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Lifts of holonomy representations and the volume of a knot complement

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1 Introduction

A fundamental invariant of a knot is its Alexander polynomial ([1]). It has been studied for a long time from many viewpoints as a fundamental and important knot invariant. In 1990, Lin ([12]) introduced the twisted Alexander polynomial associated with a knot K and a representation $\pi_1(S^3 \setminus K) \to \mathrm{GL}(m, \mathbb{F})$ using its Seifert surface, where $\mathbb{F}$ is a field. Subsequently, Wada ([19]) showed a method to define it using only a presentation of a group. Via its interpretations as the Reidemeister torsion by Kitano ([8]) and Kirk-Livingston ([7]), the twisted Alexander polynomial is a mathematical object which is investigated from various viewpoints now.

A 3-manifold which admits a complete Riemannian metric with sectional curvature -1 at each interior point is said to be *hyperbolic*. If a knot complement becomes a hyperbolic manifold, the knot is called a *hyperbolic knot*. It was shown by Thurston that a knot which is neither a torus knot nor a satellite knot is hyperbolic, and almost all knots are hyperbolic in the feeling. By the Mostow's rigidity theorem, which includes that the hyperbolic structure of a hyperbolic manifold is unique, we know the volume of a knot complement is an invariant of the knot.

There are several researches on estimates of the volume of a knot complement recently. In this note we consider the twisted Alexander polynomial of a hyperbolic knot associated with the representation given by the composition of the lift of the holonomy representation to $\mathrm{SL}(2, \mathbb{C})$ and the higher-dimensional, irreducible, complex representation of $\mathrm{SL}(2, \mathbb{C})$. Then we study a relationship between its asymptotic behavior and the volume of the knot.

2 Alexander polynomials

There are some methods to define the Alexander polynomial. Here we introduce the definition using a presentation of the fundamental group of a knot complement and the free differential calculus devised by Fox. Let K be a knot in the 3-sphere S^3. Fix a Wirtinger presentation of the knot group $G(K) = \pi_1(E(K)) = \pi_1(S^3 - \mathrm{Int}N(K))$:

$$P = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1} \rangle.$$

We denote by $\phi : F_n \to G(K)$ the epimorphism from the free group associated with P to $G(K)$, and by $\widetilde{\phi} : \mathbb{Z}F_n \to \mathbb{Z}G(K)$ the ring homomorphism which is obtained from ϕ by extending linearly. Let $\alpha : G(K) \to H_1(E(K);\mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$ be the abelianization homomorphism. It is given by $\alpha(x_1) = \cdots \alpha(x_n) = t$ since P is a Wirtinger presentation. By extending linearly, we have a homomorphism between group rings: $\widetilde{\alpha} : \mathbb{Z}G(K) \to \mathbb{Z}[t, t^{-1}]$. We denote by Φ the composed mapping $\widetilde{\alpha} \circ \widetilde{\phi}$, that is,

$$\Phi : \mathbb{Z}F_n \to \mathbb{Z}[t, t^{-1}].$$

The map $\dfrac{\partial}{\partial x_j} : \mathbb{Z}F_n \to \mathbb{Z}F_n$ is the linear extension of the map defined on the elements of F_n by (1) $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$, (2) $\dfrac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij} x_i^{-1}$, (3) $\dfrac{\partial (uv)}{\partial x_j} = \dfrac{\partial u}{\partial x_j} + u \dfrac{\partial v}{\partial x_j}$. This is called the *Fox's free differential.* We obtain a matrix whose size is $(n-1) \times n$:

$$A = \left(\Phi(\frac{\partial r_i}{\partial x_j}) \right) \quad \in M(n-1, n; \mathbb{Z}[t, t^{-1}])$$

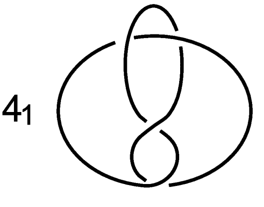
by applying the Fox's free differential to the relations $r_1, \ldots, r_{n-1}$ of the Wirtinger presentation P and composing Φ. The matrix A is called the *Alexander matrix* associated with the presentation P of the knot group $G(K)$.

