

On Another Easy Proof of Owa's Result for Starlikeness

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Abstract—Only recently, Owa [1] proved the following theorem.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $|z| < 1$,

and $|f'(z) - 1| < \alpha$ for $|z| < 1$

$$\left| \arg \frac{f(z)}{z} \right| \leq \tan^{-1} \left(\frac{\sqrt{1-\alpha^2}}{\alpha} \right) \quad \text{for } |z| < 1$$

where $\frac{2}{\sqrt{5}} < \alpha \leq 1$, then $f(z)$ is starlike in $|z| < 1$.

It is the purpose of the present paper to give an easy proof of the above result.

Keywords—Starlike and convex functions.

1. Introduction.

Let Λ denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

that are analytic in $E = \{z : |z| < 1\}$. A function $f(z) \in \Lambda$ is called starlike if and only if

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \quad \text{in } E.$$

On the other hand, a function $f(z) \in \Lambda$ is called convex if and only if

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} > 0 \quad \text{in } E$$

It is easily proved that the necessary and sufficient condition for $f(z)$ to be convex in E is that $zf'(z)$ is starlike in E .

2. Main theorem.

Theorem 1

Let $f(z) \in A$ and suppose that

$$|f'(z) - 1| < \alpha \quad \text{in } E \quad (1)$$

and

$$\left| \arg \frac{f(z)}{z} \right| \leq \tan^{-1} \left(\frac{\sqrt{1-\alpha^2}}{\alpha} \right) \quad \text{in } E \quad (2)$$

where $0 < \alpha \leq 1$.

Then $f(z)$ is starlike in E .

Proof.

For the case $\alpha=1$, it is trivial. For the case $0 < \alpha < 1$, from the hypothesis (1), we easily have

$$|\arg f'(z)| \leq \tan^{-1} \left(\frac{\alpha}{\sqrt{1-\alpha^2}} \right) \quad \text{in } E. \quad (3)$$

Then, from (2) and (3), we have

$$\begin{aligned} \left| \arg \frac{zf'(z)}{f(z)} \right| &\leq \left| \arg \frac{z}{f(z)} \right| + |\arg f'(z)| \\ &< \left| \tan^{-1} \left(\frac{\sqrt{1-\alpha^2}}{\alpha} \right) \right| + \left| \tan^{-1} \left(\frac{\alpha}{\sqrt{1-\alpha^2}} \right) \right| \\ &= \frac{\pi}{2} \end{aligned}$$

Thus,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } E.$$

This completes the proof.

Remark. Owa[1] proved Theorem 1 by using 3 Lemmas and

with the restriction of α , $\frac{2}{\sqrt{5}} < \alpha \leq 1$.

Our proof is simple, and generalized the result of [1, Theorem 2] for the case $0 < \alpha \leq 1$.

Applying Theorem 1, we have the following corollaries.

Corollary 1.

Let $f(z) \in A$ and suppose that

$$|f'(z) - 1| < \alpha \quad \text{in } E$$

and

$$\left| \frac{f(z)}{z} - 1 \right| < \sqrt{1 - \alpha^2} \quad \text{in } E$$

where $0 < \alpha \leq 1$.

Then $f(z)$ is starlike in E .

Corollary 2.

Let $f(z) \in A$ and suppose that

$$|f'(z) + zf''(z) - 1| < \alpha \quad \text{in } E$$

and

$$|\arg f'(z)| \leq \tan^{-1} \left(\frac{\sqrt{1 - \alpha^2}}{\alpha} \right) \quad \text{in } E$$

where $0 < \alpha \leq 1$.

Then $f(z)$ is convex in E .

Corollary 3.

Let $f(z) \in A$ and suppose that

$$|f'(z) + zf''(z) - 1| < \alpha \quad \text{in } E$$

and

$$|f'(z) - 1| < \sqrt{1 - \alpha^2} \quad \text{in } E$$

where $0 < \alpha \leq 1$.

Then $f(z)$ is convex in E .

REFERENCE

1. S. Owa, On the conditions of starlikeness for analytic functions, Sugaku (Math. Soc. of Japan) Vol. 45(2), 180-182, (1994), (Japanese).