

THE INDEX OF A THREEFOLD CANONICAL SINGULARITY

MASAYUKI KAWAKITA

ABSTRACT. The index of a 3-fold canonical singularity at a crepant centre is at most 6.

1. INTRODUCTION

Let $P \in X$ be a log canonical singularity. Shokurov asked if one can bound the index r_P of X at P in terms of the discrepancies of divisors over X .

Suppose that X has log canonical singularities with P a log canonical centre. In $\dim X = 2$, r_P is 1, 2, 3, 4 or 6 by the classification of singularities. In an arbitrary dimension, Ishii [5] and Fujino [2] reduced the boundedness of r_P to a conjectural boundedness of a quotient of the birational automorphism group of a variety S with $K_S \sim 0$. In particular, they proved $r_P \leq 66$ in $\dim X = 3$.

Suppose that X has canonical singularities. In $\dim X = 2$, P is a rational double point, so $r_P = 1$. The purpose of this paper is to provide an affirmative answer in $\dim X = 3$.

Theorem 1.1. *Let $P \in X$ be a 3-fold canonical singularity such that P is a crepant centre. Then the index of X at P is at most 6.*

Remark 1.2. We have such singularities P with $r_P = 1, 2, 3, 4$ in Example 4.3, but I do not know if there exists P with $r_P = 5$ or 6.

Note that no (even implicit) bound of r_P has been known before. Here a crepant centre means the centre of a divisor with discrepancy zero. The condition that P is a crepant centre is necessary even for a strictly canonical singularity, see Example 5.1. On the other hand, if once the minimal discrepancy at P is fixed, then one can bound r_P for an arbitrary 3-fold canonical singularity $P \in X$ (Theorem 5.3).

We shall prove Theorem 1.1 by using the singular Riemann–Roch formula (singRR) [14], an orbifold version of Riemann–Roch formula, due to Reid. In Sect. 2, we build a tower $Y \rightarrow X$ of crepant blow-ups with $\mathbb{Q}$ -factorial terminal Y , on which the singRR is applicable unconditionally. Then we construct a divisor F on Y which possesses the information on the index r_P . The r_P is determined by the Euler characteristics $\chi(iK_Y|_F)$, which can be explicitly computed by the singRR (Sect. 3). We derive a numerical classification of the singularities on Y together with r_P in Sect. 4, by the method [6], [7] in the classification of 3-fold divisorial contractions. The boundedness of indices in terms of minimal discrepancies is discussed in Sect. 5.

We work over an algebraically closed field k of characteristic zero. A germ $P \in X$ means an algebraic germ of a variety X at a closed point P .

2. CREPANT BLOW-UPS

Let X be a normal $\mathbb{Q}$ -Gorenstein variety.

Definition 2.1. The *index* of X at a point P is the smallest positive integer r such that rK_X is a Cartier divisor at P .

Consider a normal variety Y with a proper birational morphism $f: Y \rightarrow X$. A prime divisor E on any such Y is called a divisor *over* X , and the image $f(E)$ is called the *centre* of E on X and denoted by $c_X(E)$. The valuation v_E on the function field of X given by such E is called an *algebraic valuation* of X . If we write

$$K_Y = f^*K_X + \sum_E a_E(X)E \quad \text{with } a_E(X) \in \mathbb{Q},$$

then $a_E(X)$ is called the *discrepancy* of E . We say that X has *log canonical*, *log terminal*, *canonical*, *terminal* singularities if $a_E(X) \geq -1$, > -1 , ≥ 0 , > 0 respectively for all exceptional divisors E over X .

The notion of crepancy is crucial in this paper.

Definition 2.2. (i) A *crepant divisor* over X is an exceptional divisor E over X with $a_E(X) = 0$. A *crepant valuation* of X is the algebraic valuation v_E given by a crepant divisor E .
(ii) A *crepant centre* on X is the centre $c_X(E)$ of a crepant divisor E .
(iii) A *crepant blow-up* $f: Y \rightarrow X$ is a projective birational morphism from a normal variety Y such that $K_Y = f^*K_X$.

