\lambda \partial_\lambda \left[f_p^{\text{eq}} \left(1 + \frac{1}{T} ([L^{-1} \hat{\Pi}]_p \Pi + [L^{-1} \hat{J}^\rho]_p J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_p \pi_{\rho\sigma}) \right) \right] \\ &= \sum_p \frac{1}{p^0} (p \cdot u) f_p^{\text{eq}} [L^{-1} (\hat{\Pi}, \hat{J}^\mu, \hat{\pi}^{\mu\nu})]_p \sum_q L_{pq} \frac{1}{T} ([L^{-1} \hat{\Pi}]_q \Pi + [L^{-1} \hat{J}^\rho]_q J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_q \pi_{\rho\sigma}), \end{aligned} \quad (4.2.105)$$

where ϵ has been set equal to 1 and the bilinear terms with respect to Π , J^μ , and $\pi^{\mu\nu}$ has been neglected.

By comparing Eqs. (4.2.104) and (4.2.105) with the continuity equation (2.3.20) and the relaxation equation (2.3.21) introduced in the Maxwell-Grad moment method, we can read off the form of the deviation Φ_p and the fourteen independent quantities utilized to obtain the closed system of the differential equations as

$$\Phi_p = \frac{1}{T} ([L^{-1} \hat{\Pi}]_p \Pi + [L^{-1} \hat{J}^\nu]_p J_\nu + [L^{-1} \hat{\pi}^{\mu\nu}]_p \pi_{\mu\nu}) \equiv \Phi_p^{\text{TK}}, \quad (4.2.106)$$

and

$$1, p^\mu, [L^{-1} \hat{\Pi}]_p, [L^{-1} \hat{J}^\nu]_p, [L^{-1} \hat{\pi}^{\mu\nu}]_p, \quad (4.2.107)$$

respectively. We should stress that the deviation Φ_p^{TK} is not a polynomial form of the momentum p and differs from that proposed by Israel and Stewart, i.e., Φ_p^{IS} introduced in Eq. (2.3.16);

$$\Phi_p^{\text{TK}} \neq \Phi_p^{\text{IS}} = a + p^\mu b_\mu + p^\mu p^\nu c_{\mu\nu}. \quad (4.2.108)$$

Furthermore, it turns out that Φ_p^{TK} is different from that of Denicol et al. shown in Eq. (2.3.23);

$$\Phi_p^{\text{TK}} \neq \Phi_p^{\text{D}} = \frac{1}{p \cdot u} (a + p^\mu b_\mu + p^\mu p^\nu c_{\mu\nu}). \quad (4.2.109)$$

With the use of $[L^{-1}(\hat{\Pi}, \hat{J}^\mu, \hat{\pi}^{\mu\nu})]_p = \sum_q \mathcal{L}_{pq}^{-1}(\Pi_q, J_q^\mu, \pi_q^{\mu\nu})$ which is derived from Eqs. (2.3.7), (3.2.15), (3.3.8)-(3.3.10), we can show that Φ_p^{TK} given by Eq. (4.2.106) is equivalent to the deviation introduced in Eq. (1.3.17).

In the rest of this subsection, we show that Φ_p^{TK} can be reduced to Φ_p^{D} but not Φ_p^{IS} when the differential cross section in the transition probability $\omega(p, p_1 | p_2, p_3)$ is a constant independent of the relative momentum and the scattering angle. For this purpose, we first introduce the Ritz-Galerkin method [47] used in the calculation of the transport coefficients. The calculation requires $\psi_{\Pi p} \equiv [L^{-1} \hat{\Pi}]_p$, $\psi_{J p}^\mu \equiv [L^{-1} \hat{J}^\mu]_p$, and $\psi_{\pi p}^{\mu\nu} \equiv [L^{-1} \hat{\pi}^{\mu\nu}]_p$, which are obtained as the solutions of the algebraic equations given by

$$[L \psi_{\Pi}]_p = \hat{\Pi}_p, \quad (4.2.110)$$

$$[L \psi_J^\mu]_p = \hat{J}_p^\mu, \quad (4.2.111)$$

$$[L \psi_\pi^{\mu\nu}]_p = \hat{\pi}_p^{\mu\nu}. \quad (4.2.112)$$

Since the above equations cannot be solved exactly for an arbitrary interaction, one seeks an approximate solution by restricting the admissible functions to some appropriate finite-dimensional subspace. A standard approximation method is just the Ritz-Galerkin method [47], where $\psi_{\Pi p}$, $\psi_{J p}^\mu$, and $\psi_{\pi p}^{\mu\nu}$ are supposed to be expanded as

$$\psi_{\Pi p} \sim \hat{\Pi}_p (\alpha_0 + \alpha_1 (p \cdot u) + \cdots + \alpha_N (p \cdot u)^N), \quad (4.2.113)$$

$$\psi_{J p}^\mu \sim \hat{J}_p^\mu (\beta_0 + \beta_1 (p \cdot u) + \cdots + \beta_N (p \cdot u)^N), \quad (4.2.114)$$

$$\psi_{\pi p}^{\mu\nu} \sim \hat{\pi}_p^{\mu\nu} (\gamma_0 + \gamma_1 (p \cdot u) + \cdots + \gamma_N (p \cdot u)^N), \quad (4.2.115)$$

where N denotes a finite integer. Here, α_i , β_i , and γ_i with $i = 0, \dots, N$ denote the expansion coefficients whose values can be determined based on the variational principle for the quantities

$$\langle (L \psi_{\Pi} - \hat{\Pi}), L^{-1} (L \psi_{\Pi} - \hat{\Pi}) \rangle_{\text{eq}}, \quad (4.2.116)$$

$$\langle (L \psi_J^\mu - \hat{J}^\mu), L^{-1} (L \psi_J^\mu - \hat{J}^\mu) \rangle_{\text{eq}}, \quad (4.2.117)$$

$$\langle (L \psi_\pi^{\mu\nu} - \hat{\pi}^{\mu\nu}), L^{-1} (L \psi_\pi^{\mu\nu} - \hat{\pi}^{\mu\nu}) \rangle_{\text{eq}}, \quad (4.2.118)$$

respectively. It is well known [47] that if the differential cross section takes a constant value, the leading terms in the expansions (4.2.113)-(4.2.115) are dominant, while the dependence on the relative momentum and the scattering angle reduces the convergence of the expansions. Here, we shall take the case with the constant differential

4.2. REDUCTION OF RELATIVISTIC BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

cross section, where the choice of $N = 0$ may be validated. Thus, we have

$$\psi_{\Pi p} \sim \frac{\langle \hat{\Pi}, \hat{\Pi} \rangle_{\text{eq}}}{\langle \hat{\Pi}, L \hat{\Pi} \rangle_{\text{eq}}} \hat{\Pi}_p \equiv \hat{\Pi}_p / L_{\Pi}, \quad (4.2.119)$$

$$\psi_{Jp}^{\mu} \sim \frac{\langle \hat{J}^a, \hat{J}_a \rangle_{\text{eq}}}{\langle \hat{J}^b, L \hat{J}_b \rangle_{\text{eq}}} \hat{J}_p^{\mu} \equiv \hat{J}_p^{\mu} / L_J, \quad (4.2.120)$$

$$\psi_{\pi p}^{\mu\nu} \sim \frac{\langle \hat{\pi}^{ab}, \hat{\pi}_{ab} \rangle_{\text{eq}}}{\langle \hat{\pi}^{cd}, L \hat{\pi}_{cd} \rangle_{\text{eq}}} \hat{\pi}_p^{\mu\nu} \equiv \hat{\pi}_p^{\mu\nu} / L_{\pi}, \quad (4.2.121)$$

because

$$\alpha_0 = 1/L_{\Pi}, \quad (4.2.122)$$

$$\beta_0 = 1/L_J, \quad (4.2.123)$$

$$\gamma_0 = 1/L_{\pi}. \quad (4.2.124)$$

Then, substituting $\psi_{\Pi p}$, ψ_{Jp}^{μ} , and $\psi_{\pi p}^{\mu\nu}$ in Eqs. (4.2.119)-(4.2.121) into $\Phi_p^{\text{TK}} = (\psi_{\Pi p} \Pi + \psi_{Jp}^{\mu} J_{\mu} + \psi_{\pi p}^{\mu\nu} \pi_{\mu\nu})/T$, we obtain

$$\begin{aligned} \Phi_p^{\text{TK}} &\sim \frac{1}{T} (L_{\Pi}^{-1} \hat{\Pi}_p \Pi + L_J^{-1} \hat{J}_p^{\mu} J_{\mu} + L_{\pi}^{-1} \hat{\pi}_p^{\mu\nu} \pi_{\mu\nu}) \\ &\sim \frac{1}{p \cdot u} \frac{1}{T} (L_{\Pi}^{-1} \Pi_p \Pi + L_J^{-1} J_p^{\mu} J_{\mu} + L_{\pi}^{-1} \pi_p^{\mu\nu} \pi_{\mu\nu}), \end{aligned} \quad (4.2.125)$$

where Π_p , J_p^{μ} , and $\pi_p^{\mu\nu}$ have been given by Eqs. (3.3.8)-(3.3.10), respectively. Noting that Π_p , J_p^{μ} , and $\pi_p^{\mu\nu}$ are at most bilinear with respect to the momentum p , we conclude

$$\Phi_p^{\text{TK}} \sim \Phi_p^{\text{D}}, \quad (4.2.126)$$

where we have used the following redefinitions:

$$a = \frac{1}{T} (L_{\Pi}^{-1} \Pi (-m^2/3)), \quad (4.2.127)$$

$$b^{\mu} = \frac{1}{T} (L_{\Pi}^{-1} \Pi u^{\mu} ((\gamma - 1) T \hat{h} - \gamma T) + L_J^{-1} J^{\mu} T \hat{h}), \quad (4.2.128)$$

$$c^{\mu\nu} = \frac{1}{T} (L_{\Pi}^{-1} \Pi (4/3 - \gamma) u^{\mu} u^{\nu} - L_J^{-1} (J^{\mu} u^{\nu} + J^{\nu} u^{\mu})/2 + L_{\pi}^{-1} \pi^{\mu\nu}). \quad (4.2.129)$$

Here, one should notice that Φ_p^{D} might be validated only when interactions between each particles are treated as a hard sphere (a constant differential cross section). In fact, as mentioned in Sec. 2.3.3, Denicol et al. [67] have shown the consistency between the fourteen-moment equation derived from Φ_p^{D} and the numerical solution obtained from the relativistic Boltzmann equation in the limited case where the differential cross section is constant.

It is important to note that Φ_p^{TK} obtained in this thesis based on the RG method can be applied to generic interactions, in contrast to Φ_p^{D} . Furthermore, we stress that this thesis provides a theoretical foundation for the fact that the equation derived from Φ_p^{D} is in agreement with the relativistic Boltzmann equation with a constant differential cross section.

Chapter 5

Summary and concluding remarks

In this thesis, we obtained the first-order relativistic dissipative hydrodynamics through solving the equation for the distribution function of the relativistic Boltzmann equation in the asymptotic regime as the invariant manifold of the dynamical system, on the basis of the renormalization-group (RG) method [81, 82, 83, 84, 85, 86, 87, 88, 89, 90] in a systematic manner with no ad-hoc assumption. In the present analysis based on the RG method, the perturbative expansion of the distribution function with respect to the spatial derivative with the Knudsen number, i.e., the ratio of the average particle distance over the mean free path, is first performed with the zeroth-order being the local equilibrium distribution function; the dissipative effect is taken into account as a deformation of the distribution function made by the spatial inhomogeneity as the perturbation. We showed that the local equilibrium distribution function is identical to the Jüttner function expressed in terms of the five hydrodynamic modes, which are naturally identified with the same number of the zero modes of the linearized collision operator, i.e., the collision invariants. After defining the inner product in the function space spanned by the distribution function, the deviation from the Jüttner function that gives rise to the dissipative effects is constructed so that it is precisely orthogonal to the zero modes with respect to the inner product. Thus, any ansatz, such as the so-called conditions of fit used in the standard methods in an ad-hoc way, are not necessary, and rather the conditions of fit can be obtained as the orthogonality of the the deviation to the zero modes. In fact, the deviation turns out to satisfy the conditions of fit defining the energy frame, i.e., the Landau-Lifshitz one, provided that the macroscopic-frame vector, which defines the local rest frame of the flow velocity, is independent of the momenta of constituent particles, as it should. Our deviation thus suggests that the relativistic dissipative hydrodynamic equation must be defined in the energy frame, if it is consistent with the underlying kinetic equation. Although the relativistic Boltzmann equation which we have adopted as the kinetic equation is admittedly suitable only for a dilute gas, it is expected that the derived hydrodynamic equation itself and hence the uniqueness of the energy frame can be valid even for dense systems; this is found plausible if one recalls the universal nature of the Navier-Stokes equation beyond dilute systems, although it can be also derived [75, 87, 88] from the non-relativistic Boltzmann

equation. Furthermore, based on the positive definiteness of the inner product, we presented an analytic proof that the first-order relativistic dissipative hydrodynamic equation in the energy frame has a stable equilibrium state. We also elucidated that the problematic Burnett term does not affect the resultant equation owing to the nature of the hydrodynamic modes as the zero modes.

The second-order relativistic dissipative hydrodynamics was constructed as the mesoscopic dynamics of the relativistic Boltzmann equation. The mesoscopic dynamics occupies an intermediate level between the descriptions by hydrodynamics and kinetic theory. A basic observation presented in the extraction of the mesoscopic dynamics from the relativistic Boltzmann equation is to incorporate some excited (fast) modes of the linearized collision operator as additional components to the invariant manifold originally spanned by the zero modes for the first-order relativistic dissipative hydrodynamics. In fact, we developed the doublet scheme, i.e., a generic framework in the RG method to extract a mesoscopic dynamics from an evolution equation, on the basis of the following consistency condition and general principle of the reduction theory of the dynamics [78]: (A) the resultant dynamics should be consistent with the slow dynamics obtained by employing only the zero modes in the asymptotic regimes; (B) the resultant dynamics should be as simple as possible because we are interested to reduce the dynamics to a simpler one. Here, the principle (B) was utilized to construct the mesoscopic dynamics as described with a fewer number of dynamical variables, which is given by an equation composed of a fewer number of terms. It turns out that the number and form of the excited modes that should be included in the invariant manifold are automatically determined by the doublet scheme in the RG method. We showed that the mesoscopic dynamics of the relativistic Boltzmann equation obtained by the doublet scheme in the RG method has the same form as Israel-Stewart's [50] or Denicol et al.'s [67] fourteen-moment equation in the energy frame, i.e., typical equations of the second-order relativistic dissipative hydrodynamics, but the microscopic formulae of the coefficients, e.g., the transport coefficients and relaxation times, are different. It is found that our theory leads to the same expressions for the transport coefficients as given by the Chapman-Enskog expansion method in contrast to Israel-Stewart's or Denicol et al.'s equations, and suggests novel formulae of the relaxation times expressed in terms of relaxation functions which allow a natural physical interpretation of the relaxation times. On the basis of the positive definiteness of the inner product as in the case of the first-order relativistic dissipative hydrodynamic equation, we showed that the second-order relativistic dissipative hydrodynamic equation in the energy frame derived by the RG method has a stable equilibrium state, and respects causality, which means that the fluctuation around a static solution propagates with a finite speed that is slower than the speed of light. Furthermore, we demonstrated that the distribution function which is explicitly constructed in our theory provides a novel ansatz for the functional form of the distribution function that should be adopted in the fourteen-moment method proposed by Israel and Stewart.

It is worth emphasizing that all the equations derived in this thesis are consistent

with the underlying relativistic Boltzmann equation. This is one of the advantage in our theory because our theory explicitly gives the solution (distribution function) of the relativistic Boltzmann equation which is expressed in terms of the hydrodynamic variables and relaxation times, and thereby makes a systematic description of the time-evolution of the system from the hydrodynamic to kinetic regime. Such an overall analysis should be desirable for that of the hot and/or dense QCD matter created at RHIC, LHC, and other systems where the proper dynamics would change from the hydrodynamic to the kinetic ones or vice versa [25, 26, 27, 28, 29, 30]. Furthermore, it would be interesting to evaluate the relaxation times as well as the transport coefficients of the created matter with the use of the microscopic representations obtained in this thesis. In fact, with the use of the relaxation times (4.2.59)-(4.2.61) expressed in terms of the relaxation function, we can carry out the first-principles calculation of them base on the lattice QCD, as in the case of the transport coefficients (3.3.43)-(3.3.45). We also stress that it is interesting and important to present numerical simulations to demonstrate that a solution of the fourteen-moment causal equation obtained in this thesis is actually consistent with that of the relativistic Boltzmann equation with a differential cross section dependent on the relative momentum and the scattering angle in the mesoscopic regime.

Finally, we note that the RG method [79, 80, 81, 82, 83, 86, 87, 88, 89] itself has a universal nature and can be applied to derive a slow dynamics from kinetic equations other than the simple Boltzmann equation, say, multi-component Boltzmann equation [47] and Kadanoff-Baym equation [110]. In fact, we can derive the relativistic dissipative hydrodynamics from the multi-component relativistic Boltzmann equation, and show that the energy frame is also the most natural frame for multi-component systems [111]. Furthermore, we stress that the doublet scheme in the RG method [90] can be also applied to extract a mesoscopic dynamics from a wide class of evolution equations as far as the linearized evolution operator A is self-adjoint and has no Jordan cell. An extension of the doublet scheme to a method applicable to a more generic case where A is not self-adjoint or has Jordan cell will be studied elsewhere.

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Dr. Y. Minami and Dr. Y. Hidaka had a conjecture that the relativistic dissipative hydrodynamic equation might possibly be defined only in the energy frame, at least in the linear regime, if it should satisfy some basic thermodynamic property, which partly motivated this work. I thank the members of Nuclear Theory Group in Kyoto University and the Analysis Technology Center in Fujifilm Corporation for their discussions, supports, and encouragements.

Appendix A

Detailed derivation of formulae used in this thesis

A.1 Jüttner function as entropy-conserving distribution function

In this section, we shall present a proof that the entropy-conserving distribution function,

$$f_p^{\text{EC}} \equiv (2\pi)^{-3} \exp(a - b_\mu p^\mu), \quad (\text{A.1.1})$$

is identically the Jüttner function f_p^{eq} in Eq. (2.2.13), where the parameters (a, b^μ) can be interpreted as the chemical potential over temperature and the flow velocity over temperature, respectively; we assume that b^μ is a time-like four vector with $b^0 > 0$, where f_p^{EC} converges zero in the limit of $|\mathbf{p}| \rightarrow \infty$. This proof is mainly based on [48].

The equilibrium state in an arbitrary Lorentz frame, in which the gas moves as an isotropic body with the velocity $\mathbf{v}$ for an observer, are described by the particle current, energy-momentum tensor, and entropy current given as

$$N^{(0)\mu} = n u^\mu, \quad (\text{A.1.2})$$

$$T^{(0)\mu\nu} = e u^\mu u^\nu - p \Delta^{\mu\nu}, \quad (\text{A.1.3})$$

$$S^{(0)\mu} = s u^\mu, \quad (\text{A.1.4})$$

respectively, where n, e, p, s , and $u^\mu = (\gamma, \gamma \mathbf{v})$ with $\gamma \equiv 1/\sqrt{1 - |\mathbf{v}|^2}$ denote the particle-number density, internal energy, pressure, entropy density, and the flow velocity, respectively. We note that u^μ is a time-like four vector with $u^0 > 0$ because $u^\mu u_\mu = 1$.

Here, we note that the temperature and chemical potential of this equilibrium state are determined by the Gibbs-Duhem relation (2.2.14), which is equivalent to

$$\frac{1}{T} = \frac{\partial}{\partial e} s(e, n), \quad (\text{A.1.5})$$

$$\frac{\mu}{T} = -\frac{\partial}{\partial n} s(e, n), \quad (\text{A.1.6})$$

where the equation of state is given by $s = s(e, n)$.

Here, we assume that these above equations (A.1.2)-(A.1.6) established in the equilibrium theory can be applied to the kinetic theory based on the entropy- conserving distribution function (A.1.1). By inserting f_p^{EC} into the microscopic representations of $N^{(0)\mu}$, $T^{(0)\mu\nu}$, and $S^{(0)\mu}$, i.e.,

$$N^{(0)\mu} = \sum_p \frac{1}{p^0} p^\mu f_p^{\text{EC}}, \quad (\text{A.1.7})$$

$$T^{(0)\mu\nu} = \sum_p \frac{1}{p^0} p^\mu p^\nu f_p^{\text{EC}}, \quad (\text{A.1.8})$$

$$S^{(0)\mu} = -\sum_p \frac{1}{p^0} p^\mu f_p^{\text{EC}} (\ln (2\pi)^3 f_p^{\text{EC}} - 1), \quad (\text{A.1.9})$$

and comparing them with Eqs. (A.1.2)-(A.1.6), we shall write the parameters (a, b^μ) as a function of the hydrodynamic variables (T, μ, u^μ) .

For this purpose, let us introduce a Lorentz-scalar quantity given by

$$I \equiv \sum_p \frac{1}{p^0} f_p^{\text{EC}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} \exp(a - b_\mu p^\mu). \quad (\text{A.1.10})$$

In a frame of $b^\mu = (\sqrt{b^2}, 0, 0, 0)$, we can use spherical coordinates $d^3\mathbf{p} = |\mathbf{p}|^2 \sin\theta d|\mathbf{p}| d\theta d\varphi$ and change the variable of integration by introducing Θ such that $b_\mu p^\mu = \sqrt{b^2} p^0 = \sqrt{b^2} m \cosh\Theta$, $|\mathbf{p}|^2 = p_0^2 - m^2 = m^2 \sinh^2\Theta$, and $d|\mathbf{p}|/p^0 = d\Theta$. In this case, I in Eq. (A.1.10) reads

$$I = m^2 (2\pi)^{-3} 4\pi e^a \frac{K_1(m\sqrt{b^2})}{m\sqrt{b^2}} = I(a, \sqrt{b^2}), \quad (\text{A.1.11})$$

where $K_1(z)$ denotes the first-order modified Bessel function.

The particle current (A.1.7) reads

$$N^{(0)\mu} = -\frac{\partial}{\partial b_\mu} I = -\frac{\partial}{\partial \sqrt{b^2}} I(a, \sqrt{b^2}) \frac{b^\mu}{\sqrt{b^2}}. \quad (\text{A.1.12})$$

Now by combining Eqs. (A.1.2) and (A.1.12), we have that b^μ is given by

$$\frac{b^\mu}{\sqrt{b^2}} = u^\mu. \quad (\text{A.1.13})$$

Furthermore, the particle-number density n can be identified with

$$n = -\frac{\partial}{\partial \sqrt{b^2}} I(a, \sqrt{b^2}). \quad (\text{A.1.14})$$

A.2. DETAILED DERIVATION OF EXPLICIT FORM OF EXCITED MODES

Similarly, from Eqs. (A.1.8) and (A.1.9), the internal energy e and entropy density s are found to be

$$e = u_\mu u_\nu T^{(0)\mu\nu} = \frac{b_\mu}{\sqrt{b^2}} \frac{b_\nu}{\sqrt{b^2}} \frac{\partial^2}{\partial b_\mu \partial b_\nu} I = \frac{\partial^2}{\partial \sqrt{b^2}^2} I(a, \sqrt{b^2}), \quad (\text{A.1.15})$$

$$\begin{aligned} s &= u_\mu S^{(0)\mu} = \frac{b_\mu}{\sqrt{b^2}} \left((1-a) N^{(0)\mu} + b_\nu T^{(0)\nu\mu} \right) \\ &= (1-a) n(a, \sqrt{b^2}) + \sqrt{b^2} e(a, \sqrt{b^2}). \end{aligned} \quad (\text{A.1.16})$$

We note that n , e , and s have been obtained as a function of a and $\sqrt{b^2}$.

