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THE φ-SEMI-COMPLETE MAXIMUM PRINCIPLE

FOR REAL CONVOLUTION KERNELS

Masayuki ITÔ
伊藤 正之

1 - Let G be a locally compact, non-compact and σ-compact abelian group (1) and ξ a fixed Haar measure on G. The potential-theoritic principles for positive convolution kernels play grand roles to the study of transient convolution semi-groups (see, for example, [3] and [8]). We know that the semi-transient convolution semi-groups are resonable (see [6]) and that the semi-complete maximum principle for real convolution kernels have a very closed connection with them (see [4], [5] and [6]).

Let φ be a given non-negative continuous function on G. Similarly as in the semi-transient convolution semi-groups, we can define the φ-semi-transient convolution semi-groups, and similarly as in the semi-complete maximum principle, we can define the φ-semi-complete maximum principle.

Let $(\alpha_t)_{t \geq 0}$ be a convolution semi-group on G. If $\varphi \neq 0$ and $(\alpha_t)_{t \geq 0}$ is φ-semi-transient and recurrent, then we shall obtain that φ is exponential (2) and that $(\frac{1}{\varphi}\alpha_t)_{t \geq 0}$ is a semi-transient and recurrent convolution semi-group.

Let N be a real convolution kernel on G (i.e., N is a real Radon measure on G). If N satisfies the φ-semi-complete maximum principle (denoted by $N \in \varphi$-SCM), then, for any open set $\omega \neq \phi$, the reduced measure $\eta_{N,\omega}^{(\varphi)}$ of N sur ω with respect to (N, φ) is defined in the natural manner. In the connection with the recurrence of convolution semi-groups, it is important to examine the case of $\eta_{N,\delta}^{(\varphi)} = -\infty$, i.e., for any non-negative continuous function $f \neq 0$ on G with compact support,

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(1) We consider that G is an additive group.

(2) A continuous function φ on G is said to be exponential if for any $x, y \in G$, $\varphi(x + y) = \varphi(x)\varphi(y)$.

$$\lim_{\substack{v \uparrow G \\ v \in \mathcal{V}}} f d\eta^{(\varphi)}_{N,Cv} = -\infty \quad (^3),$$

where $\mathcal{V}$ denotes the totality of compact neighborhoods of the origin 0.

THEOREM. Let $N \in \varphi$-SCM, and assume $\varphi(0) = 1$ and that N is not pseudo-périodic (4). If $\eta^{(\varphi)}_{N,\delta} = -\infty$, then we have (1) or (2):

(1) N is a Hunt convolution kernel and φ belongs to the closure of N-potentials $N * f$ of non-negatives continuous functions f with compact support in the topology of compact convergence.

(2) φ is > 0 and exponential, and $\frac{1}{\varphi} N$ is of form

$$\frac{1}{\varphi} N = N_0 + \psi\xi,$$

where N_0 is a convolution kernel of logarithmic type and ψ is an additive continuous function on G.

In theorem, (2) implies $\eta^{(\varphi)}_{N,\delta} = -\infty$, but we do not know if (1) implies $\eta^{(\varphi)}_{N,\delta} = -\infty$. Since (1) is not essential in the case of $N \in \varphi$-SCM and $\eta^{(\varphi)}_{N,\delta} = -\infty$, the relation between φ-semi-transient convolution semi-groups and the φ-semi-complete maximum principle for real convolution kernels is essentially reduced to that between semi-transient convolution semi-groups and the semi-complete maximum principle for real convolution kernels. This result will be suggestive when we discuss the "semi-transient diffusion semi-groups".

This note is a summary of our paper [7], and we shall omit all proofs in detail.

2 - We denote by

$C = C(G)$ the usual Fréchet space of finite continuous functions on G,

$C_K = C_K(G)$ the usual topological vector space of finite continuous functions

(3) This means that for any sequence $(v_n)_{n=1}^{\infty} \subset \mathcal{V}$ with $\mathring{v}_{n+1} \supset v_n$ and $\bigcup_{n=1}^{\infty} v_n = G$, $\lim_{n \to \infty} \int f d\eta^{(\varphi)}_{N,Cv_n} = -\infty$ (Remark that G is σ-compact).