Remark 2.3. (i) Suppose that X is canonical. Then every crepant valuation is realised as a divisor on any resolution of X . In particular, the number of crepant valuations of X is finite. The complement of the union of all crepant centres is the largest terminal open subvariety of X .
(ii) If $Y \rightarrow X$ is a crepant blow-up, then X is canonical if and only if so is Y .

We have a crepant blow-up by the LMMP.

Proposition 2.4. *Let X be a variety with canonical singularities and v a crepant valuation of X . Then there exists a crepant blow-up $f: Y \rightarrow X$ such that*

- (i) Y is $\mathbb{Q}$ -factorial,
- (ii) f has exactly one exceptional divisor E , and $v_E = v$,
- (iii) $-E$ is f -nef.

Proof. Take a projective resolution of singularities $g: Z \rightarrow X$, and denote by E_Z the divisor on Z with $v_{E_Z} = v$. Take a Cartier divisor $H > 0$ on X whose support contains all the crepant centres. We write $g^*H = H_Z + F$ with the strict transform H_Z of H , and m for the coefficient of E_Z in F . Fix $\varepsilon > 0$ so that $(Z, \varepsilon(H_Z + 2(F - mE_Z)))$ is klt, and run $(K_Z + \varepsilon(H_Z + 2(F - mE_Z)))$ -LMMP over X by [1] to get a log minimal model $f: Y \rightarrow X$.

By $K_Z + \varepsilon(H_Z + 2(F - mE_Z)) \equiv_X K_Z + \varepsilon(F - 2mE_Z)$, the negativity lemma [11, Lemma 2.19] shows that this LMMP contracts exactly all the g -exceptional divisors but E_Z , and $-E$ is f -nef for the strict transform E of E_Z . Hence f is a required crepant blow-up. q.e.d.

Remark 2.5. If X is $\mathbb{Q}$ -factorial, then (ii) implies that $\rho(Y/X) = 1$ and $-E$ is f -ample.

Corollary 2.6. *Let $X = X_0$ be a variety with canonical singularities and Z a crepant centre on X . Then there exists a sequence of crepant blow-ups $f_t: X_t \rightarrow X_{t-1}$ for $1 \leq t \leq s$ such that*

- (i) X_t is $\mathbb{Q}$ -factorial for $t \geq 1$ and X_s is terminal,
- (ii) for $t \geq 1$, f_t has exactly one exceptional divisor E_t and $-E_t$ is f_t -nef,
- (iii) $f_1(E_1) = Z$.

We construct a divisor on X_s which possesses the information on the index of X .

Theorem 2.7. *Let $P \in X$ be a canonical singularity such that P is a crepant centre. Let r_P denote the index of X at P and $\mathfrak{m}_P$ the maximal ideal sheaf for P . Then there exist a crepant blow-up $f: Y \rightarrow X$ and an effective divisor F on Y supported in $f^{-1}(P)$ such that*

- (i) Y is $\mathbb{Q}$ -factorial and terminal,
- (ii) for $i \in \mathbb{Z}$,

$$f_* \mathcal{O}_Y(iK_Y - F) = \begin{cases} \mathfrak{m}_P \mathcal{O}_X(iK_X) & \text{if } r_P \mid i, \\ \mathcal{O}_X(iK_X) & \text{otherwise,} \end{cases}$$

$$R^j f_* \mathcal{O}_Y(iK_Y - F) = 0 \quad \text{for } j \geq 1.$$

Proof. We take a sequence of crepant blow-ups f_t in Corollary 2.6 with $Z = P$, and set $Y := X_s$. We will construct inductively divisors $F_t \geq 0$ on X_t such that

$$(1) \quad f_{1*} \mathcal{O}_{X_1}(iK_{X_1} - F_1) = \begin{cases} \mathfrak{m}_P \mathcal{O}_X(iK_X) & \text{if } r_P \mid i, \\ \mathcal{O}_X(iK_X) & \text{otherwise,} \end{cases}$$

$$(2) \quad R^j f_{1*} \mathcal{O}_{X_1}(iK_{X_1} - F_1) = 0 \quad \text{for } j \geq 1,$$

and for $t > 1$,

$$(3) \quad f_{t*} \mathcal{O}_{X_t}(iK_{X_t} - F_t) = \mathcal{O}_{X_{t-1}}(iK_{X_{t-1}} - F_{t-1}),$$

$$(4) \quad R^j f_{t*} \mathcal{O}_{X_t}(iK_{X_t} - F_t) = 0 \quad \text{for } j \geq 1.$$

Then Leray's spectral sequence induces that $F := F_s$ is a required divisor.