Through a straightforward manipulation with the use of the chain rules,

$$\frac{\partial}{\partial a} s(a, \sqrt{b^2}) = \frac{\partial}{\partial e} s(e, n) \frac{\partial}{\partial a} e(a, \sqrt{b^2}) + \frac{\partial}{\partial n} s(e, n) \frac{\partial}{\partial a} n(a, \sqrt{b^2}), \quad (\text{A.1.17})$$

$$\frac{\partial}{\partial \sqrt{b^2}} s(a, \sqrt{b^2}) = \frac{\partial}{\partial e} s(e, n) \frac{\partial}{\partial \sqrt{b^2}} e(a, \sqrt{b^2}) + \frac{\partial}{\partial n} s(e, n) \frac{\partial}{\partial \sqrt{b^2}} n(a, \sqrt{b^2}), \quad (\text{A.1.18})$$

we have

$$\frac{\partial}{\partial e} s(e, n) = \sqrt{b^2}, \quad (\text{A.1.19})$$

$$\frac{\partial}{\partial n} s(e, n) = -a. \quad (\text{A.1.20})$$

By comparing these equations with Eqs. (A.1.5), (A.1.6), (A.1.19), and (A.1.20), we get that

$$\sqrt{b^2} = \frac{1}{T}, \quad (\text{A.1.21})$$

$$a = \frac{\mu}{T}, \quad (\text{A.1.22})$$

and accordingly

$$b^\mu = \frac{w^\mu}{T}. \quad (\text{A.1.23})$$

Hence, we have identified a and b^μ , and f_p^{EC} in Eq. (A.1.1) can be written as the Jüttner function f_p^{eq} in Eq. (2.2.13).

A.2 Detailed derivation of explicit form of excited modes

In this section, we explicitly derive the first-excited modes $\varphi_{1p}^{\mu\alpha}$ as shown in Eq. (3.3.6).

For convenience, we introduce a dimensionless quantity [47] dependent on $z = m/T$,

$$a_\ell \equiv \frac{1}{n T^{\ell-1}} \sum_p \frac{1}{p^0} f_p^{\text{eq}} (p \cdot u)^\ell = \frac{1}{z^2 K_2(z)} \int_z^\infty d\tau (\tau^2 - z^2)^{1/2} \tau^\ell e^{-\tau}. \quad (\text{A.2.1})$$

This quantity for $\ell = 0, 1, 2, 3$ reads

$$a_0 = z^{-2} (\hat{h} - 4), \quad (\text{A.2.2})$$

$$a_1 = 1, \quad (\text{A.2.3})$$

$$a_2 = \hat{h} - 1, \quad (\text{A.2.4})$$

$$a_3 = 3\hat{h} + z^2, \quad (\text{A.2.5})$$

respectively. Here, $\hat{h}$ is defined in Eq. (3.3.11) and $K_\ell(z)$ denotes the modified Bessel function,

$$K_\ell(z) = \frac{2^\ell \ell!}{(2\ell)!} z^{-\ell} \int_z^\infty d\tau (\tau^2 - z^2)^{\ell-1/2} e^{-\tau}, \quad (\text{A.2.6})$$

which satisfies the recurrence relation

$$K_{\ell+1}(z) = K_{\ell-1}(z) + \frac{2\ell}{z} K_\ell(z). \quad (\text{A.2.7})$$

First, the metric matrix $\eta_0^{\alpha\beta}$ given by Eq. (3.2.26) reads

$$\eta_0^{\mu\nu} = n T^2 \left[a_3 u^\mu u^\nu + (z^2 a_1 - a_3) \frac{1}{3} \Delta^{\mu\nu} \right], \quad (\text{A.2.8})$$

$$\eta_0^{\mu 4} = \eta_0^{4\mu} = n T^2 z a_2 u^\mu, \quad (\text{A.2.9})$$

$$\eta_0^{44} = n T^2 z^2 a_1. \quad (\text{A.2.10})$$

By a straightforward manipulation, we have

$$\eta_{0\mu\nu}^{-1} = (n T^2)^{-1} \left[\frac{a_1}{a_3 a_1 - a_2^2} u_\mu u_\nu + \frac{3}{z^2 a_1 - a_3} \Delta_{\mu\nu} \right], \quad (\text{A.2.11})$$

$$\eta_{0\mu 4}^{-1} = \eta_{04\mu}^{-1} = (n T^2)^{-1} \left[-\frac{1}{z} \frac{a_2}{a_3 a_1 - a_2^2} u_\mu \right], \quad (\text{A.2.12})$$

$$\eta_{044}^{-1} = (n T^2)^{-1} \left[\frac{1}{z^2} \frac{a_3}{a_3 a_1 - a_2^2} \right]. \quad (\text{A.2.13})$$

Next, the inner product $\langle \varphi_0^\alpha, \tilde{\varphi}_1^{\mu\beta} \rangle$ is evaluated to be

$$\langle \varphi_0^a, \tilde{\varphi}_1^{\mu b} \rangle = n T^2 \left(a_3 u^a u^\mu u^b + (z^2 a_1 - a_3) \frac{1}{3} (u^a \Delta^{\mu b} + u^\mu \Delta^{ba} + u^b \Delta^{a\mu}) \right),$$

$$\langle \varphi_0^a, \tilde{\varphi}_1^{\mu 4} \rangle = n T^2 \left(z a_2 u^a u^\mu + z (z^2 a_0 - a_2) \frac{1}{3} \Delta^{a\mu} \right), \quad (\text{A.2.14})$$

$$\langle \varphi_0^4, \tilde{\varphi}_1^{\mu b} \rangle = n T^2 \left(z a_2 u^\mu u^b + z (z^2 a_0 - a_2) \frac{1}{3} \Delta^{\mu b} \right), \quad (\text{A.2.15})$$

$$\langle \varphi_0^4, \tilde{\varphi}_1^{\mu 4} \rangle = n T^2 z^2 a_1 u^\mu. \quad (\text{A.2.16})$$

A.3. SOLUTION TO LINEAR DIFFERENTIAL EQUATION WITH TIME-DEPENDENT INHOMOGENEOUS TERM

Finally, combining $\eta_{0\alpha\beta}^{-1}$ in Eqs. (A.2.11)-(A.2.13) and $\langle \varphi_0^\alpha, \tilde{\varphi}_1^{\mu\beta} \rangle$ in Eqs. (A.2.14)-(A.2.16) together with

$$\begin{aligned}\varphi_{1p}^{\mu\alpha} = [Q_0 \tilde{\varphi}_1^{\mu\alpha}]_p &= \tilde{\varphi}_{1p}^{\mu\alpha} - [P_0 \tilde{\varphi}_1^{\mu\alpha}]_p \\ &= \frac{1}{p \cdot u} \left(p^\mu \varphi_{0p}^\alpha - (p \cdot u) \varphi_{0p}^\beta \eta_{0\beta\gamma}^{-1} \langle \varphi_0^\gamma, \tilde{\varphi}_1^{\mu\alpha} \rangle \right),\end{aligned}\quad (\text{A.2.17})$$

we have

$$\varphi_{1p}^{\mu\alpha} = \begin{cases} \frac{1}{p \cdot u} (-\Delta^{\mu\nu} \Pi_p + \pi_p^{\mu\nu}), & \alpha = \mu, \\ \frac{1}{p \cdot u} \frac{(z^2 a_0 - a_2) z}{z^2 a_1 - a_3} J_p^\mu, & \alpha = 4. \end{cases} \quad (\text{A.2.18})$$

Here, we have introduced the following quantities

$$\Pi_p \equiv \frac{(a_2 a_0 - a_1^2) z^2 (p \cdot u)^2 - (a_3 a_0 - a_2 a_1) z m (p \cdot u) + (a_3 a_1 - a_2^2) m^2}{-3(a_3 a_1 - a_2^2)}, \quad (\text{A.2.19})$$

$$J_p^\mu \equiv \frac{\Delta^{\mu\nu} p_\nu \left[(z^2 a_0 - a_2) z (p \cdot u) - (z^2 a_1 - a_3) m \right]}{-(z^2 a_0 - a_2) z}, \quad (\text{A.2.20})$$

$$\pi_p^{\mu\nu} \equiv \Delta^{\mu\nu\rho\sigma} p_\rho p_\sigma. \quad (\text{A.2.21})$$

Substituting a_0 , a_1 , a_2 , and a_3 in Eqs. (A.2.2)-(A.2.5) into the above equations, we arrive at the explicit representation of $\varphi_{1p}^{\mu\alpha}$ shown in Eq. (3.3.6).

A.3 Solution to linear differential equation with time-dependent inhomogeneous term

In this section, we shall present formulae for obtaining appropriate special solutions of linear differential equations with a time-dependent inhomogeneous term. Let us consider the special solution of the equation given by

$$\frac{\partial}{\partial t} \mathbf{Y}(t) = A \mathbf{Y}(t) + \mathbf{R}(t - t_0). \quad (\text{A.3.1})$$

The solution reads

$$\begin{aligned}\mathbf{Y}(t) &= e^{A(t-t_0)} \mathbf{Y}(t_0) + \int_{t_0}^t ds e^{A(t-s)} \mathbf{R}(s - t_0) \\ &= e^{A(t-t_0)} \mathbf{Y}(t_0) + \int_{t_0}^t ds P_0 \mathbf{R}(s - t_0) + \int_{t_0}^t ds e^{A(t-s)} Q_0 \mathbf{R}(s - t_0),\end{aligned}\quad (\text{A.3.2})$$

where we have inserted $1 = P_0 + Q_0$ in front of $\mathbf{R}(s - t_0)$. Substituting the following Taylor expansion,

$$\mathbf{R}(s - t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} (s - t_0)^n \left[\frac{\partial}{\partial \tau} \right]^n \mathbf{R}(\tau) \Big|_{\tau=0} = e^{(s-t_0)\partial/\partial\tau} \mathbf{R}(\tau) \Big|_{\tau=0}, \quad (\text{A.3.3})$$

into Eq. (A.3.2) and carrying out integration with respect to s , we have

$$\begin{aligned} \mathbf{Y}(t) &= e^{A(t-t_0)} \mathbf{Y}(t_0) + (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &\quad + (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial\tau}) (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &= e^{A(t-t_0)} \mathbf{Y}(t_0) + (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &\quad + (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial\tau}) (P_1 + Q_1) (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &= e^{A(t-t_0)} \left[\mathbf{Y}(t_0) + Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0} \right] \\ &\quad + (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &\quad + (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial\tau}) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &\quad - e^{(t-t_0)\partial/\partial\tau} Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0}. \end{aligned} \quad (\text{A.3.4})$$

We note that the contributions from the inhomogeneous term $\mathbf{R}(t - t_0)$ are decomposed into two parts, whose time dependencies are given by $e^{A(t-t_0)}$ and $e^{(t-t_0)\partial/\partial\tau}$, respectively. The former shows a fast motion according to the vector spaces operated by itself, while the time dependence of the latter is universal due to the absence of A . Since we are interested in the motion coming from the P_0 and P_1 spaces, we eliminate the former associated with the Q_1 space with the use of the initial value $\mathbf{Y}(t_0)$ that has not yet specified as follows:

$$\mathbf{Y}(t_0) = -Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0}, \quad (\text{A.3.5})$$

which reduces Eq. (A.3.4) to

$$\begin{aligned} \mathbf{Y}(t) &= (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &\quad + (e^{A(t-t_0)} - e^{(t-t_0)\partial/\partial\tau}) P_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0} \\ &\quad - e^{(t-t_0)\partial/\partial\tau} Q_1 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{R}(\tau) \Big|_{\tau=0}. \end{aligned} \quad (\text{A.3.6})$$

Equations (A.3.5) and (A.3.6) are nothing but the formulae we wanted.

A.4 Slow dynamics as described with would-be zero modes

In this section, we shall derive the slow dynamics obtained by employing only the zero modes from the generic evolution equation (4.1.2) with the RG method for completeness, although a detailed derivation can be seen in Ref. [86, 89].

A.4. SLOW DYNAMICS AS DESCRIBED WITH WOULD-BE ZERO MODES

As mentioned in Sec. 4.1, we first try to obtain the perturbative solution $\tilde{\mathbf{X}}$ to Eq. (4.1.2) around an arbitrary initial time $t = t_0$ with the initial value $\mathbf{X}(t_0)$; $\tilde{\mathbf{X}}(t = t_0; t_0) = \mathbf{X}(t_0)$. We expand the initial value as well as the solution with respect to ϵ as shown in Eqs. (4.1.15) and (4.1.16), and obtain the series of the perturbative equations with respect to ϵ .

The zeroth-order equation is the same as Eq. (4.1.18). Since we are interested in the slow motion realized asymptotically for $t \rightarrow \infty$, we adopt the stationary solution $\mathbf{X}_{\text{st}}$ as the zeroth-order solution:

$$\tilde{\mathbf{X}}_0(t; t_0) = \mathbf{X}_{\text{st}}, \quad (\text{A.4.1})$$

which means that the zeroth-order initial value reads

$$\mathbf{X}_0(t_0) = \tilde{\mathbf{X}}_0(t_0; t_0) = \mathbf{X}_{\text{st}}. \quad (\text{A.4.2})$$

The first-order equation is

$$\frac{\partial}{\partial t} \tilde{\mathbf{X}}_1(t) = A \tilde{\mathbf{X}}_1(t) + \mathbf{F}_0, \quad (\text{A.4.3})$$

where A and $\mathbf{F}_0$ have been defined in Eqs. (4.1.6) and (4.1.26), respectively. A special solution to Eq. (A.4.3) reads

$$\begin{aligned} \tilde{\mathbf{X}}_1(t) &= e^{A(t-t_0)} \tilde{\mathbf{X}}_1(t_0) + \int_{t_0}^t ds e^{A(t-s)} \mathbf{F}_0 \\ &= e^{A(t-t_0)} \tilde{\mathbf{X}}_1(t_0) + \int_{t_0}^t ds P_0 \mathbf{F}_0 + \int_{t_0}^t ds e^{A(t-s)} Q_0 \mathbf{F}_0 \\ &= e^{A(t-t_0)} \left[\tilde{\mathbf{X}}_1(t_0) + A^{-1} Q_0 \mathbf{F}_0 \right] + (t - t_0) P_0 \mathbf{F}_0 - A^{-1} Q_0 \mathbf{F}_0, \end{aligned} \quad (\text{A.4.4})$$

where $1 = P_0 + Q_0$ has been inserted in front of $\mathbf{F}_0$. Here, P_0 denotes the projection operator onto the P_0 space spanned by the zero modes $\boldsymbol{\varphi}^{(\alpha)}$, i.e., the kernel space of A , and Q_0 the projection operator onto the Q_0 space as the complement to the P_0 space.

Since we are interested in the slow motion caused by the P_0 space, we eliminate the fast motion coming from the Q_0 space with the use of the initial value $\tilde{\mathbf{X}}_1(t_0)$ that has not yet specified as follows:

$$\mathbf{X}_1(t_0) = \tilde{\mathbf{X}}_1(t_0; t_0) = -A^{-1} Q_0 \mathbf{F}_0, \quad (\text{A.4.5})$$

which reduces Eq. (A.4.4) to

$$\tilde{\mathbf{X}}_1(t; t_0) = (t - t_0) P_0 \mathbf{F}_0 - A^{-1} Q_0 \mathbf{F}_0. \quad (\text{A.4.6})$$

The second-order equation is

$$\frac{\partial}{\partial t} \tilde{\mathbf{X}}_2(t) = A \tilde{\mathbf{X}}_2(t) + \mathbf{U}(t - t_0), \quad (\text{A.4.7})$$

with

$$\mathbf{U}(\tau) \equiv \frac{1}{2} B \left[\tau P_0 \mathbf{F}_0 - A^{-1} Q_0 \mathbf{F}_0 \right]^2 + F_1 \left[\tau P_0 \mathbf{F}_0 - A^{-1} Q_0 \mathbf{F}_0 \right], \quad (\text{A.4.8})$$

where B and F_1 have been defined in Eqs. (4.1.45) and (4.1.46), respectively. With the use of the method developed in Appendix A.3, we have a special solution to Eq. (A.4.7) as

$$\begin{aligned} \tilde{\mathbf{X}}_2(t) &= e^{A(t-t_0)} \left[\tilde{\mathbf{X}}_2(t_0) + (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0} \right] \\ &\quad + (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{U}(\tau) \Big|_{\tau=0} \\ &\quad - e^{(t-t_0)\partial/\partial\tau} (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0}. \end{aligned} \quad (\text{A.4.9})$$

As the case of the first order, we shall eliminate the fast motion caused by the Q_0 space with the use of the initial value $\tilde{\mathbf{X}}_2(t_0)$ as

$$\mathbf{X}_2(t_0) = \tilde{\mathbf{X}}_2(t_0; t_0) = -(A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0}, \quad (\text{A.4.10})$$

which leads to

$$\begin{aligned} \tilde{\mathbf{X}}_2(t; t_0) &= (1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{U}(\tau) \Big|_{\tau=0} \\ &\quad - e^{(t-t_0)\partial/\partial\tau} (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0}. \end{aligned} \quad (\text{A.4.11})$$

Summing up the solutions and initial values constructed in the perturbative analysis up to $O(\epsilon^2)$, we have

$$\mathbf{X}(t_0) = \mathbf{X}_{\text{st}} - \epsilon A^{-1} Q_0 \mathbf{F}_0 - \epsilon^2 (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0} + O(\epsilon^3), \quad (\text{A.4.12})$$

and

$$\begin{aligned} \tilde{\mathbf{X}}(t; t_0) &= \mathbf{X}_{\text{st}} + \epsilon \left[(t - t_0) P_0 \mathbf{F}_0 - A^{-1} Q_0 \mathbf{F}_0 \right] \\ &\quad + \epsilon^2 \left[(1 - e^{(t-t_0)\partial/\partial\tau}) (-\partial/\partial\tau)^{-1} P_0 \mathbf{U}(\tau) \Big|_{\tau=0} \right. \\ &\quad \left. - e^{(t-t_0)\partial/\partial\tau} (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0} \right] + O(\epsilon^3). \end{aligned} \quad (\text{A.4.13})$$

We note the appearance of the secular term proportional to $t - t_0$, which invalidates the perturbative solution when $|t - t_0|$ becomes large. For obtaining the global solution from this local perturbative solution, we apply the RG equation $d\tilde{\mathbf{X}}_1(t; t_0)/dt_0|_{t_0=t} = 0$ to Eq. (A.4.13): The RG equation reads

$$\begin{aligned} &\frac{\partial}{\partial t} (\mathbf{X}_{\text{st}} - \epsilon A^{-1} Q_0 \mathbf{F}_0) - \epsilon P_0 \mathbf{F}_0 \\ &+ \epsilon^2 \left[P_0 \mathbf{U}(0) + (-\partial/\partial\tau) (A - \partial/\partial\tau)^{-1} Q_0 \mathbf{U}(\tau) \Big|_{\tau=0} \right] + O(\epsilon^3) = 0, \end{aligned} \quad (\text{A.4.14})$$

A.5. DETAILED DERIVATION OF RELAXATION EQUATION IN FOURTEEN-MOMENT EQUATION IN ENERGY FRAME

which is the equation governing the slow motion of $C_{(\alpha)}$ in $\mathbf{X}_{\text{st}}$. By taking the inner product with the zero modes $\varphi_0^{(\alpha)}$, we can convert Eq. (A.4.14) into a simpler equation, which is found to be identically Eq. (4.1.68). Here, we have used

$$\mathbf{U}(0) = \frac{1}{2} B \left[-A^{-1} Q_0 \mathbf{F}_0 \right]^2 + F_1 \left[-A^{-1} Q_0 \mathbf{F}_0 \right], \quad (\text{A.4.15})$$

which can be derived from Eq. (A.4.8).

A.5 Detailed derivation of relaxation equation in fourteen-moment equation in energy frame

In this section, we present a detailed derivation of the relaxation equation given by Eqs. (4.2.49)-(4.2.51).

Using Eq. (C.3.29) and (4.2.40) together with the thermodynamic forces X_Π , X_J^μ , and $X_\pi^{\mu\nu}$ defined in Eqs. (3.3.23)-(3.3.25), we convert Eq. (4.2.25) into the followings:

$$\begin{aligned} & \epsilon \sum_p \frac{1}{p^0} [L^{-1} \hat{\Pi}]_p (p \cdot u) v_p^\alpha D_\alpha \left[f_p^{\text{eq}} \frac{1}{T} \left([L^{-1} \hat{\Pi}]_p \Pi \right. \right. \\ & \quad \left. \left. + [L^{-1} \hat{J}^\rho]_p J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_p \pi_{\rho\sigma} \right) \right] \\ = & \epsilon \zeta (\Pi - X_\Pi) \\ & + \epsilon^2 \frac{1}{2} \langle L^{-1} \hat{\Pi}, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} \frac{1}{T} \left(L^{-1} \hat{\Pi} \Pi + L^{-1} \hat{J}^\rho J_\rho + L^{-1} \hat{\pi}^{\rho\sigma} \pi_{\rho\sigma} \right) \right]^2 \rangle_{\text{eq}} \\ & + O(\epsilon^3), \end{aligned} \quad (\text{A.5.1})$$

$$\begin{aligned} & \epsilon \sum_p \frac{1}{p^0} [L^{-1} \hat{J}^\mu]_p (p \cdot u) v_p^\alpha D_\alpha \left[f_p^{\text{eq}} \frac{1}{T} \left([L^{-1} \hat{\Pi}]_p \Pi \right. \right. \\ & \quad \left. \left. + [L^{-1} \hat{J}^\rho]_p J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_p \pi_{\rho\sigma} \right) \right] \\ = & \epsilon T \lambda (J^\mu - X_J^\mu) \\ & + \epsilon^2 \frac{1}{2} \langle L^{-1} \hat{J}^\mu, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} \frac{1}{T} \left(L^{-1} \hat{\Pi} \Pi + L^{-1} \hat{J}^\rho J_\rho + L^{-1} \hat{\pi}^{\rho\sigma} \pi_{\rho\sigma} \right) \right]^2 \rangle_{\text{eq}} \\ & + O(\epsilon^3), \end{aligned} \quad (\text{A.5.2})$$

$$\begin{aligned} & \epsilon \sum_p \frac{1}{p^0} [L^{-1} \hat{\pi}^{\mu\nu}]_p (p \cdot u) v_p^\alpha D_\alpha \left[f_p^{\text{eq}} \frac{1}{T} \left([L^{-1} \hat{\Pi}]_p \Pi \right. \right. \\ & \quad \left. \left. + [L^{-1} \hat{J}^\rho]_p J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_p \pi_{\rho\sigma} \right) \right] \\ = & \epsilon 2 \eta (\pi^{\mu\nu} - X_\pi^{\mu\nu}) \\ & + \epsilon^2 \frac{1}{2} \langle L^{-1} \hat{\pi}^{\mu\nu}, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} \frac{1}{T} \left(L^{-1} \hat{\Pi} \Pi + L^{-1} \hat{J}^\rho J_\rho + L^{-1} \hat{\pi}^{\rho\sigma} \pi_{\rho\sigma} \right) \right]^2 \rangle_{\text{eq}} \\ & + O(\epsilon^3). \end{aligned} \quad (\text{A.5.3})$$

Here, we have introduced the differential operator given by

$$(p \cdot u) \frac{\partial}{\partial \tau} + \epsilon p \cdot \nabla = (p \cdot u) v_p^\alpha D_\alpha, \quad (\text{A.5.4})$$

where

$$v_p^\alpha \equiv \begin{cases} \Delta^{\mu\nu} p_\nu \frac{1}{p \cdot u}, & \alpha = \mu, \\ 1, & \alpha = 4, \end{cases} \quad (\text{A.5.5})$$

$$D_\alpha \equiv \begin{cases} \epsilon \nabla_\mu, & \alpha = \mu, \\ \frac{\partial}{\partial \tau}, & \alpha = 4. \end{cases} \quad (\text{A.5.6})$$