We denote by A_j obtained from A by deleting the j column of A. This becomes a square matrix and we define the *Alexander polynomial* of a knot K by

$$\Delta_K(t) = \det A_j \quad \in \mathbb{Z}[t, t^{-1}].$$

It is known that this becomes a knot invariant up to $\pm t^s$ $(s \in \mathbb{Z})$.

Example 2.1. The knot illustrated below is called the *figure eight knot* and it is known as a hyperbolic knot. The knot number is 4_1.



Its knot group has the next presentation, which is a Wirtinger presentation:

$$G(K) = \langle x,\, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle.$$

We apply the Fox's free differential to the relation: $r = xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}$, then we have:

$$\begin{aligned}
\frac{\partial}{\partial x} r &= \frac{\partial x}{\partial x} + x\frac{\partial}{\partial x}(y^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}) \\
&= 1 + x\Big(\frac{\partial y^{-1}}{\partial x} + y^{-1}\frac{\partial}{\partial x}(x^{-1}yxy^{-1}xyx^{-1}y^{-1})\Big) \\
&= 1 + xy^{-1}\Big(\frac{\partial x^{-1}}{\partial x} + x^{-1}\frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1})\Big) \\
&= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}\frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1}) = \cdots = \\
&= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}.
\end{aligned} \tag{2.1}$$

Similarly,

$$\frac{\partial}{\partial y} r = -xy^{-1} + xy^{-1}x^{-1} - xy^{-1}x^{-1}yxy^{-1} + xy^{-1}x^{-1}yxy^{-1}x - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}.$$

Since $\alpha(x) = \alpha(y) = t$, we have:

$$\begin{aligned}
\Phi\Big(\frac{\partial r}{\partial x}\Big) &= 1 - tt^{-1}t^{-1} + tt^{-1}t^{-1}t + tt^{-1}t^{-1}ttt^{-1} - tt^{-1}t^{-1}ttt^{-1}ttt^{-1} \\
&= 1 - \frac{1}{t} + 1 + 1 - t = -\frac{1}{t} + 3 - t, \\
\Phi\Big(\frac{\partial r}{\partial y}\Big) &= -1 + t^{-1} - 1 + t - 1 = \frac{1}{t} - 3 + t.
\end{aligned}$$

Thus the Alexander matrix is the matrix of 1×2: $\begin{pmatrix} -\frac{1}{t} + 3 - t & \frac{1}{t} - 3 + t \end{pmatrix}$, and the Alexander polynomial of the figure eight knot K is:

$$\Delta_K(t) = \det\Big(-\frac{1}{t} + 3 - t\Big) = -\frac{1}{t} + 3 - t$$

(up to $\pm t^s$ $(s \in \mathbb{Z})$).

Originally x and y are different generators in the knot group, but they are sent to the same element t by the map α. It makes the calculation easy while this process might reduce some information included in knot groups. The twisted Alexander polynomial, which is introduced in the following section, improves this point.

3 Twisted Alexander polynomials

We use the same nations as in the previous sections. Let $\rho : G(K) \to \mathrm{SL}(m, \mathbb{C})$ be a representation of a knot group $G(K)$. This map induces naturally the map between group rings:

$$\widetilde{\rho} : \mathbb{Z}G(K) \to M(m; \mathbb{C}),$$

moreover by taking the tensor product with the map $\widetilde{\alpha}$ induced in the previous section, we have:

$$\widetilde{\rho} \otimes \widetilde{\alpha} : \mathbb{Z}G(K) \to M(m, \mathbb{C}[t, t^{-1}]).$$

Set Φ :

$$\Phi = (\widetilde{\rho} \otimes \widetilde{\alpha}) \circ \widetilde{\phi} : \mathbb{Z}F_n \to M(m; \mathbb{C}[t, t^{-1}])$$

by composing $\widetilde{\phi}$ defined in the previous section. Suppose A_ρ is the $(n-1) \times n$ matrix whose the (i, j) element is the $m \times m$ matrix:

$$\Phi\Big(\frac{\partial r_i}{\partial x_j}\Big) \quad \in M((n-1)m \times nm; \mathbb{C}[t, t^{-1}]).$$

This matrix is called the *twisted Alexander matrix associated with* ρ. In order to make a square matrix, we delete from A_ρ 'one column' corresponding to a generator x_k in the presentation P, so that we have a $(n-1)m \times (n-1)m$ matrix, which is denoted by $A_{\rho,k}$. We define the *twisted Alexander polynomial* as:

$$\Delta_{K,\rho}(t) = \frac{\det A_{\rho,k}}{\det \Phi(x_k - 1)}.$$

Here we assume $\det \Phi(x_k - 1) \neq 0$.