We set $F_1 := E_1$. The vanishing (2) follows from Kawamata–Viehweg vanishing theorem [10, Theorem 1.2.5, Remark 1.2.6]. If $r_P \mid i$, then (1) is by the projection formula. To see (1) for $r_P \nmid i$, we regard K_X as a fixed divisor (not a divisor class), and so $K_{X_1} = f_1^* K_X$. Denote by $\mathcal{K}_X$ the constant sheaf of the function field of X . Then the inclusion $f_{1*} \mathcal{O}_{X_1}(iK_{X_1} - F_1) \subset \mathcal{O}_X(iK_X)$ is interpreted by the expressions

$$f_{1*} \mathcal{O}_{X_1}(iK_{X_1} - F_1) = \{u \in \mathcal{K}_X \mid (u)_{X_1} + i f_1^* K_X - F_1 \geq 0\},$$

$$\mathcal{O}_X(iK_X) = \{u \in \mathcal{K}_X \mid (u)_X + i K_X \geq 0\}.$$

Suppose $u \in \mathcal{K}_X$ satisfies $(u)_X + i K_X \geq 0$. If $r_P \nmid i$, then $(u)_X + i K_X$ is not Cartier at P , so there exists a divisor $D > 0$ passing through P such that $(u)_X + i K_X - D$ is an effective Cartier divisor. Then $(u)_{X_1} + i f_1^* K_X - f_1^* D \geq 0$. By $f_1^* K_X = K_{X_1}$ and $F_1 \subset \text{Supp } f_1^* D$, we obtain $(u)_{X_1} + i f_1^* K_X - F_1 \geq 0$, implying (1).

For $t > 1$, we set $F_t := \lceil f_t^* F_{t-1} \rceil$ inductively. $F_t = f_t^* F_{t-1} + c_t E_t$ with some $c_t \in [0, 1)$, so $-F_t$ is f_t -nef. The (4) is again by Kawamata–Viehweg vanishing theorem. If $c_t = 0$, then (3) is obvious. If $c_t > 0$, then the equality $iK_{X_t} - F_t = f_t^*(iK_{X_{t-1}} - F_{t-1}) - c_t E_t$ shows that $iK_{X_{t-1}} - F_{t-1}$ is not Cartier at every point in $f_t(E_t)$. Now we get (3) just as in the proof of (1) for $r_P \nmid i$. q.e.d.

3. THE SINGULAR RIEMANN–ROCH FORMULA

We shall apply the singular Riemann–Roch formula due to Reid to our crepant blow-up, and use the method [6], [7] in the classification of 3-fold divisorial contractions. We briefly recall the formula on a canonical 3-fold.

Theorem 3.1 ([14, Theorem 10.2]). *Let X be a projective 3-fold with canonical singularities and D a divisor on X such that $D \sim i_P K_X$ with $i_P \in \mathbb{Z}$ at each $P \in X$.*

(i) *There is a formula of the form*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D - K_X)(2D - K_X) + \frac{1}{12}D \cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over the singularities of $\mathcal{O}_X(D)$, and $c_P(D) \in \mathbb{Q}$ is a contribution due to the singularity at P , depending only on the analytic type.

(ii) *For a terminal cyclic quotient singularity P of type $\frac{1}{r_P}(1, -1, b_P)$,*

$$c_P(D) = -\frac{\overline{i_P} r_P^2 - 1}{12r_P} + \sum_{j=1}^{\overline{i_P}-1} \frac{j\overline{b_P}(r_P - j\overline{b_P})}{2r_P},$$

where $\overline{i} = i - \lfloor \frac{i}{r_P} \rfloor r_P$ denotes the residue of i modulo r_P .