From now on, we shall examine the left-hand sides of Eqs. (A.5.1)-(A.5.3). First, with the use of $\hat{\psi}_p = (\hat{\Pi}_p, \hat{J}_p^\mu, \hat{\pi}_p^{\mu\nu})$, we have

$$\begin{aligned} & \sum_p \frac{1}{p^0} [L^{-1} \hat{\psi}]_p (p \cdot u) v_p^\alpha D_\alpha \left[f_p^{\text{eq}} \frac{1}{T} \left([L^{-1} \hat{\Pi}]_p \Pi + [L^{-1} \hat{J}^\rho]_p J_\rho + [L^{-1} \hat{\pi}^{\rho\sigma}]_p J_{\rho\sigma} \right) \right] \\ = & \frac{1}{T} \langle L^{-1} \hat{\psi}, v^\alpha L^{-1} \hat{\Pi} \rangle_{\text{eq}} D_\alpha \Pi + \frac{1}{T} \langle L^{-1} \hat{\psi}, v^\alpha L^{-1} \hat{J}^\rho \rangle_{\text{eq}} D_\alpha J_\rho \\ & + \frac{1}{T} \langle L^{-1} \hat{\psi}, v^\alpha L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} D_\alpha \pi_{\rho\sigma} \\ & + \langle L^{-1} \hat{\psi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi}/T) \rangle_{\text{eq}} \Pi \\ & + \langle L^{-1} \hat{\psi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho/T) \rangle_{\text{eq}} J_\rho \\ & + \langle L^{-1} \hat{\psi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma}/T) \rangle_{\text{eq}} \pi_{\rho\sigma}. \end{aligned} \quad (\text{A.5.7})$$

The terms containing the derivatives of Π , J^μ , and $\pi^{\mu\nu}$ can be reduced to

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{\Pi}, v^\alpha L^{-1} \hat{\Pi} \rangle_{\text{eq}} D_\alpha \Pi &= \frac{1}{T} \langle L^{-1} \hat{\Pi}, L^{-1} \hat{\Pi} \rangle_{\text{eq}} \frac{\partial}{\partial \tau} \Pi \\ &= -\zeta \tau_\Pi \frac{\partial}{\partial \tau} \Pi, \end{aligned} \quad (\text{A.5.8})$$

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{\Pi}, v^\alpha L^{-1} \hat{J}^\rho \rangle_{\text{eq}} D_\alpha J_\rho &= \frac{1}{T} \langle L^{-1} \hat{\Pi}, v^\lambda L^{-1} \hat{J}^\rho \rangle_{\text{eq}} \epsilon \nabla_\lambda J_\rho \\ &= -\zeta \ell_{\Pi J} \Delta^{\lambda\rho} \epsilon \nabla_\lambda J_\rho, \end{aligned} \quad (\text{A.5.9})$$

$$\frac{1}{T} \langle L^{-1} \hat{\Pi}, v^\alpha L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} D_\alpha \pi_{\rho\sigma} = 0, \quad (\text{A.5.10})$$

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{J}^\mu, v^\alpha L^{-1} \hat{\Pi} \rangle_{\text{eq}} D_\alpha \Pi &= \frac{1}{T} \langle L^{-1} \hat{J}^\mu, v^\lambda L^{-1} \hat{\Pi} \rangle_{\text{eq}} \epsilon \nabla_\lambda \Pi \\ &= -T \lambda \ell_{J\Pi} \Delta^{\mu\lambda} \epsilon \nabla_\lambda \Pi, \end{aligned} \quad (\text{A.5.11})$$

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{J}^\mu, v^\alpha L^{-1} \hat{J}^\rho \rangle_{\text{eq}} D_\alpha J_\rho &= \frac{1}{T} \langle L^{-1} \hat{J}^\mu, L^{-1} \hat{J}^\rho \rangle_{\text{eq}} \frac{\partial}{\partial \tau} J_\rho \\ &= -T \lambda \tau_J \Delta^{\mu\rho} \frac{\partial}{\partial \tau} J_\rho, \end{aligned} \quad (\text{A.5.12})$$

A.5. DETAILED DERIVATION OF RELAXATION EQUATION IN FOURTEEN-MOMENT EQUATION IN ENERGY FRAME

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{J}^\mu, v^\alpha L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} D_\alpha \pi_{\rho\sigma} &= \frac{1}{T} \langle L^{-1} \hat{J}^\mu, v^\lambda L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} \epsilon \nabla_\lambda \pi_{\rho\sigma} \\ &= -T \lambda \ell_{J\pi} \Delta^{\mu\lambda\rho\sigma} \epsilon \nabla_\lambda \pi_{\rho\sigma}, \end{aligned} \quad (\text{A.5.13})$$

$$\frac{1}{T} \langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha L^{-1} \hat{\Pi} \rangle_{\text{eq}} D_\alpha \Pi = 0, \quad (\text{A.5.14})$$

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha L^{-1} \hat{J}^\rho \rangle_{\text{eq}} D_\alpha J_\rho &= \frac{1}{T} \langle L^{-1} \hat{\pi}^{\mu\nu}, v^\lambda L^{-1} \hat{J}^\rho \rangle_{\text{eq}} \epsilon \nabla_\lambda J_\rho \\ &= -2 \eta \ell_{\pi J} \Delta^{\mu\nu\lambda\rho} \epsilon \nabla_\lambda J_\rho, \end{aligned} \quad (\text{A.5.15})$$

$$\begin{aligned} \frac{1}{T} \langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} D_\alpha \pi_{\rho\sigma} &= \frac{1}{T} \langle L^{-1} \hat{\pi}^{\mu\nu}, L^{-1} \hat{\pi}^{\rho\sigma} \rangle_{\text{eq}} \frac{\partial}{\partial \tau} \pi_{\rho\sigma} \\ &= -2 \eta \tau_\pi \Delta^{\mu\nu\rho\sigma} \frac{\partial}{\partial \tau} \pi_{\rho\sigma}. \end{aligned} \quad (\text{A.5.16})$$

We note that τ_Π , τ_J , and τ_π in Eqs. (A.5.8)-(A.5.16) coincide with the relaxation times given by Eqs. (4.2.52)-(4.2.54), while $\ell_{\Pi J}$, $\ell_{J\Pi}$, $\ell_{J\pi}$, and $\ell_{\pi J}$ can be interpreted as the so-called relaxation lengths. The definitions of these quantities are given by

$$\tau_\Pi \equiv -\frac{\langle L^{-1} \hat{\Pi}, L^{-1} \hat{\Pi} \rangle_{\text{eq}}}{T \zeta} = \frac{\langle \hat{\Pi}, L^{-2} \hat{\Pi} \rangle_{\text{eq}}}{\langle \hat{\Pi}, L^{-1} \hat{\Pi} \rangle_{\text{eq}}}, \quad (\text{A.5.17})$$

$$\tau_J \equiv -\frac{\langle L^{-1} \hat{J}^\mu, L^{-1} \hat{J}_\mu \rangle_{\text{eq}}}{3 T^2 \lambda} = \frac{\langle \hat{J}^\mu, L^{-2} \hat{J}_\mu \rangle_{\text{eq}}}{\langle \hat{J}^\rho, L^{-1} \hat{J}_\rho \rangle_{\text{eq}}}, \quad (\text{A.5.18})$$

$$\tau_\pi \equiv -\frac{\langle L^{-1} \hat{\pi}^{\mu\nu}, L^{-1} \hat{\pi}_{\mu\nu} \rangle_{\text{eq}}}{10 T \eta} = \frac{\langle \hat{\pi}^{\mu\nu}, L^{-2} \hat{\pi}_{\mu\nu} \rangle_{\text{eq}}}{\langle \hat{\pi}^{\rho\sigma}, L^{-1} \hat{\pi}_{\rho\sigma} \rangle_{\text{eq}}}, \quad (\text{A.5.19})$$

$$\ell_{\Pi J} \equiv -\frac{\langle L^{-1} \hat{\Pi}, v^\mu L^{-1} \hat{J}_\mu \rangle_{\text{eq}}}{3 T \zeta} = \frac{1}{3} \frac{\langle \hat{\Pi}, L^{-1} v^\mu L^{-1} \hat{J}_\mu \rangle_{\text{eq}}}{\langle \hat{\Pi}, L^{-1} \hat{\Pi} \rangle_{\text{eq}}}, \quad (\text{A.5.20})$$

$$\ell_{J\Pi} \equiv -\frac{\langle L^{-1} \hat{J}^\mu, v_\mu L^{-1} \hat{\Pi} \rangle_{\text{eq}}}{3 T^2 \lambda} = \frac{\langle \hat{J}^\mu, L^{-1} v_\mu L^{-1} \hat{\Pi} \rangle_{\text{eq}}}{\langle \hat{J}^\rho, L^{-1} \hat{J}_\rho \rangle_{\text{eq}}}, \quad (\text{A.5.21})$$

$$\ell_{J\pi} \equiv -\frac{\langle L^{-1} \hat{J}^\mu, v^\nu L^{-1} \hat{\pi}_{\mu\nu} \rangle_{\text{eq}}}{5 T^2 \lambda} = \frac{3}{5} \frac{\langle \hat{J}^\mu, L^{-1} v^\nu L^{-1} \hat{\pi}_{\mu\nu} \rangle_{\text{eq}}}{\langle \hat{J}^\rho, L^{-1} \hat{J}_\rho \rangle_{\text{eq}}}, \quad (\text{A.5.22})$$

$$\ell_{\pi J} \equiv -\frac{\langle L^{-1} \hat{\pi}^{\mu\nu}, v_\mu L^{-1} \hat{J}_\nu \rangle_{\text{eq}}}{10 T \eta} = \frac{\langle \hat{\pi}^{\mu\nu}, L^{-1} v_\mu L^{-1} \hat{J}_\nu \rangle_{\text{eq}}}{\langle \hat{\pi}^{\rho\sigma}, L^{-1} \hat{\pi}_{\rho\sigma} \rangle_{\text{eq}}}. \quad (\text{A.5.23})$$

Then, we consider the other terms in the right-hand side of Eq. (A.5.7). We notice the following power counting with respect to ϵ :

$$\langle L^{-1} \hat{\Pi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi}/T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.24})$$

$$\langle L^{-1} \hat{\Pi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho/T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.25})$$

$$\langle L^{-1} \hat{\Pi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma}/T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.26})$$

$$\langle L^{-1} \hat{J}^\mu, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi}/T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.27})$$

$$\langle L^{-1} \hat{J}^\mu, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho / T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.28})$$

$$\langle L^{-1} \hat{J}^\mu, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma} / T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.29})$$

$$\langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi} / T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.30})$$

$$\langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho / T) \rangle_{\text{eq}} \sim O(\epsilon), \quad (\text{A.5.31})$$

$$\langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma} / T) \rangle_{\text{eq}} \sim O(\epsilon). \quad (\text{A.5.32})$$

This counting can be derived from the fact that the above terms contain the covariant temporal and spatial first-order derivatives of μ , T , and u^μ : The covariant temporal derivatives can be converted into the covariant spatial derivatives with the use of the continuity equation (4.2.31), and there exists ϵ in front of the covariant spatial derivatives. Accordingly, we can represent the above terms as the quantities of $O(\epsilon)$,

$$\langle L^{-1} \hat{\Pi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi} / T) \rangle_{\text{eq}} \equiv -\epsilon \zeta X_{\Pi\Pi}, \quad (\text{A.5.33})$$

$$\langle L^{-1} \hat{\Pi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho / T) \rangle_{\text{eq}} \equiv -\epsilon \zeta X_{\Pi J}^\rho, \quad (\text{A.5.34})$$

$$\langle L^{-1} \hat{\Pi}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma} / T) \rangle_{\text{eq}} \equiv -\epsilon \zeta X_{\Pi\pi}^{\rho\sigma}, \quad (\text{A.5.35})$$

$$\langle L^{-1} \hat{J}^\mu, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi} / T) \rangle_{\text{eq}} \equiv -\epsilon T \lambda X_{J\Pi}^\mu, \quad (\text{A.5.36})$$

$$\langle L^{-1} \hat{J}^\mu, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho / T) \rangle_{\text{eq}} \equiv -\epsilon T \lambda X_{JJ}^{\mu\rho}, \quad (\text{A.5.37})$$

$$\langle L^{-1} \hat{J}^\mu, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma} / T) \rangle_{\text{eq}} \equiv -\epsilon T \lambda X_{J\pi}^{\mu\rho\sigma}, \quad (\text{A.5.38})$$

$$\langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\Pi} / T) \rangle_{\text{eq}} \equiv -\epsilon 2 \eta X_{\Pi\Pi}^{\mu\nu}, \quad (\text{A.5.39})$$

$$\langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{J}^\rho / T) \rangle_{\text{eq}} \equiv -\epsilon 2 \eta X_{J\pi}^{\mu\nu\rho}, \quad (\text{A.5.40})$$

$$\langle L^{-1} \hat{\pi}^{\mu\nu}, v^\alpha (f^{\text{eq}})^{-1} D_\alpha (f^{\text{eq}} L^{-1} \hat{\pi}^{\rho\sigma} / T) \rangle_{\text{eq}} \equiv -\epsilon 2 \eta X_{\pi\pi}^{\mu\nu\rho\sigma}. \quad (\text{A.5.41})$$

It is noteworthy that $X_{\Pi\Pi}$, $X_{\Pi J}^\rho$, $X_{\Pi\pi}^{\rho\sigma}$, $X_{J\Pi}^\mu$, $X_{JJ}^{\mu\rho}$, $X_{J\pi}^{\mu\rho\sigma}$, $X_{\Pi\Pi}^{\mu\nu}$, $X_{J\pi}^{\mu\nu\rho}$, and $X_{\pi\pi}^{\mu\nu\rho\sigma}$, might be called also the thermodynamic forces as well as X_Π , X_J^μ , and $X_\pi^{\mu\nu}$, which are found to contain the vorticity term [50, 112].

Then, we shall examine the second terms in the right-hand side of Eqs. (A.5.1)-(A.5.3), which read

$$\begin{aligned} & \frac{1}{2} \langle L^{-1} \hat{\Pi}, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} \frac{1}{T} \left(L^{-1} \hat{\Pi} \Pi + L^{-1} \hat{J}^\rho J_\rho + L^{-1} \hat{\pi}^{\rho\sigma} \pi_{\rho\sigma} \right) \right]^2 \rangle_{\text{eq}} \\ &= -\zeta (b_{\Pi\Pi\Pi} \Pi \Pi + b_{\Pi J J} J^\mu J_\mu + b_{\Pi\pi\pi} \pi^{\mu\nu} \pi_{\mu\nu}), \end{aligned} \quad (\text{A.5.42})$$

$$\begin{aligned} & \frac{1}{2} \langle L^{-1} \hat{J}^\mu, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} \frac{1}{T} \left(L^{-1} \hat{\Pi} \Pi + L^{-1} \hat{J}^\rho J_\rho + L^{-1} \hat{\pi}^{\rho\sigma} \pi_{\rho\sigma} \right) \right]^2 \rangle_{\text{eq}} \\ &= -T \lambda (b_{J\Pi J} \Pi J^\mu + b_{J J \pi} J_\nu \pi^{\nu\mu}), \end{aligned} \quad (\text{A.5.43})$$

$$\begin{aligned} & \frac{1}{2} \langle L^{-1} \hat{\pi}^{\mu\nu}, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} \frac{1}{T} \left(L^{-1} \hat{\Pi} \Pi + L^{-1} \hat{J}^\rho J_\rho + L^{-1} \hat{\pi}^{\rho\sigma} \pi_{\rho\sigma} \right) \right]^2 \rangle_{\text{eq}} \\ &= -2 \eta (b_{\pi\Pi\pi} \Pi \pi^{\mu\nu} + b_{\pi J J} \Delta^{\mu\nu\rho\sigma} J_\rho J_\sigma). \end{aligned} \quad (\text{A.5.44})$$

We note that $b_{\Pi\Pi\Pi}$, $b_{\Pi J J}$, $b_{\Pi\pi\pi}$, $b_{J\Pi J}$, $b_{J J \pi}$, $b_{\pi\Pi\pi}$, and $b_{\pi J J}$ denote the coefficients in the non-linear terms of Π , J^μ , and $\pi^{\mu\nu}$, whose definitions are given as

$$b_{\Pi\Pi\Pi} \equiv -\frac{1}{2T^2\zeta} \langle L^{-1} \hat{\Pi}, (f^{\text{eq}})^{-1} B \left[f^{\text{eq}} L^{-1} \hat{\Pi} \right] \left[f^{\text{eq}} L^{-1} \hat{\Pi} \right] \rangle_{\text{eq}}, \quad (\text{A.5.45})$$

A.5. DETAILED DERIVATION OF RELAXATION EQUATION IN FOURTEEN-MOMENT EQUATION IN ENERGY FRAME

$$b_{\Pi JJ} \equiv -\frac{1}{6T^2\zeta} \langle L^{-1} \hat{\Pi}, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \hat{J}^\mu] [f^{\text{eq}} L^{-1} \hat{J}_\mu] \rangle_{\text{eq}}, \quad (\text{A.5.46})$$

$$b_{\Pi\pi\pi} \equiv -\frac{1}{10T^2\zeta} \langle L^{-1} \hat{\Pi}, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \hat{\pi}^{\mu\nu}] [f^{\text{eq}} L^{-1} \hat{\pi}_{\mu\nu}] \rangle_{\text{eq}}, \quad (\text{A.5.47})$$

$$b_{J\Pi J} \equiv -\frac{1}{3T^3\lambda} \langle L^{-1} \hat{J}^\mu, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \hat{\Pi}] [f^{\text{eq}} L^{-1} \hat{J}_\mu] \rangle_{\text{eq}}, \quad (\text{A.5.48})$$

$$b_{JJ\pi} \equiv -\frac{1}{5T^3\lambda} \langle L^{-1} \hat{J}^\mu, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \hat{J}^\nu] [f^{\text{eq}} L^{-1} \hat{\pi}_{\mu\nu}] \rangle_{\text{eq}}, \quad (\text{A.5.49})$$

$$b_{\pi\Pi\pi} \equiv -\frac{1}{10T^2\eta} \langle L^{-1} \hat{\pi}^{\mu\nu}, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \hat{\Pi}] [f^{\text{eq}} L^{-1} \hat{\pi}_{\mu\nu}] \rangle_{\text{eq}}, \quad (\text{A.5.50})$$

$$b_{\pi JJ} \equiv -\frac{1}{20T^2\eta} \langle L^{-1} \hat{\pi}^{\mu\nu}, (f^{\text{eq}})^{-1} B [f^{\text{eq}} L^{-1} \hat{J}_\mu] [f^{\text{eq}} L^{-1} \hat{J}_\nu] \rangle_{\text{eq}}. \quad (\text{A.5.51})$$

We note that the appearance of these non-linear terms is natural in the viewpoint of the order counting with respect to ϵ but these terms have never appeared in the existing literature.

Substituting the above equations into Eqs. (A.5.1)-(A.5.3), we have the relaxation equations as

$$\begin{aligned} & \epsilon (\Pi - X_\Pi) + \epsilon \tau_\Pi \frac{\partial}{\partial \tau} \Pi + \epsilon^2 \ell_{\Pi J} \Delta^{\lambda\rho} \nabla_\lambda J_\rho \\ &= \epsilon^2 (X_{\Pi\Pi} \Pi + X_{\Pi J}^\rho J_\rho + X_{\Pi\pi}^{\rho\sigma} \pi_{\rho\sigma}) \\ & \quad + \epsilon^2 (b_{\Pi\Pi\Pi} \Pi \Pi + b_{\Pi JJ} J^\mu J_\mu + b_{\Pi\pi\pi} \pi^{\mu\nu} \pi_{\mu\nu}) + O(\epsilon^3), \end{aligned} \quad (\text{A.5.52})$$

$$\begin{aligned} & \epsilon (J^\mu - X_J^\mu) + \epsilon^2 \ell_{J\Pi} \Delta^{\mu\lambda} \nabla_\lambda \Pi + \epsilon \tau_J \Delta^{\mu\rho} \frac{\partial}{\partial \tau} J_\rho + \epsilon^2 \ell_{J\pi} \Delta^{\mu\lambda\rho\sigma} \nabla_\lambda \pi_{\rho\sigma} \\ &= \epsilon^2 (X_{J\Pi}^\mu \Pi + X_{JJ}^{\mu\rho} J_\rho + X_{J\pi}^{\mu\rho\sigma} \pi_{\rho\sigma}) \\ & \quad + \epsilon^2 (b_{J\Pi J} \Pi J^\mu + b_{JJ\pi} J_\nu \pi^{\nu\mu}) + O(\epsilon^3), \end{aligned} \quad (\text{A.5.53})$$

$$\begin{aligned} & \epsilon (\pi^{\mu\nu} - X_\pi^{\mu\nu}) + \epsilon^2 \ell_{\pi J} \Delta^{\mu\nu\lambda\rho} \nabla_\lambda J_\rho + \epsilon \tau_\pi \Delta^{\mu\nu\rho\sigma} \frac{\partial}{\partial \tau} \pi_{\rho\sigma} \\ &= \epsilon^2 (X_{\pi\Pi}^{\mu\nu} \Pi + X_{\pi J}^{\mu\nu\rho} J_\rho + X_{\pi\pi}^{\mu\nu\rho\sigma} \pi_{\rho\sigma}) \\ & \quad + \epsilon^2 (b_{\pi\Pi\pi} \Pi \pi^{\mu\nu} + b_{\pi JJ} \Delta^{\mu\nu\rho\sigma} J_\rho J_\sigma) + O(\epsilon^3). \end{aligned} \quad (\text{A.5.54})$$

By eliminating the second-order terms with respect to ϵ , we reduce the above equations to

$$\epsilon \Pi + \epsilon \frac{\partial}{\partial \tau} \Pi = \epsilon X_\Pi + O(\epsilon^2), \quad (\text{A.5.55})$$

$$\epsilon J^\mu + \epsilon \Delta^{\mu\rho} \frac{\partial}{\partial \tau} J_\rho = \epsilon X_J^\mu + O(\epsilon^2), \quad (\text{A.5.56})$$

$$\epsilon \pi^{\mu\nu} + \epsilon \Delta^{\mu\nu\rho\sigma} \frac{\partial}{\partial \tau} \pi_{\rho\sigma} = \epsilon X_\pi^{\mu\nu} + O(\epsilon^2). \quad (\text{A.5.57})$$

We note that Eqs. (A.5.55)-(A.5.57) are also the relaxation equations that is valid up to $O(\epsilon)$.

By setting ϵ equal to 1 in Eqs. (A.5.55)-(A.5.57), we arrive at the explicit form of the relaxation equations (4.2.49)-(4.2.51).

Appendix B

Mesoscopic dynamics of 2D evolution equation

B.1 Reduction of 2D evolution equation to mesoscopic dynamics with RG method

In this section, we shall demonstrate the performance of the RG method presented in Sec. 4.1 by extracting the mesoscopic dynamics from a simple evolution equation describing the relaxation to a static solution.

The equation that we deal with reads [86]

$$\frac{\partial}{\partial t}x = y - x, \quad (\text{B.1.1})$$

$$\frac{\partial}{\partial t}y = \epsilon f(x), \quad (\text{B.1.2})$$

where t denotes time, x and y dynamical variables, $f(x)$ an arbitrary function with respect to x , and ϵ a small parameter.

Here, we shall give a brief account of the description of Eqs. (B.1.1) and (B.1.2). It is noted that the variation of y is slower than that of x due to the existence of ϵ in Eq. (B.1.2). This scale separation makes it possible to solve Eq. (B.1.1) with respect to x with y being treated as a constant. Thus, we have the asymptotic orbit as

$$x = y, \quad (\text{B.1.3})$$

for $t \rightarrow \infty$. Substituting Eq. (B.1.3) into Eq. (B.1.2), we have

$$\frac{\partial}{\partial t}y = \epsilon f(y). \quad (\text{B.1.4})$$

Thus, the dynamics described by Eqs. (B.1.1) and (B.1.2) is presented as follows: the dynamical variables x and y are rapidly attracted to the line $y = x$ from an arbitrary initial value, and then show the slow motion confined on $y = x$ governed by Eq. (B.1.4).

B.1. REDUCTION OF 2D EVOLUTION EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

From now on, by the RG method formulated in Sec. 4.1, we obtain the mesoscopic dynamics of Eqs. (B.1.1) and (B.1.2), which describes the dynamics defined on not only $y = x$ but also the neighborhood of $y = x$. We will see that the region where the mesoscopic dynamics is equivalent to the original one up to $O(\epsilon)$ lies in $|y - x| \leq O(\epsilon)$.

