

Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation

(2 サイクルを持つ擬量子多元環のホッホシルトコホモロジー)

東京理科大学 理学研究科 数学専攻 博士課程 3 年 小原 大樹 (Daiki Obara)
Department of Mathematics
Tokyo University of Science

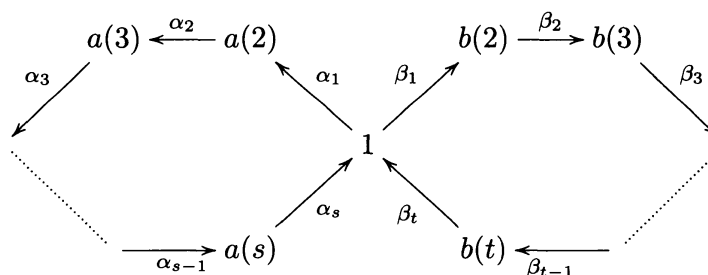
Abstract

This paper is based on my talk given at the Symposium on Cohomology Theory of Finite Groups and Related Topics held at Kyoto University, Japan, 29 August to 2 September 2011. In this paper, we consider quiver algebras A_q over a field k defined by two cycles and a quantum-like relation depending on a non-zero element q in k . We determine the ring structure of the Hochschild cohomology ring of A_q modulo nilpotence and give a necessary and sufficient condition for A_q to satisfy the finiteness conditions given in [4].

Introduction

Let A be an indecomposable finite dimensional algebra over a field k . We denote by A^e the enveloping algebra $A \otimes_k A^{\text{op}}$ of A , so that left A^e -modules correspond to A -bimodules. The Hochschild cohomology ring is given by $\text{HH}^*(A) = \text{Ext}_{A^e}^*(A, A) = \bigoplus_{n \geq 0} \text{Ext}_{A^e}^n(A, A)$ with Yoneda product. It is well-known that $\text{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \text{HH}^m(A)$ and $\theta \in \text{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\text{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\text{HH}^*(A)$, so that the maximal ideals of $\text{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\text{HH}^*(A)/\mathcal{N}$.

Let q be a non-zero element in k and s, t integers with $s, t \geq 1$. We consider the quiver algebra $A_q = kQ/I_q$ defined by the two cycles Q with $s + t - 1$ vertices and $s + t$ arrows as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^s Y^t - q Y^t X^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers i in the subscripts of $e_{a(i)}$ modulo s and j in the subscripts of $e_{b(j)}$ modulo t . In this paper, we describe the ring structure of $\text{HH}^*(A_q)/\mathcal{N}$.

In [19], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A . This led us to consider the structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [19], Snashall and Solberg conjectured that $\mathrm{HH}^*(A)/\mathcal{N}$ is always finitely generated as a k -algebra. But a counterexample to this conjecture was given by Snashall [18] and Xu [23]. This example makes us consider whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k -algebra.

On the other hand, in the theory of support varieties, it is interesting to know when the variety of a module is trivial. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the necessary and sufficient conditions on a module for it to have trivial variety under some finiteness conditions on A . In the paper, we show that A_q satisfies the finiteness conditions given in [4] if and only if q is a root of unity.

The content of the paper is organized as follows. In Section 1 we deal with the definition of the support variety given in [19] and precedent results about the Hochschild cohomology ring modulo nilpotence. In Section 2, we describe the finiteness conditions given in [4] and introduce precedent results about these conditions. In Section 3, we determine the Hochschild cohomology ring of A_q modulo nilpotence and show that A_q satisfies the finiteness conditions if and only if q is a root of unity.

1 Support variety

In [19], Snashall and Solberg defined the support variety of a finitely generated A -module M over a noetherian commutative graded subalgebra H of $\mathrm{HH}^*(A)$ with $H^0 = \mathrm{HH}^0(A)$. In this paper, we consider the case $H = \mathrm{HH}^*(A)$.