(4) This means that, for any $0 \neq x \in G$, $N * \varepsilon_x$ is not proportional to N.

on G with compact support,

$M = M(G)$ the usual topological vector space of real Radon measures on G with the weak* topology,

$M_K = M_K(G)$ the usual topological vector space of real Radon measures on G with compact support,

C^+, C_K^+, M^+, M_K^+ their subsets of non-negative elements.

Let $(\alpha_t)_{t \geq 0}$ be a family in M^+. We say that $(\alpha_t)_{t \geq 0}$ is a convolution semi-group on G if $\alpha_0 = \varepsilon$, $\alpha_t * \alpha_s = \alpha_{t+s}$ for any $t \geq 0$, $s \geq 0$ and if $t \to \alpha_t$ is continuous in M. Here ε denotes the unit measure at the origin 0. It is said to be transient (resp. recurrent) if for any $f \in C_K^+$, $\int_0^\infty dt \int f d\alpha_t < \infty$ (resp. if there exists $f \in C_K^+$ such that $\int_0^\infty dt \int f d\alpha_t = \infty$). It is said to be markovian (resp. submarkovian) if $\int d\alpha_t = 1$ (resp. $\int d\alpha_t \leq 1$). Let $\varphi \in C^+$. A convolution semi-group $(\alpha_t)_{t \geq 0}$ is said to be φ-semi-transient if for any $\mu \in M_K$ with $\varphi * \mu(0) = 0$, $(\int_0^t \alpha_s * \mu \, ds)_{t \geq 0}$ is bounded in M. In particulier, if $\varphi \equiv 1$, $(\alpha_t)_{t \geq 0}$ is said to be semi-transient.

<u>PROPOSITION 1.</u> Let $\varphi \in C^+$ with $\varphi \neq 0$ and $(\alpha_t)_{t \geq 0}$ be a φ-semi-transient convolution semi-group on G. If $(\alpha_t)_{t \geq 0}$ is recurrent, then φ is exponential, $\varphi > 0$ on G, $(\frac{1}{\varphi}\alpha_t)_{t \geq 0}$ is a markovian convolution semi-group on G and it is semi-transient.

Hence, a convolution semi-group $(\alpha_t)_{t \geq 0}$ is markovian if it is semi-transient and recurrent.

A real convolution kernel N on G is, by definition, of logarithmic type (resp. a Hunt convolution kernel) if for any $\mu \in M_K$ with $\int d\mu = 0$ (resp. for any $\mu \in M_K^+$), $N * \mu$ is of form

$$N * \mu = \int_0^\infty dt \; \alpha_t * \mu \; dt$$

(i.e., for any finite continuous function f with compact support, $\int f dN * \mu = \lim_{a \to \infty} \int_0^a dt \int f d\alpha_t * \mu$), where $(\alpha_t)_{t \geq 0}$ is a semi-transient and recurrent convolution semi-group (resp. a transient convolution semi-group).

3 - Let $\varphi \in C^+$ and N_1, N_2 two real convolution kernels on G. We denote by $(N_1, N_2) \in \varphi$-SCM if, for any $f, g \in C_K^+$ with $\varphi * f(0) = \varphi * g(0)$ and any $a \in R$,

$$N_1 * f \leq N_1 * g + a\varphi \text{ on } \mathrm{supp}(f) \rightarrow N_2 * f \leq N_2 * g + a\varphi \text{ on } G,$$

where R denotes the set of real numbers and $\mathrm{supp}(f)$ denotes the support of f. If $N = N_1 = N_2$ and $(N, N) \in \varphi$-SCM, then we write simly $N \in \varphi$-SCM and we say that N satisfies the φ-semi-complete maximum principle. In particular, if $\varphi \equiv 1$, we write $N \in$ SCM and we say that N satisfies the semi-complete maximum principle.

