

Hankel determinants and substitutions – some results and problems

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1. Introduction. Let A^* be a free monoid generated by a non-empty set A , i.e., A^* is the set of finite words over A with the empty word λ . We put $A^* := A^* \cup A^{\mathbb{N}}$, where $\mathbb{N}$ is the set of non-negative integers, so that $A^{\mathbb{N}}$ is the set of infinite words over A . Any monoid morphism $\sigma: A^* \rightarrow A^*$ can be extended to a map $\sigma: A^* \rightarrow A^*$ by $\sigma(a_0 a_1 a_2 \cdots) := \sigma(a_0) \sigma(a_1) \sigma(a_2) \cdots$ ($a_n \in A$), which is a so-called substitution (over A). We say that σ is of constant length k iff $\sigma(x)$ is a finite word of length k for all $x \in A$. A fixed point of a substitution σ is an infinite word $\varphi \in A^{\mathbb{N}}$ satisfying $\sigma(\varphi) = \varphi$.

The fixed point of a substitution σ over $\{a, b\}$ defined by

$$\sigma(a) = ab, \sigma(b) = ba \text{ (resp., } \sigma(a) = ab, \sigma(b) = a)$$

prefixed by a is referred to as the Thue-Morse word (resp., the Fibonacci word).

Let $q \geq 1$ be an integer. We denote by $\text{ord}_q(n)$ the largest integer $e \geq 0$ such that n is divisible by q^e . We say a word $w = w_1 w_2 w_3 \dots$ is a q -adic Toeplitz word iff $w_m = w_n$ holds for any positive integers m, n satisfying $\text{ord}_q(m) = \text{ord}_q(n)$. Let σ be a substitution over an infinite alphabet $A_{\infty} := \{a_0, a_1, a_2, \dots\}$ defined by

$$\sigma(a_n) = a_0 a_{n+1} \quad (n = 0, 1, 2, \dots).$$

For some of the symbols $a_0, a_1, a_2, \dots$, we also write $a_0 = a, a_1 = b, a_2 = c$, etc.

The substitution σ has a unique fixed point

$$\omega = abacabadabacabaeabacabadabacabaf\dots,$$

which is a 2-adic Toeplitz word. Any 2-adic Toeplitz word over a finite or an infinite alphabet B can be written by

$$\tau(\omega) := \tau(\omega_1) \tau(\omega_2) \tau(\omega_3) \dots,$$

where τ is a map from A_{∞} to B , and $\omega =: \omega_1 \omega_2 \omega_3 \dots$ ($\omega_i \in A_{\infty}$). In this sense the word ω

is a universal 2-adic Toeplitz word.

Our objective is to get something interesting related to determinants

$$H_n^{(m)} = H_n^{(m)}[\varphi] := \det(\varphi_{m+i+j})_{0 \leq i \leq n-1, 0 \leq j \leq n-1},$$

$$H_n = H_n[\varphi] := H_n^{(0)}[\varphi]$$

for a given infinite word $\varphi = \varphi_0 \varphi_1 \varphi_2 \dots$ ($\varphi_i \in A$) over a finite, or an infinite alphabet A , where $H_n^{(m)}$ is considered to be an element of $\mathbb{Z}[A]$, i.e., a polynomial in independent variables $\in A$ with integer coefficients. $H_n^{(m)}[\varphi]$ can be extended to $(m, n) \in \mathbb{Z} \times \mathbb{N}$ by setting $H_n^{(0)} := 1$, $\varphi_m := 0$ ($m < 0$), where $\mathbb{N}$ denotes the set of non-negative integers. In the following two sections, we give a very rough survey on the results related to $H_n^{(m)}$. In Section 2, we give some results on $H_n^{(m)}[\varphi]$ for a general word φ , cf. [K-T-W], [T2]. In Section 3, we give some of the results related to $H_n^{(m)}[\varphi]$ when φ is the Thue-Morse word, the Fibonacci word, and a Fibonacci-type word, cf. [A-P-ZXW-ZYW], [K-T-W], [T2]. In Section 4, we give a new characterization of the 2-adic Toeplitz words φ by an algebraic property (completely reducibility) of $H_n[\varphi]$, cf. [M-T-Tn]. We shall give no proofs, but state only results with minimum definition.