(iii) *For an arbitrary terminal singularity P ,*

$$c_P(D) = \sum_Q c_Q(D_Q),$$

where $\{(Q, D_Q)\}_Q$ is a flat deformation of (P, D) to the basket of terminal cyclic quotient singularities Q . Such Q is called a fictitious singularity.

Remark 3.2. The condition $D \sim i_P K_X$ always holds if X is $\mathbb{Q}$ -factorial and terminal [8, Corollary 5.2].

Our object is a germ of a crepant blow-up $f: Y \rightarrow X$ with a divisor F on Y in Theorem 2.7 at a 3-fold canonical singularity $P \in X$ with index r_P . Shrinking and compactifying it, we may assume that Y is projective and terminal (f is merely a projective morphism outside a neighbourhood of P). We shall express the function $\delta_P(i)$ below.

Definition 3.3. We define the function $\delta_P(i)$ on $\mathbb{Z}$ as

$$\delta_P(i) := \begin{cases} 1 & \text{if } r_P \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Applying (ii) in Theorem 2.7 and the vanishing $R^j f_* \mathcal{O}_Y(iK_Y) = 0$ for $j \geq 1$ to the exact sequence

$$0 \rightarrow \mathcal{O}_Y(iK_Y - F) \rightarrow \mathcal{O}_Y(iK_Y) \rightarrow \mathcal{O}_F(iK_Y|_F) \rightarrow 0,$$

we obtain

$$\begin{aligned} (5) \quad \delta_P(i) &= \dim_k f_* \mathcal{O}_Y(iK_Y) / f_* \mathcal{O}_Y(iK_Y - F) \\ &= h^0(\mathcal{O}_F(iK_Y|_F)) \\ &= \chi(\mathcal{O}_F(iK_Y|_F)) \\ &= \chi(\mathcal{O}_Y(iK_Y)) - \chi(\mathcal{O}_Y(iK_Y - F)). \end{aligned}$$

Let $I_0 := \{Q \text{ with type } \frac{1}{r_Q}(1, -1, b_Q)\}$ be the basket of fictitious singularities from singularities on Y . Note that b_Q is co-prime to r_Q . For $Q \in I_0$, let f_Q denote the smallest non-negative integer such that $F \sim f_Q K_Y$ at Q . By replacing b_Q with $r_Q - b_Q$ if necessary, we may assume $v_Q := \overline{f_Q b_Q} \leq r_Q/2$. Set $I := \{Q \in I_0 \mid f_Q \neq 0\}$.

With this notation, the singular Riemann–Roch formula computes the right-hand side of (5), to provide

$$(6) \quad \delta_P(i) = \frac{1}{6}F^3 + \frac{1}{12}F \cdot c_2(Y) + \sum_{Q \in I} (A_Q(i) - A_Q(i - f_Q)),$$

where the contribution $A_Q(i)$ is given by

$$A_Q(i) := -\bar{i} \frac{r_Q^2 - 1}{12r_Q} + \sum_{j=1}^{\bar{i}-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}.$$

The $A_Q(i)$ satisfies the formula

$$A_Q(i+1) - A_Q(i) = -\frac{r_Q^2 - 1}{12r_Q} + B_Q(ib_Q)$$

with

$$B_Q(i) := \frac{\bar{i}(r_Q - \bar{i})}{2r_Q}.$$

Therefore by (6), we have

$$(7) \quad \delta_P(i+1) - \delta_P(i) = \sum_{Q \in I} (B_Q(ib_Q) - B_Q(ib_Q - v_Q)).$$

Lemma 3.4. *The r_P equals the l.c.m. of r_Q for all $Q \in I$.*

Proof. Since $r_P K_Y = r_P f^* K_X$ is a Cartier divisor near $f^{-1}(P)$, r_Q divides r_P for all $Q \in I$. On the other hand, we see that r_P divides the l.c.m. of r_Q by (7) and the periodic properties of δ_P, B_Q . q.e.d.