We denote by $(N_1, N_2) \in \varphi$-SB (resp. $\in \varphi$-SB_g) if for any $\mu \in M_K^+$ and any relatively compact open set $\omega \neq \emptyset$ (resp. any open set $\omega \neq \emptyset$), there exist $\mu'_\omega \in M^+$ and $a_{\mu,\omega} \in R$ such that $\mathrm{supp}(\mu'_\omega) \subset \overline{\omega}$, $\varphi * \mu'_\omega(0) = \varphi * \mu(0)$, $N_1 * \mu'_\omega - a_{\mu,\omega}\varphi\xi \leq N_2 * \mu$ and $N_1 * \mu'_\omega - a_{\mu,\omega}\varphi\xi = N_2 * \mu$ in ω. If $N = N_1 = N_2$ and $(N, N) \in \varphi$-SB (resp. $\in \varphi$-SB_g), then we write simply $N \in \varphi$-SB (resp.$\in \varphi$-SB_g) and we say that N satisfies the φ-semi-balayage principle (resp. N is φ-semi-balayable). In particular, if $\varphi \equiv 1$, we write $N \in$ SB (resp. $\in SB_g$) and we say that N satisfies the semi-balayage principle (resp. N is semi-balayable).

By using the usual dual methode (see, for example, [1]), we have the following

PROPOSITION 2. Let N be a real convolution kernel on G and $\varphi \in C^+$. Then the following five statements are equivalent:

(1) $N \in \varphi$-SCM.

(2) For any positive number c, $(N + c\varepsilon, N) \in \varphi$-SCM.

(3) $\check{N} \in \check{\varphi}$-SCM.

(4) $N \in \varphi$-SB.

(5) For any positive number c, $(N + c\varepsilon, N) \in \varphi$-SB.

For any function g on G, we put $\check{g}(x) = g(-x)$ and denote by $\check{N}$ the real convolution kernel defined by $\int f d\check{N} = \int \check{f} dN$ for all $f \in C_K$.

From Proposition 2 follows the following

PROPOSITION 3. Let $N \in \varphi$-SCM. For a relatively compact open set $\omega \neq \emptyset$ in G, there exist a family $(V_{\omega,p})_{p>0}$ of universally measurable non-negative linear operators from M_K to $M_b(\omega) = \{\mu \in M(G);\ \mu(C\omega) = 0\}$ and a family $(g_{\omega,p})_{p>0}$ of universally measurable non-negative linear functionals on M_K such that:

(a) For any $p > 0$ and any $\mu \in M_K$, $(pV_{\omega,p}\mu) * \varphi(0) = \mu * \varphi(0)$ and

$$(pN + \varepsilon) * V_{\omega,p}\mu - g_{\omega,p}(\mu)\varphi\xi = N * \mu \text{ in } \omega.$$

(b) For any $p > 0$, $q > 0$ and any $\mu \in M_K$,

$$V_{\omega,p}\mu - V_{\omega,q}\mu = (q-p)V_{\omega,p}(V_{\omega,q}\mu)$$

and

$$g_{\omega,p}(\mu) - g_{\omega,q}(\mu) = (q-p)g_{\omega,p}(V_{\omega,q}\mu) = (q-p)g_{\omega,q}(V_{\omega,p}\mu).$$

In this case, $(V_{\omega,p}, g_{\omega,p})$ is uniquely determined for all $p > 0$.

Here $V_{\omega,p}$ (resp. $g_{\omega,p}$) is said to be universally measurable if for any $f \in C_K$, the function $\int f dV_{\omega,p}\varepsilon_x$ of x is universally measurable (resp. the function $g_{\omega,p}(\varepsilon_x)$ of x is universally measurable), where ε_x denotes the unit measure at x.

For an open set $\omega \neq \emptyset$ and $\mu \in M_K^+$, we put

$$P(N * \mu, \omega) = \overline{\{N * \nu;\ \nu \in M_K^+,\ \mathrm{supp}(\nu) \subset \omega,\ \varphi * \nu(0) = \varphi * \mu(0)\}},$$

where the closure is in the sense of the weak* topology, and

$$R(N * \mu, \omega) = \{\eta - a\varphi\xi;\ a \in R,\ \eta - a\varphi\xi \leq N * \mu\}.$$

PROPOSITION 4. Let $N \in \varphi$-SCM. Then there exists the maximum element $\eta^{(\varphi)}_{N*\mu,\omega} = \tilde{\eta}^{(\varphi)}_{N*\mu,\omega} - a_{\mu,\omega}\varphi\xi$ in $R(N*\mu, \omega)$, where $\tilde{\eta}^{(\varphi)}_{N*\mu,\omega} \in P(N*\mu, \omega)$ and $a_{\mu,\omega} \in R$, and $\eta^{(\varphi)}_{N*\mu,\omega}$ and $\tilde{\eta}^{(\varphi)}_{N*\mu,\omega}$ are uniquely determined.