2. General properties of $H_n[\varphi]$. The set A^* becomes a complete metric space with respect to the metric defined by

$$d(\xi, \eta) := \exp(-\inf\{n; \xi_n \neq \eta_n\}) \quad (\xi = \xi_0 \xi_1 \xi_2 \dots, \eta = \eta_0 \eta_1 \eta_2 \dots \in A^* \quad (\xi_n, \eta_n \in A)).$$

As usual, $K((Z))$ denotes the set of formal Laurent series of one variable Z over a field K . We put

$$K := \mathbb{Q}(A) \quad (\supset A).$$

The set $K((Z^{-1}))$ becomes a metric space induced by a non-Archimedean norm defined by

$$\|\varphi^{(h)}\| := \exp(-n_0 + h), \quad n_0 := \inf\{n \in \mathbb{N}; \varphi_n \neq 0\} \quad (\|0\| := 0)$$

for

$$\varphi^{(h)} = \sum_{n \geq 0} \varphi_n Z^{-n+h} \in K((Z^{-1})) \quad (1)$$

with $h \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$. Note that $\|\varphi^{(h)}\| = \exp h$ holds if $\varphi = \varphi_0 \varphi_1 \varphi_2 \dots \in A^{\mathbb{N}} (\subset$

K^* . If φ is a finite word of length k , then $\varphi_n := 0$ for $n \geq k$.) For any given $\varphi = \varphi_0 \varphi_1 \varphi_2 \dots \in K^*$, we say that $(P, Q) \in K[z]^2$ is an h -Padé pair of order m for φ iff

$$\|Q\varphi^{(h)} - P\| < \exp(-m), \quad Q \neq 0, \quad \deg Q := \deg_z Q \leq m \quad (2)$$

holds. The usual Padé pair (for a formal Laurent series) agrees with the h -Padé pair with $h = -1$ (for a word), cf. [N-S]. It is known that an h -Padé pair (P, Q) of order m for φ always exists for any $h \in \mathbb{Z}$, $m \geq 0$, $\varphi \in K^*$, cf. Lemma 1, [T2]. For h -Padé pairs (P, Q) of order m for φ , a rational function $P/Q \in K(z)$ is uniquely determined for any given $h \in \mathbb{Z}$, $m \geq 0$, $\varphi \in K^*$. The element $P/Q \in K(z)$ for an h -Padé pair (P, Q) of order m for $\varphi \in K^*$ is referred to as the h -Padé approximant of order m for φ . A number $m \in \mathbb{N}$ is called a normal h -index for $\varphi \in K^*$ if (2) implies $\deg Q = m$. A normal h -Padé pair, i.e., $\deg Q$ is a normal h -index, is said to be normalized if the leading coefficient of Q equals one. Normal (-1) -indices (resp. (-1) -Padé pairs, (-1) -Padé approximants) will be simply referred to as normal indices (resp. Padé pairs, Padé approximants). The set of all the normal h -indices for φ will be denoted by

$$\Lambda_h(\varphi) := \{m \in \mathbb{N}; m \text{ is normal } h\text{-indices for } \varphi\},$$

$$\Lambda(\varphi) := \Lambda_{-1}(\varphi).$$

We can consider the series (1) over $K = \mathbb{Q}(a, b, \dots)$ with $a, b, \dots \in \mathbb{C}$. In such a case, $\varphi^{(h)}$ defined by (1) turns out to be not only an element of $\mathbb{C}((z^{-1}))$, but also an analytic function on $\{z \in \mathbb{C}; |z| > 1\}$, and the h -Padé approximant of order m for $\varphi^{(h)}$ pointwise converges to $\varphi^{(h)}$ with respect to the usual topology on $\mathbb{C}$ for each $z \in \mathbb{C}$, $|z| > 1$ as m tends to infinity.

Proposition 1 (cf. [T1]). Let $\varphi \in K^*$ be a word over $K = \mathbb{Q}(A)$ with an alphabet A possibly consisting of infinite letters.

$$H_{h+1}^{(m)}[\varphi] = (-1)^{\lfloor m/2 \rfloor} \prod_{i=0}^{\lfloor m/2 \rfloor} P(z) \quad (h, m \in \mathbb{Z}, m \geq 0),$$

where (P, Q) is a normalized h -Padé pair of degree m for φ , $\lfloor x \rfloor$ denotes the largest integer not exceeding a real number x , and $\prod_{i=0}^{\lfloor m/2 \rfloor}$ indicates a product

taken over all the zeros z of Q with their multiplicity in any field $\widetilde{K}$ containing an algebraic closure of K .