B.1.1 Set up in doublet scheme

We can convert Eqs. (B.1.1) and (B.1.2) into the generic form (4.1.2) discussed in Sec. 4.1 with the use of the following correspondences:

$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (\text{B.1.5})$$

$$\mathbf{G}(\mathbf{X}) = \begin{pmatrix} y - x \\ 0 \end{pmatrix}, \quad (\text{B.1.6})$$

$$\mathbf{F}(\mathbf{X}) = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}. \quad (\text{B.1.7})$$

It is noted that the dimension of $\mathbf{X}$ is two, i.e., $N = 2$. Thus, the static solution $\mathbf{X}_{\text{st}}$ reads

$$\mathbf{X}_{\text{st}}(t_0) = \begin{pmatrix} c_0(t_0) \\ c_0(t_0) \end{pmatrix}, \quad (\text{B.1.8})$$

which satisfies $\mathbf{G}(\mathbf{X}_{\text{st}}) = 0$. It is noted that $c_0(t_0)$ denotes the would-be integral constant which is lifted to the dynamical variable by the RG equation.

With the use of Eqs. (4.1.6) and (B.1.8), we have

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}. \quad (\text{B.1.9})$$

We note that the eigenvalues of A read

$$\lambda_- = -1, \quad (\text{B.1.10})$$

$$\lambda_0 = 0, \quad (\text{B.1.11})$$

and the corresponding eigenvectors are given by

$$\mathbf{U}_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\text{B.1.12})$$

$$\mathbf{U}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (\text{B.1.13})$$

respectively. Since the eigenvalues λ_- and λ_0 are real, the RG method presented in Sec. 4.1 is applicable to this case. The zero mode reads

$$\varphi_0^{(\alpha=1)} = U_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (\text{B.1.14})$$

APPENDIX B. MESOSCOPIC DYNAMICS OF 2D EVOLUTION EQUATION

It is noted that the dimension of the space spanned by the zero mode is one, i.e., $M_0 = 1$.

Let us introduce the inner product defined as

$$\langle \boldsymbol{\psi}, \boldsymbol{\chi} \rangle \equiv {}^t\boldsymbol{\psi} \mathcal{M} \boldsymbol{\chi}, \quad (\text{B.1.15})$$

where ${}^t\boldsymbol{\psi} = (\psi_x, \psi_y)$ and ${}^t\boldsymbol{\chi} = (\chi_x, \chi_y)$ denote arbitrary vectors and $\mathcal{M}$ a matrix defined by

$$\mathcal{M} \equiv \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}. \quad (\text{B.1.16})$$

We note that $\mathcal{M}$ is the symmetric matrix, i.e., ${}^t\mathcal{M} = \mathcal{M}$, and the orthogonality between $\boldsymbol{U}_-$ and $\boldsymbol{U}_0$ is satisfied, i.e.,

$$\langle \boldsymbol{U}_-, \boldsymbol{U}_0 \rangle = 0. \quad (\text{B.1.17})$$

Furthermore, we find that the inner product is positive definite,

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = \psi_x^2 + 2\psi_y^2 - 2\psi_x\psi_y = (\psi_x - \psi_y)^2 + \psi_y^2 > 0, \quad (\text{B.1.18})$$

for ${}^t\boldsymbol{\psi} = (\psi_x, \psi_y) \neq 0$, and A is self-adjoint,

$$\langle A\boldsymbol{\psi}, \boldsymbol{\chi} \rangle = {}^t(A\boldsymbol{\psi}) \mathcal{M} \boldsymbol{\chi} = {}^t\boldsymbol{\psi} {}^tA \mathcal{M} \boldsymbol{\chi} = {}^t\boldsymbol{\psi} \mathcal{M} A \boldsymbol{\chi} = \langle \boldsymbol{\psi}, A\boldsymbol{\chi} \rangle. \quad (\text{B.1.19})$$

In Eq. (B.1.19), we have used

$${}^tA \mathcal{M} = \mathcal{M} A. \quad (\text{B.1.20})$$

With the use of $\boldsymbol{\varphi}_0^{(1)}$ and the inner product defined in Eqs. (B.1.14) and (B.1.15), respectively, the P_0 -space metric matrix $\eta_0^{(\alpha=1)(\beta=1)}$ reads

$$\eta_0^{(1)(1)} = \langle \boldsymbol{\varphi}_0^{(1)}, \boldsymbol{\varphi}_0^{(1)} \rangle = 1. \quad (\text{B.1.21})$$

Thus, we have the projection operator P_0 given as

$$P_0 \boldsymbol{\psi} = \boldsymbol{\varphi}_0^{(1)} \langle \boldsymbol{\varphi}_0^{(1)}, \boldsymbol{\psi} \rangle. \quad (\text{B.1.22})$$

The definition shown in Eq. (4.1.26) leads us to

$$\boldsymbol{F}_0 = \begin{pmatrix} 0 \\ f(c_0(t_0)) \end{pmatrix}, \quad (\text{B.1.23})$$

which can be decomposed as

$$\boldsymbol{F}_0 = \tilde{\boldsymbol{\varphi}}_1^{(\mu=1)} \delta \bar{X}_{(\mu=1)}, \quad (\text{B.1.24})$$

B.1. REDUCTION OF 2D EVOLUTION EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

with

$$\tilde{\varphi}_1^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{B.1.25})$$

$$\delta \bar{X}_{(1)} = f(c_0(t_0)). \quad (\text{B.1.26})$$

Through the straightforward calculation, we have

$$\varphi_1^{(1)} = Q_0 \tilde{\varphi}_1^{(1)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\mathbf{U}_-. \quad (\text{B.1.27})$$

Since we are now considering a two-dimensional model, it is natural that $\varphi_1^{(1)}$ is in accord with $\mathbf{U}_-$, which is orthogonal to the P_0 -space vector $\mathbf{U}_0$. We find that the P_1 -space vectors read

$$\Phi_1^{(0,\mu=1)} = \varphi_1^{(1)} = -\mathbf{U}_-, \quad (\text{B.1.28})$$

$$\Phi_1^{(1,\mu=1)} = A^{-1} \varphi_1^{(1)} = -A^{-1} \mathbf{U}_- = \mathbf{U}_-. \quad (\text{B.1.29})$$

We note that the P_1 space is spanned by $\mathbf{U}_-$, and the Q_1 space is null, that is,

$$Q_1 = Q_0 - P_1 = 0. \quad (\text{B.1.30})$$

With the use of $\Phi_1^{(1,1)} = A^{-1} \varphi_1^{(1)} = \mathbf{U}_-$, we have

$$\phi(t_0) = \Phi_1^{(1,1)} \delta X_{(1)} = \mathbf{U}_- c_1(t_0), \quad (\text{B.1.31})$$

where we have defined

$$\delta X_{(1)} = c_1(t_0), \quad (\text{B.1.32})$$

with

$$|c_1(t_0)| \leq O(\epsilon). \quad (\text{B.1.33})$$

It is noted that $c_1(t_0)$ is also the would-be integral constant, as well as $c_0(t_0)$, which becomes the dynamical variable by the application of the RG equation.

The definitions shown in Eqs. (4.1.45) and (4.1.46) lead us to

$$B = 0, \quad (\text{B.1.34})$$

$$F_1 = \begin{pmatrix} 0 & 0 \\ f'(c_0(t_0)) & 0 \end{pmatrix}. \quad (\text{B.1.35})$$

B.1.2 Reduced dynamics by RG method

Substituting A , B , $\mathbf{F}_0$, F_1 , $\delta\bar{X}_{(1)}$, $\delta X_{(1)}$, $\boldsymbol{\varphi}_0^{(1)}$, and $\boldsymbol{\varphi}_1^{(1)}$ defined above into Eqs. (4.1.58) and (4.1.59), we obtain the mesoscopic dynamics of Eqs. (B.1.1) and (B.1.2) as the pair of the initial value

$$x_E(t) \equiv x(t_0 = t) = c_0(t) + c_1(t), \quad (\text{B.1.36})$$

$$y_E(t) \equiv y(t_0 = t) = c_0(t), \quad (\text{B.1.37})$$

and the RG equation

$$\frac{\partial}{\partial t} c_0 + \epsilon \left[-f(c_0) - f'(c_0) c_1 \right] = 0, \quad (\text{B.1.38})$$

$$\frac{\partial}{\partial t} c_1 + \epsilon f(c_0) = -c_1, \quad (\text{B.1.39})$$

where c_0 and c_1 denotes the dynamical variables. We emphasize that the global solution is given as the initial value $(x_E(t), y_E(t))$ where the exact solution $(c_0(t), c_1(t_0))$ to Eqs. (B.1.38) and (B.1.39) is inserted.

Here, we shall explicitly show that the mesoscopic dynamics in the neighborhood of $y = x$ is extracted from the original one (B.1.1) and (B.1.2) by the RG method. Substituting Eqs. (B.1.36) and (B.1.37) into Eqs. (B.1.38) and (B.1.39), we have the following equation with respect to (x_E, y_E) :

$$\frac{\partial}{\partial t} x_E = y_E - x_E + \epsilon f'(y_E) (x_E - y_E) + O(\epsilon^2), \quad (\text{B.1.40})$$

$$\frac{\partial}{\partial t} y_E = \epsilon \left[f(y_E) + f'(y_E) (x_E - y_E) \right] + O(\epsilon^2), \quad (\text{B.1.41})$$

where $O(\epsilon^2)$ have been restored. In the region where

$$|c_1| = |x_E - y_E| \leq O(\epsilon), \quad (\text{B.1.42})$$

we can rewrite Eqs. (B.1.40) and (B.1.41) as

$$\frac{\partial}{\partial t} x_E = y_E - x_E + O(\epsilon^2), \quad (\text{B.1.43})$$

$$\frac{\partial}{\partial t} y_E = \epsilon f(x_E) + O(\epsilon^2). \quad (\text{B.1.44})$$

We emphasize that Eqs. (B.1.43) and (B.1.44) is the same as Eqs. (B.1.1) and (B.1.2) up to the first order of ϵ . Indeed, the RG method presented in Sec. C.2 works out to extract the mesoscopic dynamics validated in $|x - y| \leq O(\epsilon)$.

Appendix C

Mesoscopic dynamics of non-relativistic Boltzmann equation

Dissipative hydrodynamic equation is a powerful means for describing the long-wavelength and low-frequency dynamics of many-body systems, which are close to equilibrium state. A typical equation is the Navier-Stokes equation, whose dynamical variables are five fields consisting of temperature, density, and fluid velocity.

One of the problems inherent in the Navier-Stokes equation is instantaneous propagation of information, i.e., the lack of causality, which is attributed to parabolicity of the equation [57, 56, 58, 59]. Here, the parabolicity is a typical character of diffusion equations containing first-order (second-order) temporal-derivative (spatial-derivative) terms of dynamical variables. This character plagues a relativistically covariant extension of the Navier-Stokes equation.

In 1949, Grad [76] showed that the lack of causality could be circumvented within the framework of kinetic theory, i.e., the Boltzmann equation by employing a method of moments, where a suitable ansatz for the functional form of the distribution function provides a closed system of the differential equation. In particular, for the thirteen-moment approximation to the functional form, the resultant equation is similar to the Navier-Stokes equation but respects the causality, because the character of the equation is hyperbolic instead of parabolic, with finite propagation speeds. This thirteen-moment causal equation is called the Grad equation, whose dynamical variables are thirteen fields, i.e., temperature, density, fluid velocity, viscous pressure, and heat flux. It is noted that the Grad equation is identical to the Müller equation which is derived on the basis of the phenomenological approach [58, 59] as mentioned in Sec. 1.1. Since the advent of the Grad(-Müller) equation, a large amount of extensions to relativistic systems have been proposed [49, 50, 63, 64, 65, 66, 67, 68].

In 1996, Jou and his collaborators [60, 61] called the description by the Grad equation mesoscopic since it occupies an intermediate level between the descriptions by the Navier-Stokes equation and the Boltzmann equation. In fact, the Grad equation has been applied to various kinetic problems, e.g., in plasma and in photon transport,

whose dynamics often cannot be described by the Navier-Stokes equation since the systems are not close to the equilibrium state.

Recently, it has turned out that there exist some difficulties in the Grad equation. A typical one is the inconsistency between the dynamics described by the Grad equation and that of the Boltzmann equation in the mesoscopic regime of spatial and temporal scales. In fact, Karlin and his collaborators [113, 114] showed that some additional terms to the Grad equation should be necessary to ensure the consistency with the mesoscopic dynamics of the linearized Boltzmann equation, by employing their method of invariant manifold. Moreover, they demonstrated that these additional terms contain second-order spatial-derivative terms of viscous pressure and heat flux, which cause a significant modification in the dispersion relation of hydrodynamic modes, destroying the causality. Struchtrup and Torrilhon [115] also applied a similar approach to the non-linear Boltzmann equation for the Maxwell molecules to derive an equation, where the causality was similarly lost although a phase velocity given by their equation compared well with experiments. These facts mean that it is difficult to construct the equation that respects both of the causality and the consistency with the Boltzmann equation in the mesoscopic regime.

An extension of Grad's thirteen-moment approximation that ensures the causality has been recently proposed by Levermore [116], Torrilhon [117], and Öttinger [118], with different approaches. Unfortunately, however, the consistency between the resultant equations derived by these extensions and the mesoscopic dynamics of the Boltzmann equation remains not yet clear at all. Thus, the derivation of the causal equation consistent with the mesoscopic dynamics of the Boltzmann equation is still an open problem.

In this Appendix, we present an attempt to extract the mesoscopic dynamics from the Boltzmann equation with the RG method based on the doublet scheme developed in Sec. 4.1 and investigate whether or not the resultant equation respects the causality.

We demonstrate that the equation obtained by the RG equation contains thirteen dynamical variables and respects the causality. We show that the form of the resultant equation is the same as that of the Grad equation, but the microscopic formulae of the coefficients, e.g., the transport coefficients and relaxation times, are different. It turns out that our theory leads to the same expressions for the transport coefficients as given by the Chapman-Enskog method [75] and also novel formulae of the relaxation times in terms of relaxation functions, which allow a natural physical interpretation of the relaxation times. Moreover, the distribution function which is explicitly constructed in our theory provides a new ansatz for the functional form of the distribution function in the method of moments proposed by Grad.

This Appendix is organized as follows: In Sec. C.1, we present a summary of main results in this Appendix, i.e., an explicit form of our thirteen-moment causal equation, the microscopic formulae of the transport coefficients and relaxation times, and the novel approximation to functional form of the distribution function used in the method of moments which can reproduce our equation. In Sec. C.2, we summarize a

brief but self-contained account of the Boltzmann equation, Grad's thirteen-moment approximation in the method of moments, and the Grad equation. In Sec. C.3, with the use of the RG method developed in Sec. 4.1, we reduce the Boltzmann equation to a mesoscopic dynamics, and examine some properties of the resultant equation. In Sec. C.4 and Sec. C.5, we present a detailed derivation of some formulae used in Sec. C.3.

C.1 Main results

In this section, we shall summarize main results in this Appendix.

Our thirteen-moment causal equation consistent with the Boltzmann equation in the mesoscopic regime is given as the set of the continuity equations

$$\frac{\partial}{\partial t} n + \nabla \cdot (n \mathbf{u}) = 0, \quad (\text{C.1.1})$$

$$m n \frac{\partial}{\partial t} u^i + m n \mathbf{u} \cdot \nabla u^i = -\nabla^j (p \delta^{ji} - 2 \eta \pi^{ji}), \quad (\text{C.1.2})$$

$$n \frac{\partial}{\partial t} e + n \mathbf{u} \cdot \nabla e = \nabla^i (T \lambda J^i) + 2 \eta \pi^{ij} \nabla^i u^j - p \nabla \cdot \mathbf{u}, \quad (\text{C.1.3})$$

and the relaxation equations

$$\pi^{ij} + \tau_\pi \frac{\partial}{\partial t} \pi^{ij} = \frac{1}{2} \left(\nabla^i u^j + \nabla^j u^i - \frac{2}{3} \delta^{ij} \nabla \cdot \mathbf{u} \right), \quad (\text{C.1.4})$$

$$J^i + \tau_J \frac{\partial}{\partial t} J^i = \frac{1}{T} \nabla^i T, \quad (\text{C.1.5})$$

with i and j being the indexes of vectors or tensors in three dimensional space such as $i = 1, 2, 3$, where the dynamical variables are thirteen fields consisting of the temperature T , density n , fluid velocity u^i , heat flux J^i , and viscous pressure π^{ij} with $\pi^{ij} = \pi^{ji}$ and $\delta^{ij} \pi^{ij} = 0$. Here, p and e denote the internal energy and pressure for a dilute gas; $p = n T$ and $e = 3 T/2$.

The coefficients in Eqs. (C.1.1)-(C.1.5), i.e., η , λ , τ_π , and τ_J , denote the shear viscosity, heat conductivity, and relaxation times corresponding to the viscous pressure and heat flux, respectively. The microscopic formulae of the transport coefficients η and λ read

$$\eta = -\frac{1}{10 T} \sum_{\mathbf{v} \mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{\pi}_{\mathbf{v}}^{ij} L_{\mathbf{v} \mathbf{k}}^{-1} \hat{\pi}_{\mathbf{k}}^{ij} = \int_0^\infty ds R_\pi(s), \quad (\text{C.1.6})$$

$$\lambda = -\frac{1}{3 T^2} \sum_{\mathbf{v} \mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{J}_{\mathbf{v}}^i L_{\mathbf{v} \mathbf{k}}^{-1} \hat{J}_{\mathbf{k}}^i = \int_0^\infty ds R_J(s), \quad (\text{C.1.7})$$

and those of the relaxation times τ_π and τ_J are given by

$$\tau_\pi = -\frac{\sum_{\mathbf{v} \mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{\pi}_{\mathbf{v}}^{ij} L_{\mathbf{v} \mathbf{k}}^{-2} \hat{\pi}_{\mathbf{k}}^{ij}}{\sum_{\mathbf{v} \mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{\pi}_{\mathbf{v}}^{kl} L_{\mathbf{v} \mathbf{k}}^{-1} \hat{\pi}_{\mathbf{k}}^{kl}} = \frac{\int_0^\infty ds s R_\pi(s)}{\int_0^\infty ds R_\pi(s)}, \quad (\text{C.1.8})$$

$$\tau_J = -\frac{\sum_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{J}_{\mathbf{v}}^i L_{\mathbf{v}\mathbf{k}}^{-2} \hat{J}_{\mathbf{k}}^i}{\sum_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{J}_{\mathbf{v}}^k L_{\mathbf{v}\mathbf{k}}^{-1} \hat{J}_{\mathbf{k}}^k} = \frac{\int_0^\infty ds s R_J(s)}{\int_0^\infty ds R_J(s)}, \quad (\text{C.1.9})$$

where $\mathbf{v}$ and $\mathbf{k}$ denote the indexes of the velocity, $f_{\mathbf{v}}^{\text{eq}}$ the local equilibrium distribution function, $L_{\mathbf{v}\mathbf{k}}^{-1}$ the inverse matrix of the linearized collision operator $L_{\mathbf{v}\mathbf{k}}$, $\hat{\pi}_{\mathbf{v}}^{ij}$ and $\hat{J}_{\mathbf{v}}^i$ the microscopic representations of the viscous pressure and heat flux, and $R_\pi(s)$ and $R_J(s)$ the relaxation functions of them, respectively. We remark that η and λ agree with those obtained in the Chapman-Enskog method [75], and τ_π and τ_J can be naturally interpreted as a correlation time of $R_\pi(s)$ and $R_J(s)$. The explicit definitions of $f_{\mathbf{v}}^{\text{eq}}$, $\hat{\pi}_{\mathbf{v}}^{ij}$, $\hat{J}_{\mathbf{v}}^i$, $L_{\mathbf{v}\mathbf{k}}$, $R_\pi(s)$, and $R_J(s)$ will be shown in Eqs. (C.2.24), (C.2.38), (C.2.39), (C.2.47), (C.3.75), and (C.3.76), respectively.

A novel ansatz for the functional form of the distribution function in the method of moments, which can reproduce our thirteen-moment causal equation, reads $f_{\mathbf{v}} = f_{\mathbf{v}}^{\text{eq}}(1 + \Phi_{\mathbf{v}})$, where $\Phi_{\mathbf{v}}$ denotes the deviation given by

$$\Phi_{\mathbf{v}} = \Phi_{\mathbf{v}}^{\text{TK}} \equiv \frac{1}{T} \sum_{\mathbf{k}} (L_{\mathbf{v}\mathbf{k}}^{-1} \hat{\pi}_{\mathbf{k}}^{ij} \pi^{ij} + L_{\mathbf{v}\mathbf{k}}^{-1} \hat{J}_{\mathbf{k}}^i J^i). \quad (\text{C.1.10})$$

It is obvious that our deviation $\Phi_{\mathbf{v}}^{\text{TK}}$ is different from that proposed by Grad, i.e.,

$$\Phi_{\mathbf{v}}^{\text{TK}} \neq \Phi_{\mathbf{v}}^{\text{G}} \equiv \frac{1}{T} (L_\pi^{-1} \hat{\pi}_{\mathbf{v}}^{ij} \pi^{ij} + L_J^{-1} \hat{J}_{\mathbf{v}}^i J^i), \quad (\text{C.1.11})$$

where L_π and L_J denote the expectation values of $L_{\mathbf{v}\mathbf{k}}$ with respect to $\hat{\pi}_{\mathbf{v}}^{ij}$ and $\hat{J}_{\mathbf{v}}^i$, respectively. The explicit definitions of L_π , and L_J will be shown in Eqs. (C.2.50) and (C.2.51), respectively.

C.2 Preliminaries

In this section, we shall summarize a brief but self-contained account of basic properties of the non-relativistic Boltzmann equation, Grad's thirteen-moment approximation in the method of moments, and the Grad equation.

C.2.1 Basic properties of Boltzmann equation

The non-relativistic Boltzmann equation that we consider here reads

$$\frac{\partial}{\partial t} f_{\mathbf{v}}(t, \mathbf{x}) + \mathbf{v} \cdot \nabla f_{\mathbf{v}}(t, \mathbf{x}) = C[f]_{\mathbf{v}}(t, \mathbf{x}). \quad (\text{C.2.1})$$

Here, $f_{\mathbf{v}}(t, \mathbf{x})$ denotes the distribution function defined in the phase space $(t, \mathbf{x}, \mathbf{v})$ with t and $\mathbf{x} = (x^1, x^2, x^3)$ being the space-time coordinate and $\mathbf{v} = (v^1, v^2, v^3)$ being the velocity of the one-shell particle whose mass, momentum, and energy are

C.2. PRELIMINARIES

given as m , $m \mathbf{v}$, and $m |\mathbf{v}|^2/2$, respectively. The right-hand side of Eq. (C.2.1) is the collision integral,

$$C[f]_{\mathbf{v}}(t, \mathbf{x}) \equiv \frac{1}{2!} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \times \left(f_{\mathbf{v}_2}(t, \mathbf{x}) f_{\mathbf{v}_3}(t, \mathbf{x}) - f_{\mathbf{v}}(t, \mathbf{x}) f_{\mathbf{v}_1}(t, \mathbf{x}) \right), \quad (\text{C.2.2})$$

where $\omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3)$ denotes the transition probability due to the microscopic two particle interaction. We note that $\omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3)$ contains the delta function representing the energy-momentum conservation,

$$\omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \propto \delta^3(m \mathbf{v} + m \mathbf{v}_1 - m \mathbf{v}_2 - m \mathbf{v}_3) \times \delta(m |\mathbf{v}|^2/2 + m |\mathbf{v}_1|^2/2 - m |\mathbf{v}_2|^2/2 - m |\mathbf{v}_3|^2/2), \quad (\text{C.2.3})$$

and also has the symmetric properties due to the indistinguishability of the particles and the time reversal invariance of the microscopic transition probability,

$$\omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) = \omega(\mathbf{v}_2, \mathbf{v}_3 | \mathbf{v}, \mathbf{v}_1) = \omega(\mathbf{v}_1, \mathbf{v} | \mathbf{v}_3, \mathbf{v}_2) = \omega(\mathbf{v}_3, \mathbf{v}_2 | \mathbf{v}_1, \mathbf{v}). \quad (\text{C.2.4})$$

It should be stressed here that we have confined ourselves to the case in which the particle number is conserved in the collision process.