Wada proved the following theorem in [19].

Theorem 3.1 ([19]). *Let K be a knot and $G(K)$ the knot group. Suppose ρ is a representation of $G(K)$. The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is an invariant for the pair $(G(K), \rho)$ up to $\pm t^s$ $(s \in \mathbb{Z})$.*

Example 3.2. Let K be the figure eight knot, then the knot group $G(K)$ has the following presentation as in Example 2.1:

$$G(K) = \langle x,\, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle. \tag{3.1}$$

Define

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ \frac{-1+\sqrt{-3}}{2} & 1 \end{pmatrix}. \tag{3.2}$$

Then we can confirm that ρ becomes a representation from $G(K)$ to $\mathrm{SL}(2, \mathbb{C})$. Set $\rho(x) = X, \rho(y) = Y$, then we have : $\Phi(\dfrac{\partial r}{\partial x}) =$

$$I - \frac{1}{t}XY^{-1}X^{-1} + XY^{-1}X^{-1}Y + XY^{-1}X^{-1}YXY^{-1} - tXY^{-1}X^{-1}YXY^{-1}XYX^{-1},$$

where I is the identity matrix of size 2×2. Note that this can be obtained from (2.1) by changing $x \to X$, $y \to Y$ and $1 \to I$ with t to the appropriate power. Calculate these matrices, then we have:

$$\Delta_{K,\rho}(t) = \frac{\det \Phi\big(\frac{\partial r}{\partial x}\big)}{\det \Phi(y-1)} = \frac{1/t^2(t-1)^2(t^2-4t+1)}{(t-1)^2} \doteq t^2 - 4t + 1.$$

This seems to make up for the lack of the information caused by going through the map α. However the twisted Alexander polynomial depends on not only $G(K)$ but also ρ, so it might not be called a knot invariant, namely, it is hard to use for distinguishing two given knots. Furthermore, is is not easy to find a representation of a knot group in general. Therefore the thinkable ways to apply might be (1) to find a property of a knot satisfied for any representation or (2) to consider the restricted representation. I think an example of the former case is to determine a non-fibered knot by using any unimodular representation, i.e., we gave the theorem in [6] which states that the twisted Alexander polynomials of fibered knots are monic for any unimodular representation. See [3, 15] for the researches which followed this theorem. In the following sections, we will consider the twisted Alexander polynomial associated with the holonomy representation of a hyperbolic knot, which corresponds to the case (2) above.

For the details of basic notations and conceptions on the twisted Alexander polynomial, see [10]. See [4, 9] for its recent researches.

4 On hyperbolic knots

We refer [11, 18] for the former half in this section.

We regard the upper half space model $\mathbb{H}^3$ as a subspace of the quaternion field and set

$$\mathbb{H}^3 = \{(x + y\,i) + t\,j \ \in \mathbb{C} + \mathbb{R}\,j \mid t > 0\}$$

where $1, i, j$ are the part of basis, $i = \sqrt{-1}$, and we suppose $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. We give the metric

$$ds^2 = \frac{1}{t^2}(dx^2 + dy^2 + dt^2)$$

to $\mathbb{H}^3$ then we call this $\mathbb{H}^3$ *the 3-dimensional hyperbolic space*. It is known that the orientation preserving isometric transformation group of $\mathbb{H}^3$ is:

$$\mathrm{PSL}(2, \mathbb{C}) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \Big\} / \Big\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big\}.$$

Here the action on $\mathbb{H}^3$ of $\mathrm{PSL}(2, \mathbb{C})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = (aw + b)(cw + d)^{-1} \quad (w \in \mathbb{H}^3).$$