Remark 1. We can take $\widetilde{K} = \mathbb{C}$ in Proposition 1 in the case where A is a subset (possibly empty) of $\mathbb{C}$.

Remark 2. If m is not a normal h -index of φ , then $P, Q \in K[z]$ have common zeros. Hence, it follows from Proposition 1 that $m \notin \Lambda_n(\varphi)$ implies $H_{n+1}^{(m)} = 0$. The converse of this fact is valid, cf. Lemma 2, [T2].

In particular, Proposition 1 holds for all the fixed point $\varphi \in A^{\mathbb{N}}(CK^*)$ of a substitution over any alphabet A . The following remark is useful, while it is valid only for a word φ consisting of at most two symbols.

Remark 3. Let M be a matrix of size $n \times n$ with entries consisting of two variables a, b (symbols). Then

$$\det M = (a-b)^{n-1}(pa+qb) \in \mathbb{Z}[a, b],$$

where p, q are integers defined by

$$p = \det M \mid (a, b) = (1, 0), \quad q = \det M \mid (a, b) = (0, 1).$$

3. Thue-Morse, and Fibonacci cases.

J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen considered $H_{n,m}(\zeta)$ for the Thue-Morse sequence $\zeta = \text{abbabaab} \dots$ with $(a, b) = (1, 0)$, and showed that the 2-dimensional word $H_n^{(m)}(\zeta) \pmod{2}$ of $(n, m) \in \mathbb{N}^2$ is 2-dimensionally automatic; it is remarkable that $\Lambda(\zeta) = \mathbb{N}$ is known, cf. [A-P-ZXW-ZYW].

In general, it is very difficult to give an explicit formula of $H_n^{(m)}(\varphi)$ for a given infinite word φ that is not periodic, while explicit formulae of $H_n^{(m)}(\eta)$ are completely given for the Fibonacci word $\eta = \text{abaabab} \dots$, cf. Theorems 1-5 in [K-T-W]. By f_n we denote the n -th Fibonacci number ($f_{-1} = f_0 = 1$, $f_n = f_{n-1} + f_{n-2}$). Let

$$n = \sum_{i \geq 0} \delta_i(n) f_i \quad (\delta_i(n) \in \{0,1\}, \delta_{i+1}(n)\delta_i(n)=0 \text{ for all } i \geq 0)$$

be the representation of n in the Fibonacci base due to Zeckendorf. We write

$$m \equiv_k n$$

iff $\delta_i(m) = \delta_i(n)$ holds for all $0 \leq i < k$. We put

$$\chi(k, S) := -1 \text{ if } k \equiv s \pmod{6} \text{ for an } s \in S,$$

$$1 \text{ otherwise,}$$

for a subset S of $\{0,1,2,3,4,5\}$. Then we have, for instance,

Proposition 2 (Theorem 3, [K-T-W]). For any $k, m, i \geq 0$ integers satisfying

$$m \equiv_{k+1} i, \quad 0 \leq i \leq f_{k+1}-1,$$

the following formulae hold:

$$\begin{aligned} H_{f_k}^{(m)}[\eta] \mid (a, b) = (1, 0) &= \chi(k; 2) \chi(k; 1, 4)^i f_{k-1}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } 0 \leq i < f_{k-1}, \\ &\text{or } \delta_{k+1}(m) = 1 \text{ and } 0 \leq i < f_k, \\ &= \chi(k; 1, 2, 4) f_{k-2}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } i = f_{k-1}, \\ &\text{or } i = f_{k+1} - 1, \\ &= 0 \text{ otherwise,} \\ H_{f_k}^{(m)}[\eta] \mid (a, b) = (0, 1) &= \chi(k; 1, 2, 4) \chi(k; 1, 4)^i f_{k-2}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } 0 \leq i < f_{k-1}, \\ &\text{or } \delta_{k+1}(m) = 1 \text{ and } 0 \leq i < f_k, \\ &= \chi(k; 2) f_{k-3}, \\ &\text{if either } \delta_{k+1}(m) = 0 \text{ and } i = f_{k-1}, \\ &\text{or } i = f_{k+1} - 1, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Notice that Proposition 2 together with Remark 3 gives a part of the explicit formulae for $H_n^{(m)}[\eta]$. In comparison with the automaticity result for the 2-dimensional word $(H_n^{(m)}(\zeta) \pmod{2})_{(n,m) \in \mathbb{N}^2}$ for the Thue-Morse sequence

$\xi=10010110\cdots$ given in [A-P-ZXW-ZYW], we gave an explicit expression of the 2-dimensional word $H_n^{(m)}(\eta)$ $(n,m)\in\mathbb{N}^2$ for the Fibonacci word $\eta=abaabab\cdots$, which is rather complicated, cf. Theorem 5 in [K-T-W].