4. BOUNDEDNESS OF INDICES

We shall prove Theorem 1.1 in this section. Let r_P denote the index of X at P . We take a crepant blow-up $f: Y \rightarrow X$ with a divisor F on Y in Theorem 2.7. We restrict the possibilities of $J := \{(r_Q, v_Q)\}_{Q \in I}$ using (7) for $i = 0$.

Lemma 4.1. *J is one of the types in Table 1.*

TABLE 1

type	J	r_P	type	J	r_P
1	$(2, 1), (2, 1), (2, 1), (2, 1)$	2	8	$(2, 1), (8, 2)$	8
2	$(2, 1), (2, 1), (4, 2)$	4	9	$(3, 1), (6, 2)$	6
3	$(2, 1), (3, 1), (6, 1)$	6	10	$(5, 1), (5, 2)$	5
4	$(2, 1), (4, 1), (4, 1)$	4	11	$(8, 4)$	8
5	$(3, 1), (3, 1), (3, 1)$	3	12	$(9, 3)$	9
6	$(4, 2), (4, 2)$	4	13	$\emptyset$	1
7	$(2, 1), (6, 3)$	6			

Proof. By Lemma 3.4, r_P is determined by J , and $r_P = 1$ if and only if $J = \emptyset$. We assume $r_P > 1$ from now on. Then (7) for $i = 0$ is written as

$$(8) \quad \sum_{Q \in I} B_Q(v_Q) = 1.$$

By the definition of B_Q and $r_Q \geq 2v_Q$, we have

$$(9) \quad v_Q/4 \leq B_Q(v_Q) < v_Q/2.$$

Then $J' := \{v_Q\}_{Q \in I}$, which satisfies (8) and (9), should be one of

$$\{1, 1, 1, 1\}, \{1, 1, 2\}, \{1, 1, 1\}, \{2, 2\}, \{1, 3\}, \{1, 2\}, \{3\}, \{4\}.$$

For each of these candidates for J' , one can solve the equation (8) for r_Q ($\geq 2v_Q$) explicitly. Every solution is in Table 1. For example, suppose $J' = \{1, 2\}$. We set $J = \{(r_1, 1), (r_2, 2)\}$. Then (8) becomes $1/r_1 + 4/r_2 = 1$. Thus $(r_1, r_2) = (2, 8)$, $(3, 6)$ or $(5, 5)$, so J is of type 8, 9, 10 respectively. q.e.d.

By Lemma 4.1, we have $r_P \leq 9$, and for Theorem 1.1 it is enough to exclude types 8, 11, 12. However, we derive a finer numerical classification by determining $\tilde{J} := \{(r_Q, v_Q, b_Q)\}_{Q \in I}$.

Theorem 4.2. $\tilde{J}$ is one of the types in Table 2.

TABLE 2

type	$\tilde{J}$	r_P
1	$(2, 1, 1), (2, 1, 1), (2, 1, 1), (2, 1, 1)$	2
3	$(2, 1, 1), (3, 1, 2), (6, 1, 5)$	6
4	$(2, 1, 1), (4, 1, 3), (4, 1, 3)$	4
5	$(3, 1, 2), (3, 1, 2), (3, 1, 2)$	3
10	$(5, 1, 4), (5, 2, 3)$	5
13	$\emptyset$	1

Proof. By Lemma 4.1, there exist only finitely many candidates for $\tilde{J}$. For each candidate, one can compute the right-hand side of (7) explicitly. It must coincide with $\delta_P(i+1) - \delta_P(i)$, but such a coincidence happens only if $\tilde{J}$ is one of the types in Table 2.

Here we demonstrate for type 3. $\tilde{J} = \{(2, 1, 1), (3, 1, b_2), (6, 1, b_3)\}$ with $b_2 = 1$ or 2 and $b_3 = 1$ or 5. The (7) for $i = 1$ is $\delta_P(2) - \delta_P(1) = 1, 1/3, 2/3, 0$ when $(b_2, b_3) = (1, 1), (1, 5), (2, 1), (2, 5)$ respectively. Thus (b_2, b_3) must be $(2, 5)$, and in this case (7) surely holds for any i . q.e.d.