Definition 1.1 ([19]). *The support variety of M is given by*

$$V(M) = \{m \in \mathrm{MaxSpec} \mathrm{HH}^*(A)/\mathcal{N} \mid \mathrm{AnnExt}_A^*(M, M) \subseteq m'\}$$

where $\mathrm{AnnExt}_A^*(M, M)$ is the annihilator of $\mathrm{Ext}_A^*(M, M)$, m' is the pre-image of m for the natural epimorphism and the $\mathrm{HH}^*(A)$ -action on $\mathrm{Ext}_A^*(A, A)$ is given by the graded algebra homomorphism $\mathrm{HH}^*(A) \xrightarrow{-\otimes^M} \mathrm{Ext}_A^*(M, M)$.

Since A is indecomposable, we have that $\mathrm{HH}^0(A)$ is a local ring. Thus $\mathrm{HH}^*(A)/\mathcal{N}$ has a unique maximal graded ideal $m_{gr} = \langle \mathrm{rad} \mathrm{HH}^*(A), \mathrm{HH}^{\geq 1}(A) \rangle / \mathcal{N}$. We say that the variety of M is trivial if $V(M) = \{m_{gr}\}$.

In [18], Snashall gave the following question.

Question ([18]). *Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as a k -algebra.*

With respect to sufficient condition, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k -algebra for various classes of algebras by many authors as follows:

- (1) In [6], [22], Evens and Venkov showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a group ring of a finite group.
- (2) In [7], Friedlander and Suslin showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a finite dimensional cocommutative Hopf algebra.

- (3) In [11], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional self-injective algebras of finite representation type over an algebraically closed field.
- (4) In [12], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional monomial algebras.
- (5) In [13], Happel showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional algebras of finite global dimension.
- (6) In [17], Schroll and Snashall showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for the principal block of the Hecke algebra $H_q(S_5)$ with $q = -1$ defined by the quiver

$$Q : \varepsilon \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} 1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\bar{\alpha}} \end{array} 2 \begin{array}{c} \circlearrowright \\ \circlearrowleft \end{array} \bar{\varepsilon}$$

and the ideal I of kQ generated by

$$\alpha\bar{\varepsilon}, \bar{\alpha}\varepsilon, \bar{\varepsilon}\bar{\alpha}, \varepsilon^2 - \alpha\bar{\alpha}, \bar{\varepsilon}^2 - \bar{\alpha}\alpha.$$

- (7) In [20], Snashall and Taillefer showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for a class of special biserial algebras.
- (8) In [14], Koenig and Nagase produced many examples of finite dimensional algebras with a stratifying ideal for which $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k -algebra.
- (9) In [18] and [23], Snashall and Xu gave the example of a finite dimensional algebra for which $\mathrm{HH}^*(A)/\mathcal{N}$ is not a finitely generated k -algebra.

Example 1.2. ([18, Example 4.1]) Let $A = kQ/I$ where Q is the quiver

$$\begin{array}{c} \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \begin{array}{c} a \\ 1 \end{array} \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{b} \end{array} 2 \end{array}$$

and $I = \langle a^2, b^2, ab - ba, ac \rangle$. Then Snashall showed the following in [18, Theorem 4.5].

- (a) $\mathrm{HH}^*(A)/\mathcal{N} \cong \begin{cases} k \oplus k[a, b]b & \text{if } \mathrm{char} k = 2, \\ k \oplus k[a^2, b^2]b^2 & \text{if } \mathrm{char} k \neq 2. \end{cases}$
- (b) $\mathrm{HH}^*(A)/\mathcal{N}$ is not finitely generated as a k -algebra.

Xu showed this in the case $\mathrm{char} k = 2$ in [23].

2 Finiteness conditions

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the following two conditions **(Fg1)** and **(Fg2)** for an algebra A and a graded subalgebra H of $\mathrm{HH}^*(A)$.