In [T2], we developed a theory of analysis on words, especially for words of Fibonacci type, i.e. the fixed points of substitutions of the form

$$\sigma(a):=a^k b, \sigma(b):=a \quad (k>0). \quad (3)$$

We denote by $\varepsilon=\varepsilon(a,b;k)$ the fixed point of the substitution defined by (3), and by $|w|_x$:=the number of occurrences of an identical symbol x appearing in a word w . In Sections 2, 3 in [T2], we gave explicit formulae for the continued fraction expansion with partial denominators $\in K[z]$ ($K=\mathbb{Q}(a,b)$) and normalized Padé pairs for the Laurent series $\varepsilon^{(-1)}(z)=\varepsilon(z;a,b;k)$ defined by (1) with $\varphi=\varepsilon$, cf. Theorems 1-12, [T2]. For instance, we have

Proposition 3 (Theorem 8, [T2]). Let $k\geq 2$. The continued fraction expansion of the Laurent series $\varepsilon^{(-1)}(z)\in K((z^{-1}))$ for $\varepsilon=\varepsilon(a,b;k)\in\{a,b\}^{\mathbb{N}}$ is given by

$$\varepsilon^{(-1)}(z)=[0;a^{-1}(z-1), (-1)^m(a-b)^{-1}h_m^2b_{m-1}^*, (-1)^m(a-b)h_m^{-1}h_{m+1}^{-1}(z-1)]_{m=0}^{\infty},$$

where

$$\begin{aligned} h_n &:= |\sigma^n(a)|_a a + |\sigma^n(a)|_b b (=g_n a + g_{n-1} b \in \mathbb{Z}[a,b]), \quad f_n = f_{n,k} := |\sigma^n(a)|, \\ b_n^* &:= b_n^*(z;k) := z^{f_n} \sum_{0 \leq i \leq f_{n+1}-1} z^i \cdot \sum_{1 \leq j \leq k-1} (k-j) z^{(j-1)f_{n+1}+k} \sum_{0 \leq i \leq f_n-1} z^i \in \mathbb{Z}[z]. \end{aligned}$$

If $(a,b)\in\mathbb{C}^2$, then Proposition 3 is valid under the condition

$$a \neq b, h_n (=g_n a + g_{n-1} b) \neq 0 \text{ for all } n \geq 0. \quad (4)$$

We can give explicit formulae for the continued fraction expansion for $\varepsilon^{(-1)}(z)=\varepsilon(z;a,b;k)\in\mathbb{C}((z^{-1}))$ with $(a,b)\in\mathbb{C}^2$, which does not satisfy (4). For example,

Proposition 4 (Theorem 9, [T2]). Let $(a,b)\in\mathbb{C}^2$ with $h_1=0$, $a\neq 0$, $t\geq 0$. Then

$$\begin{aligned} \varepsilon^{(-1)}(z) &= \varepsilon^{(-1)}(z;a,-g_{t-1}^{-1}g_t a) \\ &= [0;a_1,d_{-1},c_0,d_0,\dots,c_{t-2},d_{t-2},e_1,e_2,i_m,j_m]_{m=0}^{\infty} \end{aligned}$$

holds with partial denominators $\in \mathbb{C}[z]$ given by

$$\begin{aligned} a_1 &= a^{-1}(z-1), \\ c_m &= (-1)^m(a-b)h_m^{-1}h_{m+1}^{-1}(z-1), \end{aligned}$$

$$d_m = (-1)^{m+1} (a-b)^{-1} h_{m+1}^2 b_m^*,$$

$$e_1 = (-1)^{t-1} (a-b) h_{t-1}^{-2} (z-1) b_{t-1},$$

$$e_2 = (-1)^{t-1} (a-b)^{-1} h_{t-1}^2 b_t^*,$$

$$i_m = (-1)^{m+t-1} (a-b) h_{t-1}^{-2} g_m^{-1} g_{m+1}^{-1} (z-1),$$

$$j_m = (-1)^{m+t} (a-b)^{-1} h_{t-1}^2 g_{m+1}^2 b_{m+t+1}^*.$$

Concerning such continued fractions, we studied uniform convergence in Section 5, [T2]. Related to the product formula (Proposition 1), we studied the distribution, and the simplicity of the zero points of $Q(z)$ for the Padé pairs (P, Q) for $\varepsilon(z; a, b; k)$ in Section 4, [T2].