The property of the transition probability shown in Eq. (C.2.4) leads to the following identity satisfied for an arbitrary vector $\varphi_{\mathbf{v}}(t, \mathbf{x})$,

$$\begin{aligned} & \sum_{\mathbf{v}} \varphi_{\mathbf{v}}(t, \mathbf{x}) C[f]_{\mathbf{v}}(t, \mathbf{x}) \\ &= \frac{1}{2!} \frac{1}{4} \sum_{\mathbf{v}} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \\ & \quad \times \left(\varphi_{\mathbf{v}}(t, \mathbf{x}) + \varphi_{\mathbf{v}_1}(t, \mathbf{x}) - \varphi_{\mathbf{v}_2}(t, \mathbf{x}) - \varphi_{\mathbf{v}_3}(t, \mathbf{x}) \right) \\ & \quad \times \left(f_{\mathbf{v}_2}(t, \mathbf{x}) f_{\mathbf{v}_3}(t, \mathbf{x}) - f_{\mathbf{v}}(t, \mathbf{x}) f_{\mathbf{v}_1}(t, \mathbf{x}) \right). \end{aligned} \quad (\text{C.2.5})$$

A function $\varphi_{\mathbf{v}}(t, \mathbf{x})$ is called a collision invariant when it satisfies the following equation,

$$\sum_{\mathbf{v}} \varphi_{\mathbf{v}}(t, \mathbf{x}) C[f]_{\mathbf{v}}(t, \mathbf{x}) = 0. \quad (\text{C.2.6})$$

As is easily confirmed using the identity (C.2.5) and the property (C.2.3), $\varphi_{\mathbf{v}}(t, \mathbf{x}) = 1$, $m \mathbf{v}$, and $m |\mathbf{v}|^2/2$ are collision invariants;

$$\sum_{\mathbf{v}} (1, m \mathbf{v}, m |\mathbf{v}|^2/2) C[f]_{\mathbf{v}}(t, \mathbf{x}) = 0, \quad (\text{C.2.7})$$

which represent the conservation of the particle number, momentum, and energy by the collision process, respectively. We see that the linear combination of these collision invariants as given by

$$\varphi_{\mathbf{v}}(t, \mathbf{x}) = a(t, \mathbf{x}) + \mathbf{b}(t, \mathbf{x}) \cdot m \mathbf{v} + c(t, \mathbf{x}) m |\mathbf{v}|^2/2, \quad (\text{C.2.8})$$

is also a collision invariant with $a(t, \mathbf{x})$, $\mathbf{b}(t, \mathbf{x})$, and $c(t, \mathbf{x})$ being arbitrary functions of t and $\mathbf{x}$.

Using the Boltzmann equation (C.2.1) together with the collision invariants $(1, m \mathbf{v}, m |\mathbf{v}|^2/2)$, we have the following continuity equations,

$$\sum_{\mathbf{v}} (1, m \mathbf{v}, m |\mathbf{v}|^2/2) \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f_{\mathbf{v}}(t, \mathbf{x}) = 0, \quad (\text{C.2.9})$$

which are equivalent to

$$\frac{\partial}{\partial t} \rho(t, \mathbf{x}) = -\nabla \cdot (\rho(t, \mathbf{x}) \mathbf{V}(t, \mathbf{x})), \quad (\text{C.2.10})$$

$$m \rho(t, \mathbf{x}) \frac{\partial}{\partial t} V^i(t, \mathbf{x}) = -m \rho(t, \mathbf{x}) \mathbf{V}(t, \mathbf{x}) \cdot \nabla V^i(t, \mathbf{x}) - \nabla^j P^{ji}(t, \mathbf{x}), \quad (\text{C.2.11})$$

$$\begin{aligned} \rho(t, \mathbf{x}) \frac{\partial}{\partial t} e(t, \mathbf{x}) &= -\rho(t, \mathbf{x}) \mathbf{V}(t, \mathbf{x}) \cdot \nabla e(t, \mathbf{x}) - \nabla \cdot \mathbf{Q}(t, \mathbf{x}) \\ &\quad - P^{ij}(t, \mathbf{x}) \nabla^i V^j(t, \mathbf{x}), \end{aligned} \quad (\text{C.2.12})$$

respectively. Here, we have introduced

$$\rho(t, \mathbf{x}) \equiv \sum_{\mathbf{v}} f_{\mathbf{v}}(t, \mathbf{x}), \quad (\text{C.2.13})$$

$$V^i(t, \mathbf{x}) \equiv \frac{1}{\rho(t, \mathbf{x})} \sum_{\mathbf{v}} v^i f_{\mathbf{v}}(t, \mathbf{x}), \quad (\text{C.2.14})$$

$$e(t, \mathbf{x}) \equiv \frac{1}{\rho(t, \mathbf{x})} \sum_{\mathbf{v}} \frac{m}{2} |\delta \mathbf{v}(t, \mathbf{x})|^2 f_{\mathbf{v}}(t, \mathbf{x}), \quad (\text{C.2.15})$$

$$P^{ij}(t, \mathbf{x}) \equiv \sum_{\mathbf{v}} m \delta v^i(t, \mathbf{x}) \delta v^j(t, \mathbf{x}) f_{\mathbf{v}}(t, \mathbf{x}), \quad (\text{C.2.16})$$

$$Q^i(t, \mathbf{x}) \equiv \sum_{\mathbf{v}} \frac{m}{2} |\delta \mathbf{v}(t, \mathbf{x})|^2 \delta v^i(t, \mathbf{x}) f_{\mathbf{v}}(t, \mathbf{x}), \quad (\text{C.2.17})$$

with the peculiar velocity

$$\delta \mathbf{v}(t, \mathbf{x}) \equiv \mathbf{v} - \mathbf{V}(t, \mathbf{x}). \quad (\text{C.2.18})$$

It is noted that while these equations have the same forms as the hydrodynamic equation, nothing about the dynamical properties is contained in these equations before the evolution of the distribution function $f_{\mathbf{v}}(t, \mathbf{x})$ is obtained by solving Eq. (C.2.1).

In this kinetic theory, the entropy density $s(t, \mathbf{x})$ and current $\mathbf{J}_s(t, \mathbf{x})$ may be defined by

$$s(t, \mathbf{x}) \equiv - \sum_{\mathbf{v}} f_{\mathbf{v}}(t, \mathbf{x}) \left(\ln (2\pi)^3 f_{\mathbf{v}}(t, \mathbf{x}) - 1 \right), \quad (\text{C.2.19})$$

$$\mathbf{J}_s(t, \mathbf{x}) \equiv - \sum_{\mathbf{v}} \mathbf{v} f_{\mathbf{v}}(t, \mathbf{x}) \left(\ln (2\pi)^3 f_{\mathbf{v}}(t, \mathbf{x}) - 1 \right). \quad (\text{C.2.20})$$

C.2. PRELIMINARIES

Using Eq. (C.2.1), we have

$$\frac{\partial}{\partial t} s(t, \mathbf{x}) + \nabla \cdot \mathbf{J}_s(t, \mathbf{x}) = - \sum_{\mathbf{v}} \left(\ln(2\pi)^3 f_{\mathbf{v}}(t, \mathbf{x}) \right) C[f]_{\mathbf{v}}(t, \mathbf{x}). \quad (\text{C.2.21})$$

The above equation tells us that the entropy $S(t)$ defined by

$$S(t) \equiv \int d^3 \mathbf{x} s(t, \mathbf{x}), \quad (\text{C.2.22})$$

is conserved only if $\ln(2\pi)^3 f_{\mathbf{v}}(t, \mathbf{x})$ is a collision invariant, or a linear combination of the basic collision invariants $(1, \mathbf{v}, m|\mathbf{v}|^2/2)$ as

$$\ln(2\pi)^3 f_{\mathbf{v}}(t, \mathbf{x}) = \alpha(t, \mathbf{x}) + \beta(t, \mathbf{x}) \cdot m \mathbf{v} + \gamma(t, \mathbf{x}) m |\mathbf{v}|^2/2, \quad (\text{C.2.23})$$

with $\alpha(t, \mathbf{x})$, $\beta(t, \mathbf{x})$, and $\gamma(t, \mathbf{x})$ being arbitrary functions of $(t, \mathbf{x})$. In other words, the entropy-conserving distribution function is parametrized as

$$f_{\mathbf{v}}(t, \mathbf{x}) = n(t, \mathbf{x}) \left[\frac{m}{2\pi T(t, \mathbf{x})} \right]^{\frac{3}{2}} \exp \left[- \frac{m |\mathbf{v} - \mathbf{u}(t, \mathbf{x})|^2}{2T(t, \mathbf{x})} \right] \equiv f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x}), \quad (\text{C.2.24})$$

which is identified with the local equilibrium distribution function called the Maxwellian. $T(t, \mathbf{x})$, $n(t, \mathbf{x})$, and $\mathbf{u}(t, \mathbf{x})$ in Eq. (C.2.24) should be interpreted as the local temperature, density, and flow velocity, respectively.

We note that for the Maxwellian $f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x})$, the collision integral identically vanishes,

$$C[f^{\text{eq}}]_{\mathbf{v}}(t, \mathbf{x}) = 0, \quad (\text{C.2.25})$$

due to the energy-momentum conservation implemented in the transition probability.

Substituting the Maxwellian $f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x})$ into the continuity equations (C.2.10)-(C.2.12), we have

$$\frac{\partial}{\partial t} n(t, \mathbf{x}) = - \nabla \cdot (n(t, \mathbf{x}) \mathbf{u}(t, \mathbf{x})), \quad (\text{C.2.26})$$

$$m n(t, \mathbf{x}) \frac{\partial}{\partial t} u^i(t, \mathbf{x}) = -m n(t, \mathbf{x}) \mathbf{u}(t, \mathbf{x}) \cdot \nabla u^i(t, \mathbf{x}) - \nabla^i p_0(t, \mathbf{x}), \quad (\text{C.2.27})$$

$$n(t, \mathbf{x}) \frac{\partial}{\partial t} e_0(t, \mathbf{x}) = -n(t, \mathbf{x}) \mathbf{u}(t, \mathbf{x}) \cdot \nabla e_0(t, \mathbf{x}) - p_0(t, \mathbf{x}) \nabla \cdot \mathbf{u}(t, \mathbf{x}), \quad (\text{C.2.28})$$

where

$$p_0(t, \mathbf{x}) \equiv n(t, \mathbf{x}) T(t, \mathbf{x}), \quad (\text{C.2.29})$$

$$e_0(t, \mathbf{x}) \equiv \frac{3}{2} T(t, \mathbf{x}), \quad (\text{C.2.30})$$

because

$$\rho(t, \mathbf{x}) = n(t, \mathbf{x}), \quad (\text{C.2.31})$$

$$V^i(t, \mathbf{x}) = u^i(t, \mathbf{x}), \quad (\text{C.2.32})$$

$$e(t, \mathbf{x}) = \frac{3}{2} T(t, \mathbf{x}) = e_0(t, \mathbf{x}), \quad (\text{C.2.33})$$

$$P^{ij}(t, \mathbf{x}) = \delta^{ij} n(t, \mathbf{x}) T(t, \mathbf{x}) = \delta^{ij} p_0(t, \mathbf{x}), \quad (\text{C.2.34})$$

$$Q^i(t, \mathbf{x}) = 0. \quad (\text{C.2.35})$$

It is remarkable that Eqs. (C.2.26)-(C.2.28) are identical with the Euler equation, which describes the fluid dynamics with no dissipative effects, and $p_0(t, \mathbf{x})$ and $e_0(t, \mathbf{x})$ defined by Eqs. (C.2.29) and (C.2.30) are the equations of state of the dilute gas. Since the entropy-conserving distribution function $f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x})$ reproduces the Euler equation, we find that the dissipative effect is attributable to a deviation of $f_{\mathbf{v}}(t, \mathbf{x})$ from $f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x})$.

C.2.2 Thirteen-moment approximation and Grad equation

To take into account the deviation of $f_{\mathbf{v}}(t, \mathbf{x})$ from $f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x})$, Grad's thirteen-moment approximation [76] has been often used. In this subsection, we shall first clarify the ad-hoc aspects of the ansatz made in Grad's thirteen-moment approximation for the derivation of the Grad equation, and then we shall present the microscopic representation of the transport coefficients obtained by Grad's thirteen-moment approximation, and point out that the representations are not in accord with those by the Chapman-Enskog method [75].

In Grad's thirteen-moment approximation [76], the distribution function $f_{\mathbf{v}}(t, \mathbf{x})$ around $f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x})$ is expanded as

$$f_{\mathbf{v}}(t, \mathbf{x}) = f_{\mathbf{v}}^{\text{eq}}(t, \mathbf{x}) \left(1 + \Phi_{\mathbf{v}}(t, \mathbf{x}) \right), \quad (\text{C.2.36})$$

where it is assumed that the deviation $\Phi_{\mathbf{v}}(t, \mathbf{x})$ is small and the second and higher orders can be neglected. Then, the ansatz on $\Phi_{\mathbf{v}}(t, \mathbf{x})$ is imposed as

$$\Phi_{\mathbf{v}}(t, \mathbf{x}) = \hat{\pi}_{\mathbf{v}}^{ij}(t, \mathbf{x}) \pi^{ij}(t, \mathbf{x}) + \hat{J}_{\mathbf{v}}^i(t, \mathbf{x}) J^i(t, \mathbf{x}). \quad (\text{C.2.37})$$

Here, $\hat{\pi}_{\mathbf{v}}^{ij}(t, \mathbf{x})$ and $\hat{J}_{\mathbf{v}}^i(t, \mathbf{x})$ are the $\mathbf{v}$ -dependent quantities defined by

$$\hat{\pi}_{\mathbf{v}}^{ij}(t, \mathbf{x}) \equiv m \left(\delta v^i(t, \mathbf{x}) \delta v^j(t, \mathbf{x}) - \frac{1}{3} \delta^{ij} |\delta \mathbf{v}(t, \mathbf{x})|^2 \right), \quad (\text{C.2.38})$$

$$\hat{J}_{\mathbf{v}}^i(t, \mathbf{x}) \equiv \left(\frac{m}{2} |\delta \mathbf{v}(t, \mathbf{x})|^2 - \frac{5}{2} T(t, \mathbf{x}) \right) \delta v^i(t, \mathbf{x}), \quad (\text{C.2.39})$$

with $\delta \mathbf{v}(t, \mathbf{x}) \equiv \mathbf{v} - \mathbf{u}(t, \mathbf{x})$. We can find the following properties:

$$\hat{\pi}_{\mathbf{v}}^{ij}(t, \mathbf{x}) = \hat{\pi}_{\mathbf{v}}^{ji}(t, \mathbf{x}), \quad (\text{C.2.40})$$

$$\hat{\pi}_{\mathbf{v}}^{ii}(t, \mathbf{x}) = 0, \quad (\text{C.2.41})$$

C.2. PRELIMINARIES

which can be derived from the definition shown in Eq. (C.2.38). On the other hand, $\pi^{ij}(t, \mathbf{x})$ and $J^i(t, \mathbf{x})$ are expansion coefficients whose $(t, \mathbf{x})$ -dependence is not yet determined. As will be shown in Eqs. (C.2.56) and (C.2.57), the coefficients $\pi^{ij}(t, \mathbf{x})$ and $J^i(t, \mathbf{x})$ should be interpreted as the viscous pressure and heat flux, respectively. It is noted that the total number of independent components, $T(t, \mathbf{x})$, $n(t, \mathbf{x})$, $u^i(t, \mathbf{x})$, $\pi^{ij}(t, \mathbf{x})$, and $J^i(t, \mathbf{x})$, is thirteen because we can suppose that

$$\pi^{ij}(t, \mathbf{x}) = \pi^{ji}(t, \mathbf{x}), \quad (\text{C.2.42})$$

$$\pi^{ii}(t, \mathbf{x}) = 0, \quad (\text{C.2.43})$$

without loss of generality due to the properties shown in Eqs. (C.2.40) and (C.2.41). In the following, we shall suppress $(t, \mathbf{x})$ when no misunderstanding is expected.

To determine the $(t, \mathbf{x})$ -dependence of the thirteen coefficients T , n , u^i , π^{ij} , and J^i , we utilize the thirteen-moment equations which are derived by multiplying the Boltzmann equation (C.2.1) by appropriate thirteen quantities dependent on $\mathbf{v}$, and integrating them with respect to $\mathbf{v}$. In Grad's theory, the five collision invariants $(1, m\mathbf{v}, m|\mathbf{v}|^2/2)$ and the eight quantities $\hat{\pi}_{\mathbf{v}}^{ij}$ and $\hat{J}_{\mathbf{v}}^i$ are adopted as the thirteen independent quantities: The thirteen-moment equations consist of the five continuity equations

$$\sum_{\mathbf{v}} (1, m\mathbf{v}, m|\mathbf{v}|^2/2) \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f_{\mathbf{v}}^{\text{eq}} (1 + \Phi_{\mathbf{v}}) = 0, \quad (\text{C.2.44})$$

and the eight relaxation equations

$$\sum_{\mathbf{v}} (\hat{\pi}_{\mathbf{v}}^{ij}, \hat{J}_{\mathbf{v}}^i) \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] f_{\mathbf{v}}^{\text{eq}} (1 + \Phi_{\mathbf{v}}) = \sum_{\mathbf{v}} \sum_{\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} (\hat{\pi}_{\mathbf{v}}^{ij}, \hat{J}_{\mathbf{v}}^i) L_{\mathbf{v}\mathbf{k}} \Phi_{\mathbf{k}}, \quad (\text{C.2.45})$$

In the derivation of the relaxation equations, we have used the relation

$$C[f]_{\mathbf{v}} = C[f^{\text{eq}}]_{\mathbf{v}} + \sum_{\mathbf{k}} \frac{\partial}{\partial f_{\mathbf{k}}} C[f]_{\mathbf{v}} \bigg|_{f=f^{\text{eq}}} f_{\mathbf{k}}^{\text{eq}} \Phi_{\mathbf{k}} + O(\Phi^2) = \sum_{\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} L_{\mathbf{v}\mathbf{k}} \Phi_{\mathbf{k}}, \quad (\text{C.2.46})$$

where we have utilized Eq. (C.2.25) and neglected $O(\Phi^2)$. Here, $L_{\mathbf{v}\mathbf{k}}$ denotes the linearized collision operator given by

$$L_{\mathbf{v}\mathbf{k}} \equiv (f_{\mathbf{v}}^{\text{eq}})^{-1} \frac{\partial}{\partial f_{\mathbf{k}}} C[f]_{\mathbf{v}} \bigg|_{f=f^{\text{eq}}} f_{\mathbf{k}}^{\text{eq}}. \quad (\text{C.2.47})$$

Here, let us redefine π^{ij} and J^i as

$$\pi^{ij} \rightarrow \frac{1}{T} L_{\pi}^{-1} \pi^{ij}, \quad (\text{C.2.48})$$

$$J^i \rightarrow \frac{1}{T} L_J^{-1} J^i, \quad (\text{C.2.49})$$

where

$$L_\pi \equiv \frac{\sum_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{\pi}_{\mathbf{v}}^{ij} L_{\mathbf{v}\mathbf{k}} \hat{\pi}_{\mathbf{k}}^{ij}}{\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \hat{\pi}_{\mathbf{v}}^{kl} \hat{\pi}_{\mathbf{v}}^{kl}}, \quad (\text{C.2.50})$$

$$L_J \equiv \frac{\sum_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} \hat{J}_{\mathbf{v}}^i L_{\mathbf{v}\mathbf{k}} \hat{J}_{\mathbf{k}}^i}{\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \hat{J}_{\mathbf{v}}^k \hat{J}_{\mathbf{v}}^k}. \quad (\text{C.2.51})$$

With the use of these redefined variables, the deviation $\Phi_{\mathbf{v}}$ in Eq. (C.2.37) reads

$$\Phi_{\mathbf{v}} = \frac{1}{T} (L_\pi^{-1} \hat{\pi}_{\mathbf{v}}^{ij} \pi^{ij} + L_J^{-1} \hat{J}_{\mathbf{v}}^i J^i) \equiv \Phi_{\mathbf{v}}^{\text{G}}. \quad (\text{C.2.52})$$

Substituting $\Phi_{\mathbf{v}} = \Phi_{\mathbf{v}}^{\text{G}}$ into Eqs. (C.2.44) and (C.2.45), we obtain the Grad equation as a closed system of the equations governing T , n , u^i , π^{ij} , and J^i :

$$\frac{\partial}{\partial t} n + \nabla \cdot (n \mathbf{u}) = 0, \quad (\text{C.2.53})$$

$$m n \frac{\partial}{\partial t} u^i + m n \mathbf{u} \cdot \nabla u^i = -\nabla^j (p_0 \delta^{ji} - 2 \eta^{\text{G}} \pi^{ji}), \quad (\text{C.2.54})$$

$$n \frac{\partial}{\partial t} e_0 + n \mathbf{u} \cdot \nabla e_0 = \nabla^j (T \lambda^{\text{G}} J^j) + 2 \eta^{\text{G}} \pi^{jk} \bar{X}_\pi^{jk} - p_0 \nabla \cdot \mathbf{u}, \quad (\text{C.2.55})$$

$$\pi^{ij} + \tau_\pi^{\text{G}} \frac{\partial}{\partial t} \pi^{ij} = \bar{X}_\pi^{ij} + (\text{other terms}), \quad (\text{C.2.56})$$

$$J^i + \tau_J^{\text{G}} \frac{\partial}{\partial t} J^i = \bar{X}_J^i + (\text{other terms}). \quad (\text{C.2.57})$$

In Eqs. (C.2.53)-(C.2.57), we have defined the thermodynamic forces given by

$$\bar{X}_\pi^{ij} \equiv \Delta^{ijkl} \nabla^k u^l, \quad (\text{C.2.58})$$

$$\bar{X}_J^i \equiv \frac{1}{T} \nabla^i T, \quad (\text{C.2.59})$$

with the projection matrix

$$\Delta^{ijkl} \equiv \frac{1}{2} (\delta^{il} \delta^{jk} + \delta^{il} \delta^{jk} - \frac{2}{3} \delta^{ij} \delta^{kl}). \quad (\text{C.2.60})$$

Furthermore, we have defined the transport coefficients and relaxation times as

$$\eta^{\text{G}} = -\frac{1}{10 T} \frac{\langle \hat{\pi}^{ij}, \hat{\pi}^{ij} \rangle_{\text{eq}} \langle \hat{\pi}^{kl}, \hat{\pi}^{kl} \rangle_{\text{eq}}}{\langle \hat{\pi}^{mn}, L \hat{\pi}^{mn} \rangle_{\text{eq}}}, \quad (\text{C.2.61})$$

$$\lambda^{\text{G}} = -\frac{1}{3 T^2} \frac{\langle \hat{J}^i, \hat{J}^i \rangle_{\text{eq}} \langle \hat{J}^j, \hat{J}^j \rangle_{\text{eq}}}{\langle \hat{J}^k, L \hat{J}^k \rangle_{\text{eq}}}, \quad (\text{C.2.62})$$

$$\tau_\pi^{\text{G}} = -\frac{\langle \hat{\pi}^{ij}, \hat{\pi}^{ij} \rangle_{\text{eq}}}{\langle \hat{\pi}^{kl}, L \hat{\pi}^{kl} \rangle_{\text{eq}}}, \quad (\text{C.2.63})$$

$$\tau_J^{\text{G}} = -\frac{\langle \hat{J}^i, \hat{J}^i \rangle_{\text{eq}}}{\langle \hat{J}^k, L \hat{J}^k \rangle_{\text{eq}}}, \quad (\text{C.2.64})$$

C.3. REDUCTION OF BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

where we have introduced an inner product given as

$$\langle \psi, \chi \rangle_{\text{eq}} \equiv \sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \psi_{\mathbf{v}} \chi_{\mathbf{v}}, \quad (\text{C.2.65})$$

with $\psi_{\mathbf{v}}$ and $\chi_{\mathbf{v}}$ being arbitrary functions, and used the following representations:

$$[L \hat{\pi}^{ij}]_{\mathbf{v}} = \sum_{\mathbf{k}} L_{\mathbf{v}\mathbf{k}} \hat{\pi}_{\mathbf{k}}^{ij}, \quad (\text{C.2.66})$$

$$[L \hat{J}^i]_{\mathbf{v}} = \sum_{\mathbf{k}} L_{\mathbf{v}\mathbf{k}} \hat{J}_{\mathbf{k}}^i. \quad (\text{C.2.67})$$

It is well known that the microscopic representations of the transport coefficients obtained by Grad's thirteen-moment approximation are different from those by the Chapman-Enskog method [75], i.e.,

$$\eta^{\text{CE}} = -\frac{1}{10T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{\text{eq}}, \quad (\text{C.2.68})$$

$$\lambda^{\text{CE}} = -\frac{1}{3T^2} \langle \hat{J}^i, L^{-1} \hat{J}^i \rangle_{\text{eq}}. \quad (\text{C.2.69})$$

In fact, by comparing Eqs. (C.2.68) and (C.2.68) with Eqs.(C.2.61) and (C.2.62), we can easily find that

$$\eta^{\text{CE}} \neq \eta^{\text{G}}, \quad (\text{C.2.70})$$

$$\lambda^{\text{CE}} \neq \lambda^{\text{G}}. \quad (\text{C.2.71})$$

Here, $L_{\mathbf{v}\mathbf{k}}^{-1}$ denotes the inverse matrix of $L_{\mathbf{v}\mathbf{k}}$.