We calculate the right-hand side as elements of the quaternion field. The isometric transformation of $\mathbb{H}^3$ is a conformal mapping, and the action of $\mathrm{PSL}(2, \mathbb{C})$ on $\mathbb{H}^3$ is transitive. Moreover its stabilizer of a point is $\mathrm{PSU}(2, \mathbb{C})(\cong \mathrm{SO}(3))$, that is, if $f(p)$ and the mapping between tangent spaces $T_p\mathbb{H}^3 \to T_{f(p)}\mathbb{H}^3$ are given for $f \in \mathrm{PSL}(2, \mathbb{C})$ and a point $p \in \mathbb{H}^3$, then f may be determined uniquely. Thus, if the image of the neighborhood of a point by the isometric transformation is given, then one can extend the mapping to the whole space $\mathbb{H}^3$ uniquely. Further, there exists uniquely the transformation $f \in \mathrm{PSL}(2,\mathbb{C})$ such that f transforms any 3 points $p_1, p_2, p_3 \in \partial\mathbb{H}^3$ into any 3 points $p'_1, p'_2, p'_3 \in \partial\mathbb{H}^3$.

Let M be a 3-dimensional differentiable manifold. If M has a local coordinate such that a neighborhood of each point is homeomorphic to an open set in $\mathbb{H}^3$ and the coordinate transformation can be written in an element of $\mathrm{PSL}(2,\mathbb{C})$, we call M a *hyperbolic* 3-manifold. This is the same concept defined in Section 1. For a simply-connected hyperbolic 3-manifold M', we may define the developing map from M' to $\mathbb{H}^3$ as follows. Give a local coordinate of a neighborhood of the base point in M'. For any point p we take a path from the base point to p and a sequence of local coordinates along the path. We may have the image of p by the developing map by determining the image of the coordinate functions corresponding to the sequence in order. (It does not depend on the way to take a path.) Let γ be an element of the fundamental group of a hyperbolic 3-manifold M and $\widetilde{\gamma}$ a lift of γ to a universal covering $\widetilde{M}$ of M. We call the homomorphism $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ the *holonomy representation* of M if $\rho(\gamma)$ is the element ($\in \mathrm{PSL}(2,\mathbb{C})$) which maps the image by the developing map of the neighborhood of the base point of $\widetilde{\gamma}$ to that of the neighborhood of the end point of $\widetilde{\gamma}$. Let Γ be the image $\rho(\pi_1(M))$ for the holonomy representation ρ of a complete hyperbolic 3-manifold M, then Γ acts $\mathbb{H}^3$ naturally and M is homeomorphic to $\mathbb{H}^3/\Gamma$. Therefore the classification of complete hyperbolic 3-manifolds is equivalent essentially to that of a kind of discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$, so we may think the geometrical information of a complete hyperbolic 3-manifold is included in Γ. It is shown by Thurston that a holonomy representation can be lift to $\mathrm{SL}(2,\mathbb{C})$ representation, and in [2] it is proved that the lift has a one-to-one correspondence to the spin structure of M. Let η be a spin structure of M. Then we have the following homomorphism:

$$\mathrm{Hol}_{(M,\eta)} : \pi_1(M,\eta) \to \mathrm{SL}(2,\mathbb{C}).$$

If a submanifold of a hyperbolic 3-manifold is homeomorphic to the direct product of the 2-dimensional torus and the half-line, the submanifld is called a *cusp*. A 3-manifold M with a cusp is non-compact, and we obtain from M a compact 3-manifold whose boundary is a torus by getting rid of a neighborhood of the cusp. It is known that a complete hyperbolic 3-manifold with finite volume is a closed 3-manifold or a 3-manifold with cusps. In particular, a knot K (L resp.) is said to be *hyperbolic* if $S^3 - K$ ($S^3 - L$ resp.) admits the structure of the hyperbolic 3-manifold with a cusp (cusps resp.). A knot which is neither a torus knot nor a satellite knot is hyperbolic.