It is an interesting problem which asks whether we can do the same for the Thue-Morse word ξ . The fact $\Lambda(\xi) = N$ (cf. [A-P-ZXW-ZYW]), we mentioned, says that all the denominators of the continued fraction for $\xi^{(-1)}(z)$ are of degree 1. It is of special interest to find the the continued fraction expansion for $\xi^{(-1)}(z)$ in a closed form.

We could not give a completely explicit formula for $H_n^{(m)}[\varepsilon(z; a, b; k)]$ when $k \geq 2$; we gave the following

Proposition 5 (cf. Corollary 6, [T2]).

$$H_{f_n}^{(0)}[\varepsilon(z; a, b; k)] = r_n (g_n a + g_{n-1} b) (a-b)^{f_n-1},$$

$$H_{f_{n+1}-1}^{(0)}[\varepsilon(z; a, b; k)] = s_n (g_n a + g_{n-1} b) (a-b)^{f_{n+1}-2} \quad (n \geq 0);$$

and $H_m^{(0)} = 0$ for all $m \neq f_n$ and $m \neq f_{n+1}-1$ ($n \geq 0$), where $r_n \neq 0$, $s_n \neq 0$ are integers independent of a , b .

4. Toeplitz cases. In this section, we consider Hankel determinants for 2-adic Toeplitz words

$$w = w_1 w_2 w_3 \dots \quad (w_i \in A).$$

Note that the numbering of the symbols starts from 1 (not from 0), cf. Sections 2, 3. Recall that a word $w = w_1 w_2 w_3 \dots$ ($w_i \in A$) is a 2-adic Toeplitz word iff

$$\text{ord}_2(m) = \text{ord}_2(n) \implies w_m = w_n.$$

Without loss of generality, we may suppose $A = \{a_0, a_1, a_2, \dots\}$. In some cases, we use symbols $a, b, c, \dots$ instead of $a_0, a_1, a_2, \dots$ as before. Recall also the universal 2-adic Toeplitz word

$$\omega = \text{abacabadabacabaeabacabadabacabaf} \dots$$

and the notation

$$H_n[\omega] := H_n^{(0)}[\omega]$$

defined in Section 1. For example, by direct calculation, we have

$$\begin{aligned} H_7[\omega] &= \begin{vmatrix} a & b & a & c & a & b & a \\ b & a & c & a & b & a & d \\ a & c & a & b & a & d & a \\ c & a & b & a & d & a & b \\ a & b & a & d & a & b & a \\ b & a & d & a & b & a & c \\ a & d & a & b & a & c & a \end{vmatrix} \\ &= -4ab^2c^4 + abc^5 - ac^6 + 16ab^2c^3d - 12abc^4d + 2ac^5d - 24ab^2c^2d^2 \\ &\quad + 8abc^3d^2 + ac^4d^2 + 16ab^2cd^3 + 8abc^2d^3 - 4ac^3d^3 - 4ab^2d^4 - 12abcd^4 \\ &\quad + ac^2d^4 + 4abd^5 + 2acd^5 - ad^6 \\ &= -a(2b-c-d)^2(c-d)^4. \end{aligned}$$

We say that a form (i.e., a homogeneous polynomial) $P \in \mathbb{Z}[A]$ is completely reducible iff $P=0$, or P can be factorized into linear forms $\in \mathbb{Z}[A]$, i.e.,

$$P = P_1 P_2 \cdots P_k \quad (\deg P_i = 1 \text{ for all } 1 \leq i \leq k).$$

One can check that $H_n[\omega]$ are non-zero completely reducible forms for small n (for $n \leq 30$ or so) by using the soft "Mathematica". This is a curious phenomenon, since, for instance, $H_2[w]$ (resp., $H_3[w]$, etc.) is not completely reducible for any word w having abc (resp., $abacd$, etc.) as its prefix of w . Related to such a phenomenon, we can show the following

Proposition 6 (Main Theorem in [M-T-Tn]). Let w be a fixed point of a substitution of constant length 2. Suppose w is a word strictly over an alphabet

consisting of at least 3 symbols. Then $H_n[w]$ is completely reducible for all $n \geq 1$ if and only if w is a 2-adic Toeplitz word.

The proof of this proposition together with something more interesting (probably) will appear in the forthcoming paper [M-T-Tn].

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