C.3 Reduction of Boltzmann equation to mesoscopic dynamics with RG method

In this section, we apply the RG method developed in Sec. 4.1 to extract the mesoscopic dynamics from the Boltzmann equation.

Since we are interested in the mesoscopic solution whose space-time scales are coarse-grained from those in the kinetic regime, we try to solve the Boltzmann equation (C.2.1) in the mesoscopic regime where the space-time variation of $f_{\mathbf{v}}(t, \mathbf{x})$ is small. To make a coarse graining in a systematic manner, we shall convert Eq. (C.2.1) into

$$\frac{\partial}{\partial t} f_{\mathbf{v}}(t, \mathbf{x}) = C[f]_{\mathbf{v}}(t, \mathbf{x}) - \epsilon \mathbf{v} \cdot \nabla f_{\mathbf{v}}(t, \mathbf{x}), \quad (\text{C.3.1})$$

where a parameter ϵ has been introduced to express that the space derivatives are small for the system that we are interested in. Here, ϵ is identified with the ratio of the average particle distance over the mean free path, i.e., the Knudsen number.

Since ϵ appears in front of the second term of the right-hand side of Eq. (C.3.1), the Boltzmann equation has a form to which the RG method developed in Sec. 4.1 is applicable.

C.3.1 Set up in doublet scheme

By comparing the Boltzmann equation (C.3.1) with the evolution equation (4.1.2) discussed in Sec. 4.1, we find the following correspondences:

$$\mathbf{X} = \{f_{\mathbf{v}}(t, \mathbf{x})\}_{\mathbf{v}}, \quad (\text{C.3.2})$$

$$\mathbf{G}(\mathbf{X}) = \{C[f]_{\mathbf{v}}(t, \mathbf{x})\}_{\mathbf{v}}, \quad (\text{C.3.3})$$

$$\mathbf{F}(\mathbf{X}) = \{-\mathbf{v} \cdot \nabla f_{\mathbf{v}}(t, \mathbf{x})\}_{\mathbf{v}}. \quad (\text{C.3.4})$$

It is noted that the velocity $\mathbf{v}$ is interpreted as an index of the vector, while we treat the space coordinate $\mathbf{x}$ solely as a parameter, in accordance with the past works by Kuramoto [78] and Hatta and Kunihiro [87]. From now, let us omit $\{\cdot\}_{\mathbf{v}}$.

With the use of Eq. (4.1.4), we find that the static solution reads

$$\mathbf{X}_{\text{st}}(t_0) = f_{\mathbf{v}}^{\text{eq}}(\mathbf{x}; t_0) = n(\mathbf{x}; t_0) \left[\frac{m}{2\pi T(\mathbf{x}; t_0)} \right]^{\frac{3}{2}} \exp \left[-\frac{m|\mathbf{v} - \mathbf{u}(\mathbf{x}; t_0)|^2}{2T(\mathbf{x}; t_0)} \right], \quad (\text{C.3.5})$$

which is nothing but the Maxwellian (C.2.24) and satisfies $C[f^{\text{eq}}]_{\mathbf{v}} = 0$ as discussed in Eq. (C.2.25). We note that the five would-be integral constants $n(\mathbf{x}; t_0)$, $T(\mathbf{x}; t_0)$, and $\mathbf{u}(\mathbf{x}; t_0)$ corresponding to $C_{(\alpha)}(t_0)$ in Sec. 4.1 are lifted to the dynamical variables by applying the RG equation. In the following, we shall suppress $(\mathbf{x}; t_0)$ when no misunderstanding is expected.

Using Eqs. (4.1.6) and (C.3.3), we have the linearized evolution operator A as

$$\begin{aligned} A &= \left. \frac{\partial}{\partial f_{\mathbf{k}}} C[f]_{\mathbf{v}} \right|_{f=f^{\text{eq}}} \\ &= \frac{1}{2!} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \\ &\quad \times (\delta_{\mathbf{v}_2 \mathbf{k}} f_{\mathbf{v}_3}^{\text{eq}} + f_{\mathbf{v}_2}^{\text{eq}} \delta_{\mathbf{v}_3 \mathbf{k}} - \delta_{\mathbf{v} \mathbf{k}} f_{\mathbf{v}_1}^{\text{eq}} - f_{\mathbf{v}}^{\text{eq}} \delta_{\mathbf{v}_1 \mathbf{k}}). \end{aligned} \quad (\text{C.3.6})$$

Here, let us examine the property of A . We define the inner product by

$$\langle \psi, \chi \rangle \equiv \sum_{\mathbf{v}} (f_{\mathbf{v}}^{\text{eq}})^{-1} \psi_{\mathbf{v}} \chi_{\mathbf{v}}, \quad (\text{C.3.7})$$

with $\psi_{\mathbf{v}}$ and $\chi_{\mathbf{v}}$ being arbitrary vectors. This inner product is different from one defined in Eq. (C.2.65). We note that the norm through this inner product is positive definite,

$$\langle \psi, \psi \rangle = \sum_{\mathbf{v}} (f_{\mathbf{v}}^{\text{eq}})^{-1} (\psi_{\mathbf{v}})^2 > 0, \quad \psi \neq 0. \quad (\text{C.3.8})$$

It is noteworthy that with the inner product A is self-adjoint,

$$\begin{aligned} \langle \psi, A\chi \rangle &= -\frac{1}{2!} \frac{1}{4} \sum_{\mathbf{v}} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) (\psi_{\mathbf{v}} + \psi_{\mathbf{v}_1} - \psi_{\mathbf{v}_2} - \psi_{\mathbf{v}_3}) \\ &\quad \times (\chi_{\mathbf{v}} + \chi_{\mathbf{v}_1} - \chi_{\mathbf{v}_2} - \chi_{\mathbf{v}_3}) \\ &= \langle A\psi, \chi \rangle, \end{aligned} \quad (\text{C.3.9})$$

C.3. REDUCTION OF BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

and real semi-negative definite,

$$\begin{aligned} \langle \psi, A\psi \rangle &= -\frac{1}{2!} \frac{1}{4} \sum_{\mathbf{v}} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \\ &\quad \times (\psi_{\mathbf{v}} + \psi_{\mathbf{v}_1} - \psi_{\mathbf{v}_2} - \psi_{\mathbf{v}_3})^2 \leq 0. \end{aligned} \quad (\text{C.3.10})$$

Owing to the property of A , we can apply the method presented in Sec. 4.1 to extract the mesoscopic dynamics from Eq. (C.3.1).

The zero modes of A are found to be

$$\varphi_0^{(\alpha)} = f_{\mathbf{v}}^{\text{eq}} \varphi_{0\mathbf{v}}^\alpha = [f_{\mathbf{v}}^{\text{eq}} \varphi_0^\alpha]_{\mathbf{v}}, \quad \alpha = 0, 1, 2, 3, 4, \quad (\text{C.3.11})$$

with the diagonal matrix $f_{\mathbf{v}\mathbf{k}}^{\text{eq}} \equiv \delta_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}}$. In Eq. (C.3.11), we have introduced the following five vectors:

$$\varphi_{0\mathbf{v}}^\alpha \equiv \begin{cases} \frac{1}{\sqrt{n}}, & \alpha = 0, \\ \frac{1}{\sqrt{n}} \sqrt{\frac{m}{T}} \delta v^i, & \alpha = i, \\ \frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \left(\frac{m}{2T} |\delta \mathbf{v}|^2 - \frac{3}{2} \right), & \alpha = 4, \end{cases} \quad (\text{C.3.12})$$

with the peculiar velocity $\delta \mathbf{v} = \mathbf{v} - \mathbf{u}$. It is noted that $\varphi_{0\mathbf{v}}^\alpha$ with $\alpha = 0, \dots, 4$ coincide with the collision invariants shown in Eq. (C.2.7), and the dimension of the kernel space of A is five, i.e., $M_0 = 5$. In fact, using the particle-number and energy-momentum conservation laws presented in Eq. (C.2.3), we can show that

$$\begin{aligned} [A f_{\mathbf{v}}^{\text{eq}} \varphi_0^\alpha]_{\mathbf{v}} &= -\frac{1}{2!} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) f_{\mathbf{v}_2}^{\text{eq}} f_{\mathbf{v}_3}^{\text{eq}} \\ &\quad \times (\varphi_{0\mathbf{v}}^\alpha + \varphi_{0\mathbf{v}_1}^\alpha - \varphi_{0\mathbf{v}_2}^\alpha - \varphi_{0\mathbf{v}_3}^\alpha) = 0. \end{aligned} \quad (\text{C.3.13})$$

With the use of $\varphi_{0\mathbf{v}}^\alpha$ and the inner product defined in Eqs. (C.3.11) and (C.3.7), respectively, we have the P_0 -space metric matrix as follows,

$$\eta_0^{(\alpha)(\beta)} = \langle f_{\mathbf{v}}^{\text{eq}} \varphi_0^\alpha, f_{\mathbf{v}}^{\text{eq}} \varphi_0^\beta \rangle = \sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \varphi_{0\mathbf{v}}^\alpha \varphi_{0\mathbf{v}}^\beta = \delta^{\alpha\beta}. \quad (\text{C.3.14})$$

Thus, we have the projection operators P_0 and Q_0 given as

$$[P_0 \psi]_{\mathbf{v}} = \sum_{\alpha=0}^4 f_{\mathbf{v}}^{\text{eq}} \varphi_{0\mathbf{v}}^\alpha \langle f_{\mathbf{v}}^{\text{eq}} \varphi_0^\alpha, \psi \rangle, \quad (\text{C.3.15})$$

and $Q_0 = 1 - P_0$.

The definition shown in Eq. (4.1.26) leads to

$$\begin{aligned} \mathbf{F}_0 &= -\mathbf{v} \cdot \nabla f_{\mathbf{v}}^{\text{eq}} \\ &= -v^i f_{\mathbf{v}}^{\text{eq}} \left[\frac{1}{n} \nabla^i n + \left(\frac{m}{2T} |\delta \mathbf{v}|^2 - \frac{3}{2} \right) \frac{1}{T} \nabla^i T + \frac{m}{T} \delta v^j \nabla^i u^j \right]. \end{aligned} \quad (\text{C.3.16})$$

APPENDIX C. MESOSCOPIC DYNAMICS OF NON-RELATIVISTIC
BOLTZMANN EQUATION

By comparing Eq. (C.3.16) with Eq. (4.1.26), we can read $\tilde{\varphi}_1^{(\mu)}$ and $\delta\bar{X}_{(\mu)}$ as

$$\tilde{\varphi}_1^{(\mu)} = f_{\mathbf{v}}^{\text{eq}} \tilde{\varphi}_{1\mathbf{v}}^{i\alpha}, \quad (\text{C.3.17})$$

$$\delta\bar{X}_{(\mu)} = \delta\bar{X}^{i\alpha}, \quad (\text{C.3.18})$$

respectively, where

$$\tilde{\varphi}_{1\mathbf{v}}^{i\alpha} \equiv v^i \varphi_{0\mathbf{v}}^\alpha = (\delta v^i + u^i) \varphi_{0\mathbf{v}}^\alpha, \quad (\text{C.3.19})$$

and

$$\delta\bar{X}^{i\alpha} \equiv \begin{cases} -\frac{1}{\sqrt{n}} \nabla^i n, & \alpha = 0, \\ -\sqrt{n} \sqrt{\frac{m}{T}} \nabla^i u^j, & \alpha = j, \\ -\sqrt{n} \sqrt{\frac{3}{2}} \frac{1}{T} \nabla^i T, & \alpha = 4. \end{cases} \quad (\text{C.3.20})$$

We can check that $\mathbf{F}_0$ in Eq. (C.3.16) is expressed in terms of $\delta\bar{X}^{i\alpha}$ as $\mathbf{F}_0 = \sum_{\alpha=0}^4 f_{\mathbf{v}}^{\text{eq}} \tilde{\varphi}_{1\mathbf{v}}^{i\alpha} \delta\bar{X}^{i\alpha}$.

Through the straightforward calculation shown in Appendix C.4, we have the vectors belonging to the P_1 space as

$$\varphi_1^{(\mu)} = f_{\mathbf{v}}^{\text{eq}} \varphi_{1\mathbf{v}}^{i\alpha}, \quad (\text{C.3.21})$$

with

$$\varphi_{1\mathbf{v}}^{i\alpha} = \begin{cases} 0, & \alpha = 0, \\ \frac{1}{\sqrt{n}} \sqrt{\frac{m}{T}} \frac{1}{m} \hat{\pi}_{\mathbf{v}}^{ij}, & \alpha = j, \\ \frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \frac{1}{T} \hat{J}_{\mathbf{v}}^i, & \alpha = 4. \end{cases} \quad (\text{C.3.22})$$

Here, we have defined $\hat{J}_{\mathbf{v}}^i$ and $\hat{\pi}_{\mathbf{v}}^{ij}$ as

$$\hat{J}_{\mathbf{v}}^i \equiv \left(\frac{m}{2} |\delta\mathbf{v}|^2 - \frac{5}{2} T \right) \delta v^i, \quad (\text{C.3.23})$$

$$\hat{\pi}_{\mathbf{v}}^{ij} \equiv m \left(\delta v^i \delta v^j - \frac{1}{3} \delta^{ij} |\delta\mathbf{v}|^2 \right) = m \Delta^{ijkl} \delta v^k \delta v^l, \quad (\text{C.3.24})$$

respectively, where the projection matrix Δ^{ijkl} introduced in Eq. (C.2.60) has been used; $\Delta^{ijkl} = 1/2 (\delta^{ik} \delta^{jl} + \delta^{ik} \delta^{jl} - 2/3 \delta^{ij} \delta^{kl})$. We remark that $\hat{J}_{\mathbf{v}}^i$ and $\hat{\pi}_{\mathbf{v}}^{ij}$ are the same as the vectors defined in Eqs. (C.2.38) and (C.2.39), respectively, introduced in the method of moments. Thus, the number of independent components of $\varphi_{1\mathbf{v}}^{i\alpha}$ is eight, i.e., $M_1 = 8$. We stress that the number and form of the excited modes $\varphi_1^{(\mu)}$ have been automatically determined in the construction of the mesoscopic dynamics

C.3. REDUCTION OF BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

from the Boltzmann equation based on the doublet scheme in the RG method. Here, we call $\hat{\pi}_{\mathbf{v}}^{ij}$ and $\hat{J}_{\mathbf{v}}^i$ the microscopic representation of the viscous pressure and heat flux, respectively.

Using the P_1 -space vectors

$$\Phi_1^{(n,\mu)} = [A^{-n} f^{\text{eq}} \varphi_1^{i\alpha}]_{\mathbf{v}}, \quad n = 0, 1, \quad (\text{C.3.25})$$

we have the P_1 -space metric matrix

$$\eta_1^{(n,\mu)(m,\nu)} = \langle A^{-n} f^{\text{eq}} \varphi_1^{i\alpha}, A^{-m} f^{\text{eq}} \varphi_1^{j\beta} \rangle \equiv \eta_1^{ni\alpha, mj\beta}, \quad (\text{C.3.26})$$

which leads us to the projection operators P_1 and Q_1 given as

$$[P_1 \psi]_{\mathbf{v}} = \sum_{n=0,1} \sum_{\alpha=0}^4 \sum_{m=0,1} \sum_{\beta=0}^4 [A^{-n} f^{\text{eq}} \varphi_1^{i\alpha}]_{\mathbf{v}} \eta_{1ni\alpha, mj\beta}^{-1} \langle A^{-m} f^{\text{eq}} \varphi_1^{j\beta}, \psi \rangle, \quad (\text{C.3.27})$$

and $Q_1 = Q_0 - P_1$.

We shall introduce the integral constants that represents the deviation from $f_{\mathbf{v}}^{\text{eq}}$ as follows:

$$\delta X_{(\mu)}(t_0) = \delta X^{i\alpha}(\mathbf{x}; t_0). \quad (\text{C.3.28})$$

As shown in Sec. 4.1, $\delta X^{i\alpha}$ appears in the form of $\phi = \sum_{\mu=1}^{M_1} A^{-1} \varphi_1^{(\mu)} \delta X_{(\mu)}$ in the resultant equation. In this case, ϕ reads

$$\phi = \sum_{\mathbf{k}} A_{\mathbf{v}\mathbf{k}}^{-1} \sum_{\alpha=0}^4 f_{\mathbf{k}}^{\text{eq}} \varphi_{1\mathbf{k}}^{i\alpha} \delta X^{i\alpha} = \frac{1}{T} \sum_{\mathbf{k}} A_{\mathbf{v}\mathbf{k}}^{-1} f_{\mathbf{k}}^{\text{eq}} (\hat{\pi}_{\mathbf{k}}^{ij} \pi^{ij} + \hat{J}_{\mathbf{k}}^i J^i), \quad (\text{C.3.29})$$

where

$$\pi^{ij} \equiv \frac{1}{\sqrt{n}} \sqrt{\frac{T}{m}} \Delta^{ijkl} \delta X^{kl}, \quad (\text{C.3.30})$$

$$J^i \equiv \frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \delta X^{i4}. \quad (\text{C.3.31})$$

Instead of $\delta X^{i\alpha}$, we shall use π^{ij} and J^i , i.e., the viscous pressure and heat flux, as the fundamental quantities. It is noted that the definition of π^{ij} ensures that

$$\pi^{ij} = \pi^{ji}, \quad (\text{C.3.32})$$

$$\pi^{ii} = 0. \quad (\text{C.3.33})$$

Owing to the properties shown in Eqs. (C.3.32) and (C.3.33), the number of independent components of π^{ij} is five, and the total number of the would-be integral constants T , n , u^i , π^{ij} and J^i are thirteen. Although this number is the same as that of the dynamical variables introduced in the thirteen-moment approximation proposed by Grad, we emphasize that this number and form of the dynamical variables have been

automatically determined from the Boltzmann equation by the doublet scheme in the RG method developed in Sec. 4.1, which dose not demand any ansatz at all in contrast to the traditional approaches.

For convenience, let us introduce the following quantities:

$$\bar{X}_\pi^{ij} = -\frac{1}{\sqrt{n}} \sqrt{\frac{T}{m}} \Delta^{ijkl} \delta \bar{X}^{kl} = \Delta^{ijkl} \nabla^k u^l, \quad (\text{C.3.34})$$

$$\bar{X}_J^i = -\frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \delta \bar{X}^{i4} = \frac{1}{T} \nabla^i T, \quad (\text{C.3.35})$$

which are identical with the thermodynamic forces defined in Eqs. (C.2.58) and (C.2.59). With the use of $\bar{X}_\pi^{ij}$ and $\bar{X}_J^i$, we have

$$\sum_\mu \varphi_1^{(\mu)} \delta \bar{X}_{(\mu)} = \sum_{\alpha=0}^4 f_{\mathbf{v}}^{\text{eq}} \varphi_{1\mathbf{v}}^{i\alpha} \delta \bar{X}^{i\alpha} = -\frac{1}{T} f_{\mathbf{v}}^{\text{eq}} (\hat{\pi}_{\mathbf{v}}^{ij} \bar{X}_\pi^{ij} + \hat{J}_{\mathbf{v}}^i \bar{X}_J^i). \quad (\text{C.3.36})$$

The definitions presented in Eqs. (4.1.45) and (4.1.46) lead to

$$\begin{aligned} B &= \left. \frac{\partial^2}{\partial f_{\mathbf{k}} \partial f_{\mathbf{l}}} C[f]_{\mathbf{v}} \right|_{f=f^{\text{eq}}} \\ &= \frac{1}{2!} \sum_{\mathbf{v}_1} \sum_{\mathbf{v}_2} \sum_{\mathbf{v}_3} \omega(\mathbf{v}, \mathbf{v}_1 | \mathbf{v}_2, \mathbf{v}_3) \\ &\quad \times (\delta_{\mathbf{v}_2 \mathbf{k}} \delta_{\mathbf{v}_3 \mathbf{l}} + \delta_{\mathbf{v}_2 \mathbf{l}} \delta_{\mathbf{v}_3 \mathbf{k}} - \delta_{\mathbf{v} \mathbf{k}} \delta_{\mathbf{v}_1 \mathbf{l}} - \delta_{\mathbf{v} \mathbf{l}} \delta_{\mathbf{v}_1 \mathbf{k}}), \end{aligned} \quad (\text{C.3.37})$$

$$F_1 = -\mathbf{v} \cdot \nabla \delta_{\mathbf{v} \mathbf{k}}. \quad (\text{C.3.38})$$

C.3.2 Reduced dynamics by RG method

Substituting A , B , $\mathbf{F}_0$, F_1 , $\delta \bar{X}_{(\mu)}$, $\delta X_{(\mu)}$, $\varphi_0^{(\alpha)}$, and $\varphi_1^{(\mu)}$ obtained above into Eqs. (4.1.58)/(4.1.63), (4.1.59)/(4.1.64), and (4.1.66), we can obtain the mesoscopic dynamics of the Boltzmann equation.

Equations (4.1.58) and (4.1.59) read

$$\begin{aligned} &\sum_{\mathbf{v}} (1, m \mathbf{v}, m |\mathbf{v}|^2/2) \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] \left[f_{\mathbf{v}}^{\text{eq}} \left(1 + \frac{1}{T} ([L^{-1} \hat{\pi}^{kl}]_{\mathbf{v}} \pi^{kl} \right. \right. \\ &\quad \left. \left. + [L^{-1} \hat{J}^k]_{\mathbf{v}} J^k) \right) \right] \\ &= \frac{1}{2} \sum_{\mathbf{v}} (1, m \mathbf{v}, m |\mathbf{v}|^2/2) \sum_{\mathbf{k}} \sum_{\mathbf{l}} B_{\mathbf{v} \mathbf{k} \mathbf{l}} f_{\mathbf{k}}^{\text{eq}} \frac{1}{T} ([L^{-1} \hat{\pi}^{kl}]_{\mathbf{k}} \pi^{kl} + [L^{-1} \hat{J}^k]_{\mathbf{k}} J^k) \\ &\quad \times f_{\mathbf{l}}^{\text{eq}} \frac{1}{T} ([L^{-1} \hat{\pi}^{mn}]_{\mathbf{l}} \pi^{mn} + [L^{-1} \hat{J}^m]_{\mathbf{l}} J^m), \end{aligned} \quad (\text{C.3.39})$$

and

$$\sum_{\mathbf{v}} [L^{-1} (\hat{\pi}^{ij}, \hat{J}^i)]_{\mathbf{v}} \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] \left[f_{\mathbf{v}}^{\text{eq}} \left(1 + \frac{1}{T} ([L^{-1} \hat{\pi}^{kl}]_{\mathbf{v}} \pi^{kl} + [L^{-1} \hat{J}^k]_{\mathbf{v}} J^k) \right) \right]$$

C.3. REDUCTION OF BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

$$\begin{aligned}
&= \sum_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} [L^{-1}(\hat{\pi}^{ij}, \hat{J}^i)]_{\mathbf{v}} L_{\mathbf{v}\mathbf{k}} \frac{1}{T} ([L^{-1}\hat{\pi}^{kl}]_{\mathbf{k}} \pi^{kl} + [L^{-1}\hat{J}^k]_{\mathbf{k}} J^k) \\
&\quad + \frac{1}{2} \sum_{\mathbf{v}} [L^{-1}(\hat{\pi}^{ij}, \hat{J}^i)]_{\mathbf{v}} \sum_{\mathbf{k}} \sum_{\mathbf{l}} B_{\mathbf{v}\mathbf{k}\mathbf{l}} f_{\mathbf{k}}^{\text{eq}} \frac{1}{T} ([L^{-1}\hat{\pi}^{kl}]_{\mathbf{k}} \pi^{kl} + [L^{-1}\hat{J}^k]_{\mathbf{k}} J^k) \\
&\quad \times f_{\mathbf{l}}^{\text{eq}} \frac{1}{T} ([L^{-1}\hat{\pi}^{mn}]_{\mathbf{l}} \pi^{mn} + [L^{-1}\hat{J}^m]_{\mathbf{l}} J^m), \tag{C.3.40}
\end{aligned}$$

where ϵ has been set equal to 1 and we have used the identity

$$L_{\mathbf{v}\mathbf{k}}^{-1} = f_{\mathbf{v}}^{\text{eq}-1} A_{\mathbf{v}\mathbf{k}}^{-1} f_{\mathbf{k}}^{\text{eq}}. \tag{C.3.41}$$

It is notable that $L = f^{\text{eq}-1} A f^{\text{eq}}$ denotes the linearized collision operator defined in Eq. (C.2.47).