We say that $\eta^{(\varphi)}_{N*\mu,\omega}$ is the reduced measure of $N*\mu$ on ω with respect to (N, φ) and that $\tilde{\eta}^{(\varphi)}_{N*\mu,\omega}$ is the pseudo-reduced measure of $N*\mu$ on ω with respect to (N, φ).

Let $N \in \varphi$-SCM and $(v_n)_{n=1}^{\infty} \subset \mathcal{V}$ is an exhaustion of G, i.e., $\mathring{v}_{n+1} \supset v_n$ and $\bigcup_{n=1}^{\infty} v_n = G$, where $\mathring{v}_{n+1}$ denotes the interior of v_{n+1}. Then $(\eta^{(\varphi)}_{N*\mu,Cv_n})_{n=1}^{\infty}$ is decreasing for all $\mu \in M_K^+$ and $\lim_{n\to\infty} {}^{(\varphi)}_{N*\mu,Cv_n}$ is a real Radon measure or for any $0 \neq f \in C_K^+$, $\lim_{n\to\infty}\int f d\eta^{(\varphi)}_{N*\mu,Cv_n} = -\infty$. Put

$$\eta^{(\varphi)}_{N*\mu,\delta} = \begin{cases} \lim_{n\to\infty}\eta^{(\varphi)}_{N*\mu,Cv_n} & \text{if for } f \subset C_K^+,\ \lim_{n\to\infty}\int f d\eta^{(\varphi)}_{N*\mu,Cv_n} > -\infty \\ -\infty & \text{if for } 0 \neq f \in C_K^+,\ \lim_{n\to\infty}\int f d\eta^{(\varphi)}_{N*\mu,Cv_n} = -\infty \end{cases};$$

then $\eta^{(\varphi)}_{N*\mu,\delta}$ is independent from the choice of $(v_n)_{n=1}^{\infty}$.

Let $B_K(\xi)$ be the space of bounded ξ-measurable functions on G with compact support and $B_K^+(\xi)$ its subset of non-negative functions.

PROPOSITION 5. Let $\varphi \in C^+$ with $\varphi(0) > 0$. Then $\eta^{(\varphi)}_{N,\delta} = -\infty$ if and only if, for any $g \in B_K^+(\xi)$, $\eta^{(\varphi)}_{N*(g\xi),\delta} = -\infty$.

PROPOSITION 6. Let $\varphi \in C^+$ and $N \in \varphi$-SCM. If for any $f \in B_K^+(\xi)$, $\eta^{(\varphi)}_{N*(f\xi),\delta} \neq -\infty$, then $\eta^{(\varphi)}_{N*(f\xi),\delta}$ is absolutely continuous with respect to ξ. Denote by $R_\delta f$ its density. Then

$$V_N : B_K(\xi) \ni f \to N*f - R_\delta f^+ + R_\delta f^-$$

satisfies the domination principle, i.e., for any $f, g \in B_K^+(\xi)$, $V_N f \leq V_N g$ ξ-a.e. on $\mathrm{supp}(f\xi) \to V_N f \leq V_N g$ ξ-a.e. on G. Here $N*f$ denotes the density of $N*(f\xi)$.

In the case of $\eta^{(\varphi)}_{N*(f\xi),\delta} \neq -\infty$ for all $f \in B_K^+(\xi)$, the φ-semi-complete maximum principle for N results principally from the domination principle for

V_N. But we omit the precise discussion of this case.