In the case of a knot in S^3, we let $A_1, \ldots, A_n$ be the images of generators $a_1, \ldots, a_n$ of a Wirtinger presentation of $G(K)$ by the holonomy representation ρ, then their lifts to $\mathrm{SL}(2,\mathbb{C})$ are $A_1, \ldots, A_n$ or $-A_1, \ldots, -A_n$ (Corollary 2.3 in [14]). We denote by $\rho^{\pm}(a_i) = \pm A_i(\in$

$\mathrm{SL}(2, \mathbb{C})$) for the lifts of the holonomy representation $\rho(a_i) = A_i (\in \mathrm{PSL}(2, \mathbb{C}))$.

5 Irreducible SL$(m, \mathbb{C})$-representations of SL$(2, \mathbb{C})$

We review irreducible representations of $\mathrm{SL}(2, \mathbb{C})$ briefly. The vector space $\mathbb{C}$ has the standard action of $\mathrm{SL}(2, \mathbb{C})$. It is known that the symmetric product $\mathrm{Sym}^{m-1}(\mathbb{C}^2)$ and the induced action by $\mathrm{SL}(2, \mathbb{C})$ give an m-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$. We can identify $\mathrm{Sym}^{m-1}(\mathbb{C}^2)$ with the vector space of homogeneous polynomials on $\mathbb{C}^2$ with degree $m-1$, namely,

$$V_m = \mathrm{span}_{\mathbb{C}} \langle x^{m-1}, x^{m-2}y, \ldots, xy^{m-2}, y^{m-1} \rangle.$$

The action of $A \in \mathrm{SL}(2, \mathbb{C})$ is expressed as

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p\left(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where $p \begin{pmatrix} x \\ y \end{pmatrix}$ is a homogeneous polynomial and the variables in the right-hand side are determined by the action of A^{-1} on the column vector as a matrix multiplication. We denote by (V_m, σ_m) the representation given by the above action of $\mathrm{SL}(2, \mathbb{C})$ where σ_m denotes the homomorphism from $\mathrm{SL}(2, \mathbb{C})$ into $\mathrm{GL}(V_m)$. It is known that (1) each representation (V_m, σ_m) turns into an irreducible $\mathrm{SL}(m, \mathbb{C})$-representation and (2) every irreducible m-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$ is equivalent to (V_m, σ_m).

Let M be a complete hyperbolic 3-manifold and $\mathrm{Hol}_{(M,\eta)}$ the homomorphism defined in Section 4. By composing $\mathrm{Hol}_{(M,\eta)}$ and σ_m, we have the representation:

$$\rho_m : \pi_1(M) \to \mathrm{SL}(m, \mathbb{C}).$$

Example 5.1. It is known that the map given by (3.2) in Example 3.2 is the holonomy representation of the figure eight knot K. To avoid the reduplication, let a and b be generators of $G(K)$ instead of x and y in the group presentation (3.1) and set:

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ \frac{-1+\sqrt{-3}}{2} & 1 \end{pmatrix}.$$

Note that these are the elements in $\mathrm{SL}(2, \mathbb{C})$. Since $(x-y)^2 = x^2 - 2xy + y^2$, $(x-y)y = xy - y^2$, $y^2 = y^2$, we have the next matrix by taking the coefficients:

$$\rho_3(a) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^T.$$

By setting $u = \dfrac{-1+\sqrt{-3}}{2}$ and calculating similarly, we obtain:

$$\rho_3(b) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ u^2 & 2u & 1 \end{pmatrix}^T, \ \rho_4(a) = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T, \ \rho_4(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & 1 & 0 & 0 \\ u^2 & 2u & 1 & 0 \\ u^3 & 3u^2 & 3u & 1 \end{pmatrix}^T.$$

Here $(\cdot)^T$ means the transposed matrix.

6 Main Theorem and the outline of the proof

Let K be a hyperbolic knot, and ρ_m the $\mathrm{SL}(m,\mathbb{C})$ -representation which is obtained from the holonomy representation of $G(K)$ by the method described in Sections 4 and 5. Set:

$$\mathcal{A}_{K,2k}(t) = \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}; \quad \mathcal{A}_{K,2k+1}(t) = \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}. \tag{6.1}$$

Our main result is the following:

Theorem 6.1 ([5]).

$$\lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2} = \frac{\mathrm{Vol}(K)}{4\pi}.$$

As in (6.1), $\mathcal{A}_{K,\rho_m}$ is defined by dividing the principal part, but it is inessential, especially in the case of m even. We may describe as follows if there is no corrections:

- $\displaystyle\lim_{k\to\infty} \frac{\log|\Delta_{K,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}$;
- $\displaystyle\lim_{k\to\infty} \frac{1}{(2k+1)^2}\Big(\log\Big(\lim_{t\to 1}\Big|\frac{\Delta_{K,2k+1}(t)}{t-1}\Big|\Big)\Big) = \frac{\mathrm{Vol}(K)}{4\pi}$.