We can reduce Eq. (C.3.39) to

$$\begin{aligned}
&\sum_{\mathbf{v}} (1, m\mathbf{v}, m|\mathbf{v}|^2/2) \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] \left[f_{\mathbf{v}}^{\text{eq}} \left(1 + \frac{1}{T} ([L^{-1}\hat{\pi}^{kl}]_{\mathbf{v}} \pi^{kl} \right. \right. \\
&\quad \left. \left. + [L^{-1}\hat{J}^k]_{\mathbf{v}} J^k) \right) \right] = 0, \tag{C.3.42}
\end{aligned}$$

because

$$\begin{aligned}
&\frac{1}{2} \sum_{\mathbf{v}} (1, m\mathbf{v}, m|\mathbf{v}|^2/2) \sum_{\mathbf{k}} \sum_{\mathbf{l}} B_{\mathbf{v}\mathbf{k}\mathbf{l}} f_{\mathbf{k}}^{\text{eq}} \frac{1}{T} ([L^{-1}\hat{\pi}^{kl}]_{\mathbf{k}} \pi^{kl} + [L^{-1}\hat{J}^k]_{\mathbf{k}} J^k) \\
&\quad \times f_{\mathbf{l}}^{\text{eq}} \frac{1}{T} ([L^{-1}\hat{\pi}^{mn}]_{\mathbf{l}} \pi^{mn} + [L^{-1}\hat{J}^m]_{\mathbf{l}} J^m) \\
&= \sum_{\mathbf{v}} (1, m\mathbf{v}, m|\mathbf{v}|^2/2) C[f^{\text{eq}} \frac{1}{T} (L^{-1}\hat{\pi}^{kl} \pi^{kl} + L^{-1}\hat{J}^k J^k)]_{\mathbf{v}} \\
&= 0, \tag{C.3.43}
\end{aligned}$$

where we have used the fact that $(1, m\mathbf{v}, m|\mathbf{v}|^2/2)$ are collision invariants shown in Eq. (C.2.7). In contrast to Eq. (C.3.39), the term associated with $B_{\mathbf{v}\mathbf{k}\mathbf{l}}$ remains in Eq. (C.3.40), because $[L^{-1}(\hat{\pi}^{ij}, \hat{J}^i)]_{\mathbf{v}}$ are not collision invariants. We note that this term produces non-linear terms with respect to π^{ij} and J^i , the presence of which are natural on account of the order counting with respect to ϵ , although they have been omitted in the existing literature. For the sake of the comparison with the equations in the literature, we shall neglect this term and consider the equation

$$\begin{aligned}
&\sum_{\mathbf{v}} [L^{-1}(\hat{\pi}^{ij}, \hat{J}^i)]_{\mathbf{v}} \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] \left[f_{\mathbf{v}}^{\text{eq}} \left(1 + \frac{1}{T} ([L^{-1}\hat{\pi}^{kl}]_{\mathbf{v}} \pi^{kl} + [L^{-1}\hat{J}^k]_{\mathbf{v}} J^k) \right) \right] \\
&= \sum_{\mathbf{v}\mathbf{k}} f_{\mathbf{v}}^{\text{eq}} [L^{-1}(\hat{\pi}^{ij}, \hat{J}^i)]_{\mathbf{v}} L_{\mathbf{v}\mathbf{k}} \frac{1}{T} ([L^{-1}\hat{\pi}^{kl}]_{\mathbf{k}} \pi^{kl} + [L^{-1}\hat{J}^k]_{\mathbf{k}} J^k). \tag{C.3.44}
\end{aligned}$$

By comparing Eqs. (C.3.42) and (C.3.44) with the continuity equation (C.2.44) and the relaxation equation (C.2.45) introduced in the method of moments, we can read

the form of the derivation $\Phi_{\mathbf{v}}$ and the thirteen quantities utilized to obtain the closed system of the differential equations as

$$\Phi_{\mathbf{v}} = \frac{1}{T} ([L^{-1} \hat{\pi}^{ij}]_{\mathbf{v}} \pi^{ij} + [L^{-1} \hat{J}^i]_{\mathbf{v}} J^i) \equiv \Phi_{\mathbf{v}}^{\text{TK}}, \quad (\text{C.3.45})$$

and

$$1, \quad m \mathbf{v}, \quad \frac{m}{2} |\mathbf{v}|^2, \quad [L^{-1} \hat{J}^i]_{\mathbf{v}}, \quad [L^{-1} \hat{\pi}^{ij}]_{\mathbf{v}}, \quad (\text{C.3.46})$$

respectively. We should stress that the deviation $\Phi_{\mathbf{v}}^{\text{TK}}$ is different from that proposed by Grad, i.e., $\Phi_{\mathbf{v}}^{\text{G}}$ introduced in Eq. (C.2.52);

$$\Phi_{\mathbf{v}}^{\text{TK}} \neq \Phi_{\mathbf{v}}^{\text{G}} = \frac{1}{T} (L_{\pi}^{-1} \hat{\pi}_{\mathbf{v}}^{ij} \pi^{ij} + L_J^{-1} \hat{J}_{\mathbf{v}}^i J^i). \quad (\text{C.3.47})$$

We also stress that $\Phi_{\mathbf{v}}^{\text{TK}}$ provides a new ansatz for the functional form of the distribution function in the method of moments, which is compatible with the Boltzmann equation in the mesoscopic regime.

It is important to notice the following relation between the Grad equation and our equations (C.3.42) and (C.3.44): By replacing

$$L_{\pi}^{-1} \hat{\pi}_{\mathbf{v}}^{ij} \rightarrow [L^{-1} \hat{\pi}^{ij}]_{\mathbf{v}}, \quad (\text{C.3.48})$$

$$L_J^{-1} \hat{J}_{\mathbf{v}}^i \rightarrow [L^{-1} \hat{J}^i]_{\mathbf{v}}, \quad (\text{C.3.49})$$

in the Grad equation, one can obtain Eqs. (C.3.42) and (C.3.44) without mathematical ambiguity. As will be shown explicitly in the next subsection, this replacement changes the values of the coefficients, e.g., the transport coefficients and relaxation times, included in the Grad equation, but the form of the resultant equation is the same as that of the Grad equation. Thus, our equations (C.3.42) and (C.3.44) holds the hyperbolic character, i.e., the causality, as the Grad equation dose. We emphasize that Eqs. (C.3.42) and (C.3.44) are identical with the thirteen-moment causal equation consistent with the Boltzmann equation, which has been long sought for.

In the rest of this subsection, we present an explicit form of the invariant/ attractive manifold: Equation (4.1.66) reads

$$f_{\mathbf{E}\mathbf{v}}(t, \mathbf{x}) = f_{\mathbf{v}}^{\text{eq}} (1 + \Phi_{\mathbf{v}}^{\text{TK}})(\mathbf{x}; t_0 = t) + \Delta f_{\mathbf{v}}(\mathbf{x}; t_0 = t). \quad (\text{C.3.50})$$

Here, $\Delta f_{\mathbf{v}}$ denotes the contribution from the Q_1 space given as

$$\Delta f_{\mathbf{v}} \equiv -[Q_1 f^{\text{eq}} (L - \partial/\partial\tau)^{-1} (f^{\text{eq}})^{-1} Q_0 \mathbf{K}(\tau)]_{\tau=0} \mathbf{v}, \quad (\text{C.3.51})$$

where

$$\mathbf{K}(\tau) = \frac{1}{2} \sum_{\alpha=0}^4 \sum_{\beta=0}^4 B \left[f^{\text{eq}} e^{L\tau} L^{-1} \varphi_1^{i\alpha} \delta X^{i\alpha} + \tau P_0 f^{\text{eq}} \tilde{\varphi}_1^{i\alpha} \delta \bar{X}^{i\alpha} \right]$$

C.3. REDUCTION OF BOLTZMANN EQUATION TO MESOSCOPIC DYNAMICS WITH RG METHOD

$$\begin{aligned}
& + f^{\text{eq}} (e^{L\tau} - 1) L^{-1} \varphi_1^{i\alpha} \delta \bar{X}^{i\alpha} \Big] \times \Big[f^{\text{eq}} e^{L\tau} L^{-1} \varphi_1^{j\beta} \delta X^{j\beta} \\
& + \tau P_0 f^{\text{eq}} \tilde{\varphi}_1^{j\beta} \delta \bar{X}^{j\beta} + f^{\text{eq}} (e^{L\tau} - 1) L^{-1} \varphi_1^{j\beta} \delta \bar{X}^{j\beta} \Big] \\
& + \sum_{\alpha=0}^4 F_1 \Big[f^{\text{eq}} e^{L\tau} L^{-1} \varphi_1^{i\alpha} \delta X^{i\alpha} + \tau P_0 f^{\text{eq}} \tilde{\varphi}_1^{i\alpha} \delta \bar{X}^{i\alpha} \\
& + f^{\text{eq}} (e^{L\tau} - 1) L^{-1} \varphi_1^{i\alpha} \delta \bar{X}^{i\alpha} \Big].
\end{aligned} \tag{C.3.52}$$

We note that the global solution to the Boltzmann equation (C.3.1) in the mesoscopic regime can be obtained by substituting the exact solution to Eqs. Eq. (C.3.42) and Eq. (C.3.44), i.e., $n(\mathbf{x}; t)$, $T(\mathbf{x}; t)$, $u^i(\mathbf{x}; t)$, $\pi^{ij}(\mathbf{x}; t)$, and $J^i(\mathbf{x}; t)$, into $f_{\text{E}\mathbf{v}}(t, \mathbf{x})$ in Eq. (C.3.50).

C.3.3 Causal equation for thirteen moments

We shall derive an explicit form of the causal equation for the thirteen moments by integrating the continuity equations (C.3.42) and the relaxation equations (C.3.44) with respect to $\mathbf{v}$.

First, we shall investigate Eq. (C.3.42). By carrying out the calculation shown in Appendix C.5, we find that Eq. (C.3.42) reads

$$\frac{\partial}{\partial t} n + \nabla \cdot (n \mathbf{u}) = 0, \tag{C.3.53}$$

$$m n \frac{\partial}{\partial t} u^i + m n \mathbf{u} \cdot \nabla u^i = -\nabla^j (p \delta^{ji} - 2\eta \pi^{ji}), \tag{C.3.54}$$

$$n \frac{\partial}{\partial t} e + n \mathbf{u} \cdot \nabla e = \nabla^j (T \lambda J^j) + 2\eta \pi^{jk} \bar{X}_{\pi}^{jk} - p \nabla \cdot \mathbf{u}, \tag{C.3.55}$$

with $\bar{X}_{\pi}^{ij} = \Delta^{ijkl} \nabla^k u^l$ defined in Eq. (C.3.34). In Eqs. (C.3.54) and (C.3.55), p and e denote the internal energy and pressure, and η and λ the transport coefficients, i.e., the shear viscosity and heat conductivity, respectively. These quantities have the following forms,

$$p \equiv n T, \tag{C.3.56}$$

$$e \equiv \frac{3}{2} T, \tag{C.3.57}$$

$$\eta \equiv -\frac{1}{10 T} \langle \hat{\pi}^{ij}, L^{-1} \hat{\pi}^{ij} \rangle_{\text{eq}}, \tag{C.3.58}$$

$$\lambda \equiv -\frac{1}{3 T^2} \langle \hat{J}^i, L^{-1} \hat{J}^i \rangle_{\text{eq}}, \tag{C.3.59}$$

respectively, where we have used the inner product defined in Eq. (C.2.65). We note that the forms of these quantities are inherently for the dilute gas. This is natural because our model equation treated in this thesis is the Boltzmann equation which represents the dynamics of the dilute gas in the kinetic regime. We should stress that the microscopic representations of the transport coefficients shown in Eqs. (C.3.58)

APPENDIX C. MESOSCOPIC DYNAMICS OF NON-RELATIVISTIC BOLTZMANN EQUATION

and (C.3.59) perfectly agree with those derived in the Chapman-Enskog method, not in Grad's thirteen-moment approximation:

$$\eta \equiv \eta^{\text{TK}} = \eta^{\text{CE}} \neq \eta^{\text{G}}, \quad (\text{C.3.60})$$

$$\lambda \equiv \lambda^{\text{TK}} = \lambda^{\text{CE}} \neq \lambda^{\text{G}}. \quad (\text{C.3.61})$$

Then, we shall examine the properties of Eq. (C.3.44). Through the straightforward calculation shown in Appendix C.5, it turns out that Eq. (4.1.64) can be written as

$$\pi^{ij} + \tau_\pi \frac{\partial}{\partial t} \pi^{ij} = \bar{X}_\pi^{ij}, \quad (\text{C.3.62})$$

$$J^i + \tau_J \frac{\partial}{\partial t} J^i = \bar{X}_J^i, \quad (\text{C.3.63})$$

where we have neglected the second-order terms with respect to ϵ for the sake of the simplicity. We note that the full expressions of the relaxation equations are shown in Appendix C.5. Here, τ_π and τ_J denote the relaxation times. We note that these quantities are time constants characterizing the corresponding relaxation process, where π^{ij} and J^i can be reduced to $\bar{X}_\pi^{ij} = \Delta^{ijkl} \nabla^k u^l$ and $\bar{X}_J^i = \nabla^i T/T$ asymptotically, respectively;

$$\pi^{ij} \rightarrow \bar{X}_\pi^{ij}, \quad (\text{C.3.64})$$

$$J^i \rightarrow \bar{X}_J^i. \quad (\text{C.3.65})$$

The definitions of the relaxation times read

$$\tau_\pi \equiv \frac{1}{10T\eta} \langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle_{\text{eq}}, \quad (\text{C.3.66})$$

$$\tau_J \equiv \frac{1}{3T^2\lambda} \langle \hat{J}^i, L^{-2} \hat{J}^i \rangle_{\text{eq}}. \quad (\text{C.3.67})$$

It is noteworthy that we obtain the microscopic representations of τ_π and τ_J as well as the transport coefficients η and λ shown in Eqs. (C.3.58) and (C.3.59).

Here, we shall convert the definitions of the transport coefficients η and λ and the relaxation times τ_π and τ_J into a more familiar form, i.e., the Green-Kubo formula in the linear response theory. For this purpose, we utilize the following identities satisfied for $\hat{\psi}_{\mathbf{v}} = (\hat{\pi}_{\mathbf{v}}^{ij}, \hat{J}_{\mathbf{v}}^i)$:

$$\begin{aligned} [L^{-1} \hat{\psi}]_{\mathbf{v}} &= (f_{\mathbf{v}}^{\text{eq}})^{-1} [A^{-1} f^{\text{eq}} \hat{\psi}]_{\mathbf{v}} \\ &= -(f_{\mathbf{v}}^{\text{eq}})^{-1} \int_0^\infty ds [e^{sA} f^{\text{eq}} \hat{\psi}]_{\mathbf{v}} \\ &= -\int_0^\infty ds [e^{sL} \hat{\psi}]_{\mathbf{v}} \\ &= -\int_0^\infty ds \hat{\psi}_{\mathbf{v}}(s), \\ [L^{-2} \hat{\psi}]_{\mathbf{v}} &= \int_0^\infty ds_1 \int_0^\infty ds_2 [e^{(s_1+s_2)L} \hat{\psi}]_{\mathbf{v}} \end{aligned} \quad (\text{C.3.68})$$

C.4. DETAILED DERIVATION OF EXPLICIT FORM OF EXCITED MODES

$$\begin{aligned}
&= \int_0^\infty ds_1 \int_0^\infty ds_2 \hat{\psi}_{\mathbf{v}}(s_1 + s_2) \\
&= \int_0^\infty ds s \hat{\psi}_{\mathbf{v}}(s),
\end{aligned} \tag{C.3.69}$$

where we have defined

$$\hat{\psi}_{\mathbf{v}}(s) \equiv [e^{sL} \hat{\psi}]_{\mathbf{v}}. \tag{C.3.70}$$

It is noted that $\hat{\psi}_{\mathbf{v}}(s)$ could be interpreted as a “time-evolved” vector of $\hat{\psi}_{\mathbf{v}}$ by $L_{\mathbf{v}\mathbf{k}}$.

With the use of the above identities, we can obtain the compact forms for the transport coefficients and relaxation times as follows:

$$\eta = \int_0^\infty ds R_\pi(s), \tag{C.3.71}$$

$$\lambda = \int_0^\infty ds R_J(s), \tag{C.3.72}$$

$$\tau_\pi = \frac{\int_0^\infty ds s R_\pi(s)}{\int_0^\infty ds R_\pi(s)}, \tag{C.3.73}$$

$$\tau_J = \frac{\int_0^\infty ds s R_J(s)}{\int_0^\infty ds R_J(s)}, \tag{C.3.74}$$

where $R_\pi(s)$ and $R_J(s)$ have been defined by

$$R_\pi(s) \equiv \frac{1}{10T} \langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s) \rangle_{\text{eq}}, \tag{C.3.75}$$

$$R_J(s) \equiv \frac{1}{3T^2} \langle \hat{J}^i(0), \hat{J}^i(s) \rangle_{\text{eq}}. \tag{C.3.76}$$

It is noted that $R_\pi(s)$ and $R_J(s)$ are identical with the relaxation function introduced in the linear response theory. We remark that τ_π and τ_J can be interpreted as a correlation time of $R_\pi(s)$ and $R_J(s)$, respectively, whose physical meaning is clear. We should stress that it is for the first time that the microscopic representations of τ_π and τ_J based on $R_\pi(s)$ and $R_J(s)$ are obtained, and the RG method presented in Sec. 4.1 plays an essential role in the derivation of Eqs. (C.3.71)-(C.3.74).

C.4 Detailed derivation of explicit form of excited modes

In this section, we shall show a detailed derivation of $\varphi_{1\mathbf{v}}^{i\alpha}$ as shown in Eq. (C.3.22). The definitions in Eqs. (C.3.17) and (C.3.21) lead to

$$f_{\mathbf{v}}^{\text{eq}} \varphi_{1\mathbf{v}}^{i\alpha} \equiv [Q_0 f^{\text{eq}} \tilde{\varphi}_1^{i\alpha}]_{\mathbf{v}} = f_{\mathbf{v}}^{\text{eq}} \tilde{\varphi}_{1\mathbf{v}}^{i\alpha} - \sum_{\beta=0}^4 f_{\mathbf{v}}^{\text{eq}} \varphi_{0\mathbf{v}}^\beta \langle f^{\text{eq}} \varphi_0^\beta, f^{\text{eq}} \tilde{\varphi}_1^{i\alpha} \rangle. \tag{C.4.1}$$

APPENDIX C. MESOSCOPIC DYNAMICS OF NON-RELATIVISTIC
BOLTZMANN EQUATION

Then, we can convert Eq. (C.4.1) into

$$\varphi_{1\mathbf{v}}^{i\alpha} = \delta v^i \varphi_{0\mathbf{v}}^\alpha - \sum_{\beta=0}^4 \varphi_{0\mathbf{v}}^\beta M^{\beta i\alpha}, \quad (\text{C.4.2})$$

where

$$M^{\alpha i\beta} \equiv \sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \varphi_{0\mathbf{v}}^\alpha \delta v^i \varphi_{0\mathbf{v}}^\beta = \langle f^{\text{eq}} \varphi_0^\beta, f^{\text{eq}} \tilde{\varphi}_1^{i\alpha} \rangle - u^i \delta^{\beta\alpha}. \quad (\text{C.4.3})$$

We note that $M^{\alpha i\beta}$ read

$$M^{0i0} = 0, \quad (\text{C.4.4})$$

$$M^{0ij} = \sqrt{\frac{T}{m}} \delta^{ij}, \quad (\text{C.4.5})$$

$$M^{0i4} = 0, \quad (\text{C.4.6})$$

$$M^{ji0} = \sqrt{\frac{T}{m}} \delta^{ji}, \quad (\text{C.4.7})$$

$$M^{jik} = 0, \quad (\text{C.4.8})$$

$$M^{ji4} = \sqrt{\frac{2}{3}} \sqrt{\frac{T}{m}} \delta^{ji}, \quad (\text{C.4.9})$$

$$M^{4i0} = 0, \quad (\text{C.4.10})$$

$$M^{4ij} = \sqrt{\frac{2}{3}} \sqrt{\frac{T}{m}} \delta^{ij}, \quad (\text{C.4.11})$$

$$M^{4i4} = 0, \quad (\text{C.4.12})$$

which have been derived straightforwardly with the use of

$$\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} = n, \quad (\text{C.4.13})$$

$$\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \delta v^i = 0, \quad (\text{C.4.14})$$

$$\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \delta v^i \delta v^j = n \frac{T}{m} \delta^{ij}, \quad (\text{C.4.15})$$

$$\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \delta v^i \delta v^j \delta v^k = 0, \quad (\text{C.4.16})$$

$$\sum_{\mathbf{v}} f_{\mathbf{v}}^{\text{eq}} \delta v^i \delta v^j \delta v^k \delta v^l = n \left(\frac{T}{m} \right)^2 (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}). \quad (\text{C.4.17})$$

Substituting $M^{\alpha i\beta}$ into Eq. (C.4.2), we obtain the explicit form of $\varphi_{1\mathbf{v}}^{i\alpha}$ shown in Eq. (C.3.22).

C.5 Detailed derivation of explicit form of causal equation for thirteen moments

In this section, we shall show a detailed derivation of the explicit form of the thirteen-moment causal equation given by the set of Eqs. (C.3.53)-(C.3.55), (C.3.62), and (C.3.63).