Example 4.3. In simple cases, Theorem 1.1 is known by the classification.

- (i) (Morrison [13], Ishida–Iwashita [4]) If P is a cyclic quotient singularity, then $r_P = 1$ except $\frac{1}{4n}(1, 2n+1, -2)$ ($n \geq 2$), $\frac{1}{14}(1, 9, 11)$, $\frac{1}{9}(1, 4, 7)$, with $r_P = 2, 2, 3$ respectively.
- (ii) (Hayakawa–Takeuchi [3]) If P is an isolated singularity which is a cyclic quotient of a hypersurface singularity, then $r_P \leq 4$. The only case when $r_P = 4$ is $o \in (x_1x_2 + x_3^2 + x_4^2 = 0) \subset \mathbb{A}_{x_1x_2x_3x_4}^4/\mathbb{Z}_8(1, 5, 3, 7)$.

5. MINIMAL DISCREPANCIES

To begin with, we provide an example which explains the need of P being a crepant centre in Theorem 1.1 even for a strictly canonical singularity. A similar example exists also for a 3-fold strictly log canonical singularity [2, Example 6.1].

Example 5.1. Let $r \in \mathbb{N}$. Let $P \in X$ be the germ

$$o \in (x_1x_2 + x_3^2 = 0) \subset \mathbb{A}_{x_1x_2x_3x_4}^4 / \mathbb{Z}_r(1, -1, 0, 1),$$

which is singular along the x_4 -axis C . Let $f: Y \rightarrow X$ be the weighted blow-up with weights $\text{wt}(x_1, x_2, x_3, x_4) = \frac{1}{r}(1, r-1, r, 1)$. Then $K_Y = f^*K_X + \frac{1}{r}E$ with the exceptional divisor E , and Y has 2 terminal quotient singularities of types $\frac{1}{r-1}(1, -1, 1)$ and $\frac{1}{r}(1, -1, 1)$ outside the strict transform C_Y of C . Let $g: Z \rightarrow Y$ be the blow-up with centre C_Y . The g is a crepant blow-up and Z is smooth near $g^{-1}(C_Y)$. Hence X has canonical singularities with a crepant centre C , but P is not a crepant centre. The index of X at P is r .

We focus on the minimal discrepancy to grasp this phenomenon. For a normal $\mathbb{Q}$ -Gorenstein singularity $P \in X$, the *minimal discrepancy* $\text{md}_P X$ of X at P is the infimum of discrepancies $a_E(X)$ for all divisors E over X with $c_X(E) = P$. Note that $\text{md}_P X \in \{-\infty\} \cup [-1, \infty)$, and $P \in X$ is log canonical if and only if $\text{md}_P X \geq -1$.

In Example 5.1, we have $\text{md}_P X = 1/r$. Shokurov formulated a question on the boundedness of indices in terms of minimal discrepancies.

Question 5.2 (Shokurov). *For each $(n, a) \in \mathbb{N} \times [-1, \infty)$, does there exist a number $r(n, a)$ such that the index of an arbitrary n -fold log canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r(n, a)$?*

He raised its weaker variant for canonical singularities.

Question 5.2'. *For each $(n, a) \in \mathbb{N} \times [0, \infty)$, does there exist a number $r'(n, a)$ such that the index of an arbitrary n -fold canonical singularity $P \in X$ with $\text{md}_P X = a$ is at most $r'(n, a)$?*

The result of Ishii and Fujino gives $r(3, -1) = 66$ for Question 5.2. Theorem 1.1 gives $r'(3, 0) = 6$ for Question 5.2'. Further, we provide an affirmative answer to Question 5.2' for $n = 3$.

Theorem 5.3. *Question 5.2' is true for $n = 3$. More precisely, the minimal discrepancy of a 3-fold canonical singularity is 0, $1/r$ ($r \in \mathbb{N}$) or 2, and one can take*

$$r'(3, 0) = 6, \quad r'(3, 1/r) = r!, \quad r'(3, 2) = 1.$$

Proof. Let $P \in X$ be a 3-fold canonical singularity with index r_P . We shall verify the statement for any such P . We take a crepant blow-up $f: Y \rightarrow X$ with Y terminal by Corollary 2.6.