In the next section, we give sample calculations of the figure eight knot. As shown there the volume of a knot complement can be approximated using a kind of a combinatorial method. The crucial points are the results of Müller: one of them states the analytic torsion and the Reidemeister torsion are the same essentially for unimodular representations ([16]) and the other gives the volume formula using the analytic torsion ([17]) for a closed complete hyperbolic 3-manifold. Thus, combing them, we are able to have a volume formula for a closed complete hyperbolic 3-manifold using the Reidemeister torsion. Applying the Thurston's hyperbolic Dehn surgery theorem to these Müller's works, Menal-Ferrer and Porti gave a volume formula for a complete hyperbolic 3-manifold with cusps in [14] (see Theorem 6.4), so we have only to make clear the relation between the Reidemeister torsion and the twisted Alexander polynomial.

Let us review some results of Menal-Ferrer and Porti. Let M be an oriented complete hyperbolic 3-manifold whose boundary is one torus cusp, i.e., we will consider M with $\partial\overline{M} = T^2$.

Proposition 6.2 ([13]). (1) *If m is even, then* $\dim_{\mathbb{C}} H_i(M;\rho_m) = 0$ *for any i.*

(2) *If m is odd, then* $\dim_{\mathbb{C}} H_0(M;\rho_m) = 0$ *and* $\dim_{\mathbb{C}} H_i(M;\rho_m) = 1$ *for $i = 1, 2$.*

Proposition 6.3 ([14]). *Suppose m is odd and let $G < \pi_1(M)$ be some fixed realization of the fundamental group of T as a subgroup of $\pi_1(M)$. Choose a non-trivial cycle $\theta \in H_1(T;\mathbb{Z})$, and a non-trivial vector $v \in V_m$ fixed by $\rho_m(G)$. If $i : T \to M$ denotes the inclusion, then the following assertions hold.*

(1) *A basis for $H_1(M;\rho_m)$ is given by $i_*([v \otimes \theta])$.*

(2) *Let $[T] \in H_2(T;\mathbb{Z})$ be a fundamental class of T. A basis for $H_2(M;\rho_m)$ is given by $i_*([v \otimes T])$.*

Using the above notations, we set:

$$\mathcal{T}_{2k+1}(M) = \frac{\mathrm{Tor}(M;\rho_{2k+1};\theta)}{\mathrm{Tor}(M;\rho_3;\theta)};$$
$$\mathcal{T}_{2k}(M) = \frac{\mathrm{Tor}(M;\rho_{2k})}{\mathrm{Tor}(M;\rho_2)}.$$

Here $\mathrm{Tor}(\cdot)$ means the Reidemeister torsion.

Theorem 6.4 ([14]).

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k}(M)|}{(2k)^2} = \frac{\mathrm{Vol}(M)}{4\pi}.$$

As in Proposition 6.2 (1), the twisted homology vanishes in the case that m is even. In such a case the corresponding chain complex is said to be *acyclic* and it is easy relatively to discuss the Reidemeister torsion. Let M be the complement $E(K)$ of a knot K. It is proved by Kitano ([8]) that the Reidemeister torsion can be obtained from the twisted Alexander polynomial by evaluating $t = 1$ in this case, that is,

$$\mathrm{Tor}(M;\rho_{2k}) = \Delta_{K,\rho_{2k}}(1).$$

Thus we get the even case of our main result via Theorem 6.4

The representation obtained from the adjoint action of the $\mathrm{SL}(2,\mathbb{C})$-representation of a fundamental group is the same as ρ_3 in our setting essentially. The next proposition is a generalization of the Yamaguchi's theorem ([20, 21]) which treats the adjoint action of the $\mathrm{SL}(2,\mathbb{C})$-representation of a fundamental group. We restrict the base θ in Proposition 6.3 to a longitude λ and handle it well, so that we have this proposition:

Proposition 6.5 ([5]). *Let λ be a longitude of a knot K and M the complement of K, then the following equation holds:*

$$|\mathrm{Tor}(M;\rho_{2k+1};\lambda)| = \lim_{t\to 1} \frac{|\Delta_{K,\rho_{2k+1}}(t)|}{t-1}.$$

The odd case in our main result follows from the proposition.