PROPOSITION 7. Let $N \in \varphi$-SCM and assume that $\eta^{(\varphi)}_{N * (f\xi),\delta} = -\infty$ for all $0 \neq f \in B^+_K(\xi)$. Then for an open exhaustion $(\omega_n)_{n=1}^{\infty}$ of G (5) and any $f \in B_K(\xi)$, $\lim_{n\to\infty} V_{\omega_n,p}(f\xi)$ and $\lim_{n\to\infty} g_{\omega_n,p}(f\xi)$ exist for all $p > 0$. Put

$$V_p(f\xi) = \lim_{n\to\infty} V_{\omega_n,p}(f\xi) \quad\text{and}\quad g_p(f\xi) = \lim_{n\to\infty} g_{\omega_n,p}(f\xi);$$

then, for any $p > 0$, $q > 0$ and any $f \in B_K(\xi)$,

$$(pN + \varepsilon) * V_p(f\xi) - g_p(f\xi)\varphi\xi = N * (f\xi),\ V_p(f\xi) - V_q(f\xi) = (q - p)V_p(V_q(f\xi))$$

and

$$g_p(f\xi) - g_q(f\xi) = (q - p)g_p(V_q(f\xi)) = (q - p)g_q(V_p(f\xi)).$$

PROPOSITION 8. Let $\varphi \in C^+$ with $\varphi(0) = 1$ and $N \in \varphi$-SCM. If there exist $g \in B^+_K(\xi)$ and an exhaustion $(v_n)_{n=1}^{\infty}$ of G contained in $\mathcal{V}$ such that $\lim_{n\to\infty} \tilde{\eta}^{(\varphi)}_{N * (g\xi),Cv_n} = -\infty$ (i.e., for any $0 \neq f \in C^+_K$, $\lim_{n\to\infty} \int f d\tilde{\eta}^{(\varphi)}_{N,Cv_n} = -\infty$), then φ is exponential and for any $p > 0$, V_p is a convolution kernel on G, i.e., there exists a positive Radon measure N_p on G such that for any $\mu \in M^+_K$, $V_p\mu = N_p * \mu$.

For the proof of this proposition, we use the well-known Choquet-Deny theorem (see [2]) and the following lemma.

LEMMA 9. Under the same conditions in Proposition 8, we have, for any $p > 0$ and ξ-almost all $x \in G$ with $\varphi(x) > 0$,

$$\check{\varphi}(x) = p\ \varphi * V_p\varepsilon_x(0).$$

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(5) This means that ω_n is a relatively compact open set, $\overline{\omega_n} \subset \omega_{n+1}$ and $\bigcup_{n=1}^{\infty} \omega_n = G$.

PROPOSITION 10. Let $\varphi \in C^{+}$ with $\varphi(0) > 0$ and $N \in \varphi$-SCM. Assume that $\eta_{N,\delta}^{(\varphi)} = -\infty$ and that $(\tilde{\eta}_{N,Cv}^{(\varphi)})_{v \in \mathcal{V}}$ is bounded in M. Then, for any $f \in B_K^{+}(\xi)$, $V_p(f\xi) \uparrow N * (f\xi)$ as $p \downarrow 0$.

4 - Our main theorem is followed from Proposition 8 and Proposition 10. Assume that $(\tilde{\eta}_{N,Cv}^{(\varphi)})_{v \in \mathcal{V}}$ is unbounded. Then $N \in \varphi$-SCM shows that the hypothese in Proposition 9 is verified. Applying théorème 2 in [5] to $\frac{1}{\varphi}N$, we have (2) in Theorem. In this case, $\eta_{N,Cv}^{(\varphi)} = \tilde{\eta}_{N,Cv}^{(\varphi)}$ (see [4] or [5]). If $(\tilde{\eta}_{N,Cv}^{(\varphi)})_{v \in \mathcal{V}}$ is bounded, Proposition 10 gives (1) in Theorem. Our Theorem, Proposition 8 and Proposition 9 give the following

COROLLARY 11. Let $\varphi \in C^{+}$ with $\varphi(0) = 1$ and N be a real convolution kernel on G. Then, N is of form in Theorem, (2), if and only if $N \in \varphi$-SCM, N is not pseudo-periodic and $\lim\limits_{\substack{v \uparrow G \\ v \in \mathcal{V}}} \tilde{\eta}_{N,Cv}^{(\varphi)} = -\infty$.

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Masayuki ITÔ

Department of Mathematics
Faculty of Sciences
Nagoya University
Furo-cho, chikusa-ku
Nagoya, 464, Japan