Instead of Eqs. (C.3.42) and (C.3.44), we shall start with Eqs. (4.1.63) and (4.1.64):

$$\begin{aligned}
& \langle f^{\text{eq}} \varphi_0^\alpha, \frac{\partial}{\partial t} f^{\text{eq}} \rangle / c^\alpha + \epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_0^\alpha, f^{\text{eq}} \tilde{\varphi}_1^{i\beta} \rangle \delta \bar{X}^{i\beta} / c^\alpha \\
&= -\epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_0^\alpha, K^\mu \partial_\mu (A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta}) \rangle / c^\alpha + O(\epsilon^3), \quad (\text{C.5.1}) \\
&\quad \epsilon \sum_{\beta=0}^4 \langle A^{-1} f^{\text{eq}} \varphi_1^{i\alpha}, K^\mu \partial_\mu (A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta}) \rangle / c^\alpha \\
&= \epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_1^{i\alpha}, A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \rangle (\delta X^{j\beta} + \delta \bar{X}^{j\beta}) / c^\alpha \\
&\quad + \epsilon^2 \frac{1}{2} \sum_{\beta=0}^4 \sum_{\gamma=0}^4 \langle A^{-1} f^{\text{eq}} \varphi_1^{i\alpha}, B [f^{\text{eq}} \varphi_1^{j\beta}] [f^{\text{eq}} \varphi_1^{k\gamma}] \rangle \delta X^{j\beta} \delta X^{k\gamma} / c^\alpha \\
&\quad + O(\epsilon^3), \quad (\text{C.5.2})
\end{aligned}$$

where the explicit forms of A , B , $\mathbf{F}_0$, F_1 , $\delta \bar{X}_{(\mu)}$, $\delta X_{(\mu)}$, $\varphi_0^{(\alpha)}$, and $\varphi_1^{(\mu)}$ have been utilized. Here, we have divided the obtained equations by c^α given by

$$c^\alpha \equiv \begin{cases} \frac{1}{\sqrt{n}}, & \alpha = 0, \\ \frac{1}{\sqrt{n}} \sqrt{\frac{m}{T}} \frac{1}{m}, & \alpha = i, \\ \frac{1}{\sqrt{n}} \sqrt{\frac{2}{3}} \frac{1}{T}, & \alpha = 4, \end{cases} \quad (\text{C.5.3})$$

and have used the relation

$$\left[\frac{\partial}{\partial t} - \epsilon F_1 \right]_{\mathbf{v}\mathbf{k}} = \left[\frac{\partial}{\partial t} + \epsilon \mathbf{v} \cdot \nabla \right] \delta_{\mathbf{v}\mathbf{k}} = K_{\mathbf{v}\mathbf{k}}^\mu \partial_\mu, \quad (\text{C.5.4})$$

with the definitions

$$(\partial_0, \partial_1, \partial_2, \partial_3) \equiv (\partial/\partial t, \epsilon \nabla^1, \epsilon \nabla^2, \epsilon \nabla^3), \quad (\text{C.5.5})$$

$$(K_{\mathbf{v}\mathbf{k}}^0, K_{\mathbf{v}\mathbf{k}}^1, K_{\mathbf{v}\mathbf{k}}^2, K_{\mathbf{v}\mathbf{k}}^3) \equiv (1, v^1, v^2, v^3) \delta_{\mathbf{v}\mathbf{k}}. \quad (\text{C.5.6})$$

APPENDIX C. MESOSCOPIC DYNAMICS OF NON-RELATIVISTIC
BOLTZMANN EQUATION

From now on, we show the explicit form of each terms in Eqs. (C.5.1) and (C.5.2) one by one: The first and second terms in the left-hand side of Eq. (C.5.1) read

$$\langle f^{\text{eq}} \varphi_0^\alpha, \frac{\partial}{\partial t} f^{\text{eq}} \rangle / c^\alpha = \begin{cases} \frac{\partial}{\partial t} n, & \alpha = 0, \\ m n \frac{\partial}{\partial t} u^i, & \alpha = i, \\ n \frac{\partial}{\partial t} e, & \alpha = 4, \end{cases} \quad (\text{C.5.7})$$

and

$$\epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_0^\alpha, f^{\text{eq}} \tilde{\varphi}_1^{i\beta} \rangle \delta \bar{X}^{i\beta} / c^\alpha = \begin{cases} -\epsilon \nabla \cdot (n \mathbf{u}), & \alpha = 0, \\ -\epsilon m n \mathbf{u} \cdot \nabla u^i - \epsilon \nabla^i p, & \alpha = i, \\ -\epsilon n \mathbf{u} \cdot \nabla e - \epsilon p \nabla \cdot \mathbf{u}, & \alpha = 4, \end{cases} \quad (\text{C.5.8})$$

respectively. Here, we have defined

$$e \equiv \frac{3}{2} T, \quad (\text{C.5.9})$$

$$p \equiv n T, \quad (\text{C.5.10})$$

which are consistent with the equations of state of the dilute gas shown in Eqs. (C.3.56) and (C.3.57).

The term in the right-hand side of Eq. (C.5.2) reads

$$\begin{aligned} & \epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_1^{i\alpha}, A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \rangle (\delta X^{j\beta} + \delta \bar{X}^{j\beta}) / c^\alpha \\ &= \begin{cases} 0, & \alpha = 0, \\ \epsilon 2 \eta (-\pi^{ij} + \bar{X}_\pi^{ij}), & \alpha = j, \\ \epsilon T \lambda (-J^i + \bar{X}_J^i), & \alpha = 4, \end{cases} \end{aligned} \quad (\text{C.5.11})$$

where we have defined

$$\eta \equiv -\frac{1}{10 T} \langle f^{\text{eq}} \hat{\pi}^{ij}, A^{-1} f^{\text{eq}} \hat{\pi}^{ij} \rangle, \quad (\text{C.5.12})$$

$$\lambda \equiv -\frac{1}{3 T^2} \langle f^{\text{eq}} \hat{J}^i, A^{-1} f^{\text{eq}} \hat{J}^i \rangle, \quad (\text{C.5.13})$$

and have utilized the following identities:

$$\langle f^{\text{eq}} \hat{\pi}^{ij}, A^{-1} f^{\text{eq}} \hat{\pi}^{kl} \rangle = \frac{1}{5} \Delta^{ijkl} \langle f^{\text{eq}} \hat{\pi}^{ab}, A^{-1} f^{\text{eq}} \hat{\pi}^{ab} \rangle, \quad (\text{C.5.14})$$

$$\langle f^{\text{eq}} \hat{J}^i, A^{-1} f^{\text{eq}} \hat{J}^j \rangle = \frac{1}{3} \delta^{ij} \langle f^{\text{eq}} \hat{J}^a, A^{-1} f^{\text{eq}} \hat{J}^a \rangle. \quad (\text{C.5.15})$$

C.5. DETAILED DERIVATION OF EXPLICIT FORM OF CAUSAL EQUATION FOR THIRTEEN MOMENTS

We note that the transport coefficients, i.e., η and λ , given by Eqs. (C.5.12) and (C.5.13) accord with those in Eqs. (C.3.58) and (C.3.59), respectively, with the use of the inner product defined in Eq. (C.2.65) and $A^{-1} = f^{\text{eq}} L^{-1} (f^{\text{eq}})^{-1}$.

The term in the right-hand side of Eq. (C.5.1) is more complicated than the other terms. First, we expand this term as

$$\begin{aligned}
& \epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_0^\alpha, K^\mu \partial_\mu (A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta}) \rangle / c^\alpha \\
&= \epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} K^\mu (\varphi_0^\alpha / c^\alpha), \partial_\mu (A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta}) \rangle \\
&= \epsilon \sum_{\beta=0}^4 \left[\partial_\mu \langle f^{\text{eq}} K^\mu (\varphi_0^\alpha / c^\alpha), A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta} \rangle \right. \\
&\quad \left. - \langle f^{\text{eq}} K^\mu \partial_\mu (\varphi_0^\alpha / c^\alpha), A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta} \rangle \right] \\
&= \epsilon \sum_{\beta=0}^4 \left[\partial_\mu \langle Q_0 f^{\text{eq}} K^\mu (\varphi_0^\alpha / c^\alpha), A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta} \rangle \right. \\
&\quad \left. - \langle Q_0 f^{\text{eq}} K^\mu \partial_\mu (\varphi_0^\alpha / c^\alpha), A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta} \rangle \right]. \tag{C.5.16}
\end{aligned}$$

We note that inserting Q_0 as the final line in the above equation does not change the values of the inner product, because $A^{-1} f^{\text{eq}} \varphi_1^{j\beta}$ belongs to the Q_0 space.

Then, substituting

$$[Q_0 f^{\text{eq}} K^\mu (\varphi_0^\alpha / c^\alpha)]_{\mathbf{v}} = f_{\mathbf{v}}^{\text{eq}}(0, \varphi_{1\mathbf{v}}^{1\alpha}, \varphi_{1\mathbf{v}}^{2\alpha}, \varphi_{1\mathbf{v}}^{3\alpha}) / c^\alpha, \tag{C.5.17}$$

and

$$[Q_0 f^{\text{eq}} K^\mu \partial_\mu (\varphi_0^\alpha / c^\alpha)]_{\mathbf{v}} = \begin{cases} 0, & \alpha = 0, \\ 0, & \alpha = i, \\ -\epsilon f_{\mathbf{v}}^{\text{eq}} \hat{\pi}_{\mathbf{v}}^{jk} \bar{X}_{\pi}^{jk}, & \alpha = 4, \end{cases} \tag{C.5.18}$$

into Eq. (C.5.16), we have

$$\begin{aligned}
& \epsilon \sum_{\beta=0}^4 \langle f^{\text{eq}} \varphi_0^\alpha, K^\mu \partial_\mu (A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta}) \rangle / c^\alpha \\
&= \begin{cases} 0, & \alpha = 0, \\ -\epsilon^2 \nabla^j (2 \eta \pi^{ji}), & \alpha = i, \\ -\epsilon^2 \nabla^j (T \lambda J^j) - \epsilon^2 2 \eta \pi^{jk} \bar{X}_{\pi}^{jk}, & \alpha = 4. \end{cases} \tag{C.5.19}
\end{aligned}$$

It is noted that Eqs. (C.5.17) and (C.5.18) have been derived with the direct use of the definitions.

APPENDIX C. MESOSCOPIC DYNAMICS OF NON-RELATIVISTIC
BOLTZMANN EQUATION

Substituting Eqs. (C.5.7), (C.5.8), and (C.5.19) into Eq. (C.5.1), we have the continuity equations as

$$\frac{\partial}{\partial t} n = -\epsilon \nabla \cdot (n \mathbf{u}) + O(\epsilon^3), \quad (\text{C.5.20})$$

$$m n \frac{\partial}{\partial t} u^i = \epsilon m n \mathbf{u} \cdot \nabla u^i - \epsilon \nabla^j (p \delta^{ji} - \epsilon 2 \eta \pi^{ji}) + O(\epsilon^3), \quad (\text{C.5.21})$$

$$n \frac{\partial}{\partial t} e = -\epsilon n \mathbf{u} \cdot \nabla e - \epsilon p \nabla \cdot \mathbf{u} + \epsilon^2 \nabla^j (T \lambda J^j) + \epsilon^2 2 \eta \pi^{jk} \bar{X}_\pi^{jk} + O(\epsilon^3). \quad (\text{C.5.22})$$

We emphasize that Eqs. (C.5.20)-(C.5.22) describe the slow motion of n , u^i , and $T = 2e/3$, because the time derivative of them is the first order of ϵ , as is manifest.

The term in the left-hand side of Eq. (C.5.2) reads

$$\begin{aligned} & \epsilon \sum_{\beta=0}^4 \langle A^{-1} f^{\text{eq}} \varphi_1^{i\alpha}, K^\mu \partial_\mu (A^{-1} f^{\text{eq}} \varphi_1^{j\beta} \delta X^{j\beta}) \rangle / c^\alpha \\ &= \begin{cases} 0, & \alpha = 0, \\ \epsilon T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle, & \alpha = j, \\ \epsilon T \langle \tilde{J}^i, K^\mu \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle, & \alpha = 4, \end{cases} \end{aligned} \quad (\text{C.5.23})$$

where

$$\tilde{\pi}_{\mathbf{v}}^{ij} \equiv \frac{1}{T} [A^{-1} f^{\text{eq}} \hat{\pi}^{ij}]_{\mathbf{v}}, \quad (\text{C.5.24})$$

$$\tilde{J}_{\mathbf{v}}^i \equiv \frac{1}{T} [A^{-1} f^{\text{eq}} \hat{J}^i]_{\mathbf{v}}. \quad (\text{C.5.25})$$

On the basis of the expansions

$$\begin{aligned} & T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle \\ &= T \langle \tilde{\pi}^{ij}, K^\mu \tilde{\pi}^{kl} \rangle \partial_\mu \pi^{kl} + T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu \tilde{\pi}^{kl} \rangle \pi^{kl} \\ & \quad + T \langle \tilde{\pi}^{ij}, K^\mu \tilde{J}^k \rangle \partial_\mu J^k + T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu \tilde{J}^k \rangle J^k, \end{aligned} \quad (\text{C.5.26})$$

$$\begin{aligned} & T \langle \tilde{J}^i, K^\mu \partial_\mu (\tilde{\pi}^{kl} \pi^{kl} + \tilde{J}^k J^k) \rangle \\ &= T \langle \tilde{J}^i, K^\mu \tilde{\pi}^{kl} \rangle \partial_\mu \pi^{kl} + T \langle \tilde{J}^i, K^\mu \partial_\mu \tilde{\pi}^{kl} \rangle \pi^{kl} \\ & \quad + T \langle \tilde{J}^i, K^\mu \tilde{J}^k \rangle \partial_\mu J^k + T \langle \tilde{J}^i, K^\mu \partial_\mu \tilde{J}^k \rangle J^k, \end{aligned} \quad (\text{C.5.27})$$

we shall proceed to the further analysis of Eq. (C.5.23). First, the first and third terms in the right-hand side of Eqs. (C.5.26) and (C.5.27) read

$$T \langle \tilde{\pi}^{ij}, K^\mu \tilde{\pi}^{kl} \rangle \partial_\mu \pi^{kl} = 2 \eta \tau_\pi \left(\frac{\partial}{\partial t} + \epsilon \mathbf{u} \cdot \nabla \right) \pi^{ij}, \quad (\text{C.5.28})$$

$$T \langle \tilde{\pi}^{ij}, K^\mu \tilde{J}^k \rangle \partial_\mu J^k = \epsilon 2 \eta \ell_{\pi J} \Delta^{ijmk} \nabla^m J^k, \quad (\text{C.5.29})$$

$$T \langle \tilde{J}^i, K^\mu \tilde{\pi}^{kl} \rangle \partial_\mu \pi^{kl} = \epsilon T \lambda \ell_{J\pi} \Delta^{imkl} \nabla^m \pi^{kl}, \quad (\text{C.5.30})$$

$$T \langle \tilde{J}^i, K^\mu \tilde{J}^k \rangle \partial_\mu J^k = T \lambda \tau_J \left(\frac{\partial}{\partial t} + \epsilon \mathbf{u} \cdot \nabla \right) J^i, \quad (\text{C.5.31})$$

C.5. DETAILED DERIVATION OF EXPLICIT FORM OF CAUSAL EQUATION FOR THIRTEEN MOMENTS

respectively. In Eqs. (C.5.28)-(C.5.31), we have used the definitions given by

$$\tau_\pi \equiv \frac{T}{10\eta} \langle \tilde{\pi}^{ij}, \tilde{\pi}^{ij} \rangle = \frac{1}{10T\eta} \langle f^{\text{eq}} \hat{\pi}^{ij}, A^{-2} f^{\text{eq}} \hat{\pi}^{ij} \rangle, \quad (\text{C.5.32})$$

$$\tau_J \equiv \frac{1}{3\lambda} \langle \tilde{J}^i, \tilde{J}^i \rangle = \frac{1}{3T^2\lambda} \langle f^{\text{eq}} \hat{J}^i, A^{-2} f^{\text{eq}} \hat{J}^i \rangle, \quad (\text{C.5.33})$$

$$\ell_{\pi J} \equiv \frac{T}{10\eta} \langle \tilde{\pi}^{ij}, \delta K^i \tilde{J}^j \rangle = \frac{1}{10T\eta} \langle f^{\text{eq}} \hat{\pi}^{ij}, A^{-1} \delta K^i A^{-1} f^{\text{eq}} \hat{J}^j \rangle, \quad (\text{C.5.34})$$

$$\ell_{J\pi} \equiv \frac{1}{5\lambda} \langle \tilde{J}^i, \delta K^j \tilde{\pi}^{ij} \rangle = \frac{1}{5T^2\lambda} \langle f^{\text{eq}} \hat{J}^i, A^{-1} \delta K^j A^{-1} f^{\text{eq}} \hat{\pi}^{ij} \rangle, \quad (\text{C.5.35})$$

and the following identities:

$$\langle \tilde{\pi}^{ij}, \tilde{\pi}^{kl} \rangle = \frac{1}{5} \Delta^{ijkl} \langle \tilde{\pi}^{ab}, \tilde{\pi}^{ab} \rangle, \quad (\text{C.5.36})$$

$$\langle \tilde{\pi}^{ij}, \delta K^m \tilde{\pi}^{kl} \rangle = 0, \quad (\text{C.5.37})$$

$$\langle \tilde{\pi}^{ij}, \tilde{J}^k \rangle = 0, \quad (\text{C.5.38})$$

$$\langle \tilde{\pi}^{ij}, \delta K^m \tilde{J}^k \rangle = \frac{1}{5} \Delta^{ijmk} \langle \tilde{\pi}^{ab}, \delta K^a \tilde{J}^b \rangle, \quad (\text{C.5.39})$$

$$\langle \tilde{J}^i, \tilde{\pi}^{kl} \rangle = 0, \quad (\text{C.5.40})$$

$$\langle \tilde{J}^i, \delta K^m \tilde{\pi}^{kl} \rangle = \frac{1}{5} \Delta^{imkl} \langle \tilde{J}^a, \delta K^b \tilde{\pi}^{ab} \rangle, \quad (\text{C.5.41})$$

$$\langle \tilde{J}^i, \tilde{J}^k \rangle = \frac{1}{3} \delta^{ik} \langle \tilde{J}^a, \tilde{J}^a \rangle, \quad (\text{C.5.42})$$

$$\langle \tilde{J}^i, \delta K^m \tilde{J}^k \rangle = 0, \quad (\text{C.5.43})$$

where

$$\delta K_{\mathbf{v}\mathbf{k}}^i \equiv \delta v^i \delta_{\mathbf{v}\mathbf{k}}. \quad (\text{C.5.44})$$

We note that the relaxation times, i.e., τ_π and τ_J , given by Eqs. (C.5.32) and (C.5.33) agree with those in Eqs. (C.3.66) and (C.3.67), respectively, using the inner product (C.2.65) and $A^{-1} = f^{\text{eq}} L^{-1} (f^{\text{eq}})^{-1}$, as well as the transport coefficients (C.5.12) and (C.5.13). Furthermore, we note that $\ell_{\pi J}$ and $\ell_{J\pi}$ defined in Eqs. (C.5.34) and (C.5.35) denote the relaxation lengths.

Next, we consider the second and fourth terms in the right-hand side of Eqs. (C.5.26) and (C.5.27). We notice the following power counting with respect to ϵ :

$$T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu \tilde{\pi}^{kl} \rangle \sim O(\epsilon), \quad (\text{C.5.45})$$

$$T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu \tilde{J}^k \rangle \sim O(\epsilon), \quad (\text{C.5.46})$$

$$T \langle \tilde{J}^i, K^\mu \partial_\mu \tilde{\pi}^{kl} \rangle \sim O(\epsilon), \quad (\text{C.5.47})$$

$$T \langle \tilde{J}^i, K^\mu \partial_\mu \tilde{J}^k \rangle \sim O(\epsilon). \quad (\text{C.5.48})$$

This order counting can be derived from the fact that the above terms contain the temporal and spatial first-order derivatives of n , T , and u^i : The temporal derivatives

can be converted into the spatial derivatives with the use of the continuity equations (C.5.20)-(C.5.22), and there exists ϵ in front of the spatial derivatives. Accordingly, we can represent the above terms as the quantities of $O(\epsilon)$,

$$T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu \tilde{\pi}^{kl} \rangle \equiv -\epsilon 2 \eta \bar{X}_{\pi\pi}^{ijkl}, \quad (\text{C.5.49})$$

$$T \langle \tilde{\pi}^{ij}, K^\mu \partial_\mu \tilde{J}^k \rangle \equiv -\epsilon 2 \eta \bar{X}_{\pi J}^{ijk}, \quad (\text{C.5.50})$$

$$T \langle \tilde{J}^i, K^\mu \partial_\mu \tilde{\pi}^{kl} \rangle \equiv -\epsilon T \lambda \bar{X}_{J\pi}^{ikl}, \quad (\text{C.5.51})$$

$$T \langle \tilde{J}^i, K^\mu \partial_\mu \tilde{J}^k \rangle \equiv -\epsilon T \lambda \bar{X}_{JJ}^{ik}. \quad (\text{C.5.52})$$

It is noteworthy that $\bar{X}_{\pi\pi}^{ijkl}$, $\bar{X}_{\pi J}^{ijk}$, $\bar{X}_{J\pi}^{ikl}$, and $\bar{X}_{JJ}^{ik}$ might be also called the thermodynamic forces as $\bar{X}_\pi^{ij} = \Delta^{ijkl} \nabla^k u^l$ and $\bar{X}_J^i = \nabla^i T/T$ are, which are found to contain the vorticity term [112] as in the case of the relativistic system [50].

Next, we shall examine the second term in the right-hand side of Eq. (C.5.2), which reads

$$\begin{aligned} & \epsilon^2 \frac{1}{2} \sum_{\beta=0}^4 \sum_{\gamma=0}^4 \langle A^{-1} f^{\text{eq}} \varphi_1^{i\alpha}, B [f^{\text{eq}} \varphi_1^{j\beta}] [f^{\text{eq}} \varphi_1^{k\gamma}] \rangle \delta X^{j\beta} \delta X^{k\gamma} / c^\alpha \\ &= \begin{cases} 0, & \alpha = 0, \\ \epsilon^2 2 \eta b_{\pi JJ} \Delta^{ijkl} J^k J^l, & \alpha = j, \\ \epsilon^2 T \lambda b_{J\pi J} \pi^{ik} J^k, & \alpha = 4. \end{cases} \end{aligned} \quad (\text{C.5.53})$$

We note that $b_{\pi JJ}$ and $b_{J\pi J}$ denote the coefficients in the non-linear terms of π^{ij} and J^i , whose definitions are given by

$$b_{\pi JJ} \equiv \frac{1}{20 \eta} \Delta^{ijkl} \langle f^{\text{eq}} \hat{\pi}^{ij}, B \left[\frac{1}{T} A^{-1} f^{\text{eq}} \hat{J}^k \right] \left[\frac{1}{T} A^{-1} f^{\text{eq}} \hat{J}^l \right] \rangle, \quad (\text{C.5.54})$$

$$b_{J\pi J} \equiv \frac{1}{5 T \lambda} \Delta^{ijkl} \langle f^{\text{eq}} \hat{J}^i, B \left[\frac{1}{T} A^{-1} f^{\text{eq}} \hat{\pi}^{kl} \right] \left[\frac{1}{T} A^{-1} f^{\text{eq}} \hat{J}^j \right] \rangle. \quad (\text{C.5.55})$$

By substituting Eqs. (C.5.11) and (C.5.23) together with Eqs. (C.5.26)-(C.5.31), (C.5.49)-(C.5.52), and (C.5.53) into Eq. (C.5.2), we have the relaxation equations as

$$\begin{aligned} & \epsilon \pi^{ij} + \epsilon \tau_\pi \left(\frac{\partial}{\partial t} + \epsilon \mathbf{u} \cdot \nabla \right) \pi^{ij} + \epsilon^2 \ell_{\pi J} \Delta^{ijmk} \nabla^m J^k \\ &= \epsilon \bar{X}_\pi^{ij} + \epsilon^2 \bar{X}_{\pi\pi}^{ijkl} \pi^{kl} + \epsilon^2 \bar{X}_{\pi J}^{ijk} J^k + \epsilon^2 b_{\pi JJ} \Delta^{ijkl} J^k J^l + O(\epsilon^3), \end{aligned} \quad (\text{C.5.56})$$

$$\begin{aligned} & \epsilon J^i + \epsilon \tau_J \left(\frac{\partial}{\partial t} + \epsilon \mathbf{u} \cdot \nabla \right) J^i + \epsilon^2 \ell_{J\pi} \Delta^{imkl} \nabla^m \pi^{kl} \\ &= \epsilon \bar{X}_J^i + \epsilon^2 \bar{X}_{J\pi}^{ikl} \pi^{kl} + \epsilon^2 \bar{X}_{JJ}^{ik} J^k + \epsilon^2 b_{J\pi J} \pi^{ik} J^k + O(\epsilon^3). \end{aligned} \quad (\text{C.5.57})$$

By eliminating the second-order terms with respect to ϵ , we reduce the above equations to

$$\epsilon \pi^{ij} + \epsilon \tau_\pi \frac{\partial}{\partial t} \pi^{ij} = \epsilon \bar{X}_\pi^{ij} + O(\epsilon^2), \quad (\text{C.5.58})$$

$$\epsilon J^i + \epsilon \tau_J \frac{\partial}{\partial t} J^i = \epsilon \bar{X}_J^i + O(\epsilon^2). \quad (\text{C.5.59})$$

C.5. DETAILED DERIVATION OF EXPLICIT FORM OF CAUSAL EQUATION FOR THIRTEEN MOMENTS

We note that Eqs. (C.5.58) and (C.5.59) are also the relaxation equations that can be validated up to $O(\epsilon)$.

By setting ϵ equal to 1 in Eqs. (C.5.20)-(C.5.22), (C.5.58) and (C.5.59), we arrive at the explicit form of the thirteen-moment causal equation consisting the continuity equations (C.3.53)-(C.3.55) and the relaxation equations (C.3.62) and (C.3.63).

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