7 Some calculations

Here we give some calculations on the figure eight knot K. It is known that the volume of the complement of K is equal to $2.0298832\cdots$.

We use the lifts $\rho^+(a)=\begin{pmatrix}1&1\\0&1\end{pmatrix}, \rho^+(b)=\begin{pmatrix}1&0\\\frac{-1+\sqrt{-3}}{2}&1\end{pmatrix}$, stated in Example 5.1, and we proceed the calculation in Example 3.2, then we have:

$\Delta_{K,\rho_2^+}(t)=\frac{1}{t^2}(t^2-4t+1), \Delta_{K,\rho_3^+}(t)=-\frac{1}{t^3}(t-1)(t^2-5t+1), \Delta_{K,\rho_4^+}(t)=\frac{1}{t^4}(t^2-4t+1)^2.$

In the same way, we can have:

$$\Delta_{K,\rho_5^+}(t)=-\frac{1}{t^5}(t-1)(t^4-9t^3+44t^2-9t+1).$$

We denote by $\mathcal{A}^+_{K,m}$ the corresponding $\mathcal{A}_{K,m}$ with ρ^+, so we obtain:

$$\frac{4\pi\log|\mathcal{A}^+_{K,4}(t)|}{4^2}=\frac{\pi\log|t^2-4t+1|}{4}\xrightarrow{t=1}\frac{\pi\log 2}{4}\approx 0.544397\cdots;$$
$$\frac{4\pi\log|\mathcal{A}^+_{K,5}(t)|}{5^2}=\frac{\pi\log\left|\frac{t^4-9t^3+44t^2-9t+1}{t^2-5t+1}\right|}{5^2}\xrightarrow{t=1}\frac{4\pi\log\frac{28}{3}}{5^2}\approx 1.12273\cdots.$$

The following is the results using by a computer. The symbol $\mathcal{A}^-_{K,m}$ corresponds to the lift of the holonomy representation of K:

$$\rho^-(a)=-\begin{pmatrix}1&1\\0&1\end{pmatrix};\quad \rho^-(b)=-\begin{pmatrix}1&0\\\frac{-1+\sqrt{-3}}{2}&1\end{pmatrix}.$$

Note that $\mathcal{A}^+_{K,m}(t)=\mathcal{A}^-_{K,m}(t)$ when m is odd. Mr. Tetsuya Takahashi helped me to calculate these and we used the softwares Wolfram Mathematica and MathWorks Matlab. It took about $4\sim5$ hours to compute in the degree 33 case.

m(even)	$\frac{4\pi\log\lvert\mathcal{A}^+_{K,m}(1)\rvert}{m^2}$	$\frac{4\pi\log\lvert\mathcal{A}^-_{K,m}(1)\rvert}{m^2}$	m(odd)	$\frac{4\pi\log\lvert\mathcal{A}_{K,m}(1)\rvert}{m^2}$
4	$0.54439\cdots$	$1.40724\cdots$	5	$1.12273\cdots$
8	$1.66441\cdots$	$1.84668\cdots$	9	$1.76436\cdots$
12	$1.86678\cdots$	$1.94781\cdots$	13	$1.90158\cdots$
16	$1.93822\cdots$	$1.98381\cdots$	17	$1.95494\cdots$
20	$1.97121\cdots$	$2.00039\cdots$	21	$1.98076\cdots$
24	$1.98914\cdots$	$2.00940\cdots$	25	$1.99522\cdots$
28	$1.99994\cdots$	$2.01483\cdots$	29	$2.00412\cdots$
32	$2.00696\cdots$	$2.01836\cdots$	33	$2.00999\cdots$

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