Suppose $\dim f^{-1}(P) = 0$, that is, P is terminal. Then it suffices to recall $\text{md}_P X = 1/r_P$ [9], [12] for terminal P except for smooth P .

Suppose $\dim f^{-1}(P) = 1$. For any curve $C \subset f^{-1}(P)$, the blow-up of Y with centre C generates a divisor E with $a_E(X) = 1$. Together with the mentioned result [9], [12], we see that $\text{md}_P X$ is the minimum of $1/r_Q$ for all $Q \in f^{-1}(P)$, where r_Q denotes the index of Y at Q . Hence $\text{md}_P X = 1/r$ with $r \in \mathbb{N}$ and $r_Q \leq r$ for all $Q \in f^{-1}(P)$. Thus $r!K_Y$ is a Cartier divisor near $f^{-1}(P)$, so $r_P \mid r!$ by [8, Corollary 1.5].

Suppose $\dim f^{-1}(P) = 2$. Then P is a crepant centre, that is, $\mathrm{md}_P X = 0$. The statement holds by Theorem 1.1. q.e.d.

Acknowledgements. I was asked the question on the boundedness of indices by Professor V. V. Shokurov at Johns Hopkins University in 2005. This question was raised again in the workshop at American Institute of Mathematics in 2012. I am grateful to him for the discussions. The research was partially supported by Grant-in-Aid for Young Scientists (A) 24684003.

REFERENCES

- [1] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, *J. Am. Math. Soc.* **23**, No. 2, 405-468 (2010)
- [2] O. Fujino, The indices of log canonical singularities, *Am. J. Math.* **123**, No. 2, 229-253 (2001)
- [3] T. Hayakawa and K. Takeuchi, On canonical singularities of dimension three, *Jap. J. Math., New Ser.* **13**, No. 1, 1-46 (1987)
- [4] M.-N. Ishida and N. Iwashita, Canonical cyclic quotient singularities of dimension three, *Complex analytic singularities*, Tsukuba 1984, *Adv. Stud. Pure Math.* **8**, 135-151 (1987)
- [5] S. Ishii, The quotients of log-canonical singularities by finite groups, *Singularities*, Sapporo 1998, *Adv. Stud. Pure Math.* **29**, 135-161 (2000)
- [6] M. Kawakita, Divisorial contractions in dimension three which contract divisors to smooth points, *Invent. Math.* **145**, No. 1, 105-119 (2001)
- [7] M. Kawakita, Three-fold divisorial contractions to singularities of higher indices, *Duke Math. J.* **130**, No. 1, 57-126 (2005)
- [8] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, *Ann. Math. (2)* **127**, No. 1, 93-163 (1988)
- [9] Y. Kawamata, The minimal discrepancy coefficients of terminal singularities in dimension 3, Appendix to V. V. Shokurov, Three-dimensional log perestroikas, *Izv. Ross. Akad. Nauk Ser. Mat.* **56**, No. 1, 105-201, Appendix 201-203 (1992)
- [10] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, *Algebraic geometry*, Sendai 1985, *Adv. Stud. Pure Math.* **10**, 283-360 (1987)
- [11] J. Kollár (ed.), *Flips and abundance for algebraic threefolds*, *Astérisque* **211** (1992)
- [12] D. Markushevich, Minimal discrepancy for a terminal cDV singularity is 1, *J. Math. Sci. Tokyo* **3**, No. 2, 445-456 (1996)
- [13] D. R. Morrison, Canonical quotient singularities in dimension three, *Proc. Am. Math. Soc.* **93**, No. 3, 393-396 (1985)
- [14] M. Reid, Young person's guide to canonical singularities, *Algebraic geometry*, Bowdoin 1985, *Proc. Symp. Pure Math.* **46**, Part 1, 345-414 (1987)

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: masayuki@kurims.kyoto-u.ac.jp