(Fg1) H is a commutative Noetherian algebra with $H^0 = \mathrm{HH}^0(A)$.

(Fg2) $\text{Ext}_A^*(A/\text{rad } A, A/\text{rad } A)$ is a finitely generated H -module.

In [4], under the finiteness conditions above, some geometric properties of the support variety and some representation theoretic properties are related. In particular, the following theorem hold.

Theorem 2.1 ([4, Theorem 2.5]). *Suppose that A satisfies the finiteness conditions.*

- (a) *A is Gorenstein, that is, A has finite injective dimension both as a left A -module and as a right A -module.*
- (b) *The following are equivalent for an A -module M .*
 - (i) *The variety of M is trivial.*
 - (ii) *The projective dimension of M is finite.*
 - (iii) *The injective dimension of M is finite.*

There are some papers which deal with the finiteness conditions as follows.

- (1) In [2], Bergh and Oppermann show that a codimension n quantum complete intersection satisfies the finiteness conditions if and only if all the commutators q_{ij} are roots of unity.

Definition 2.2. *A codimension n quantum complete intersection is defined by*

$$k\langle x_1, \dots, x_n \rangle / I$$

where I generated by

$$x_i^{a_i}, x_j x_i - q_{ij} x_i x_j \quad \text{for } 1 \leq i < j \leq n, a_i \geq 2, q_{ij} \in k.$$

- (2) In [5], Erdmann and Solberg gave the necessary and sufficient conditions on a Koszul algebra for it to satisfy the finiteness conditions.

Theorem 2.3 ([5, Theorem 1.3]). *Let A be a finite dimensional Koszul algebra over an algebraically closed field, and let $E(A) = \text{Ext}_A^*(A/\text{rad } A, A/\text{rad } A)$. A satisfies the finiteness conditions if and only if $Z_{gr}(E(A))$ is Noetherian and $E(A)$ is a finitely generated $Z_{gr}(E(A))$ -module.*

- (3) In [9], Furuya and Snashall provided examples of (D, A) -stacked monomial algebras which are not self-injective but satisfy the finiteness conditions.

Example 2.4. ([9, Example 3.2]) Let Q be the quiver

$$\begin{array}{ccc} 1 & \xrightarrow{\alpha} & 2 \\ \delta \uparrow & & \downarrow \beta \\ 4 & \xleftarrow{\gamma} & 3 \end{array}$$

and I the ideal of kQ generated by

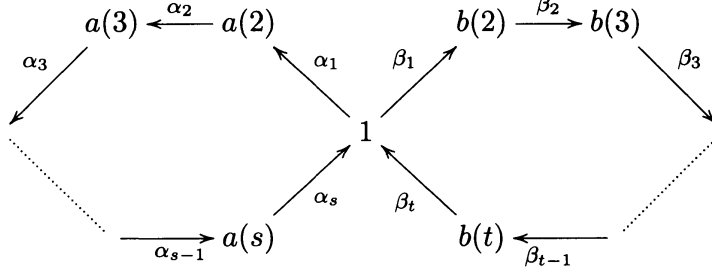
$$\alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta.$$

Then, $A = kQ/I$ is not self-injective but satisfies the finiteness conditions.

- (4) In [17], Schroll and Snashall show that the finiteness conditions hold for the principal block of the Hecke algebra $H_q(S_5)$ with $q = -1$.

3 Hochschild cohomology ring of quiver algebras defined by two cycles and a quantum-like relation

In this section, we consider the quiver algebras $A_q = kQ/I_q$ defined by the quiver Q as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^s Y^t - q Y^t X^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, and q is non-zero element in k . Paths are written from right to left. We will determine the Hochschild cohomology ring of A_q modulo nilpotence $\text{HH}^*(A_q)/\mathcal{N}$ and show that A_q satisfies the finiteness conditions if and only if q is a root of unity.

First, we note that the following elements in A_q form a k -basis of A_q .

$$\begin{aligned} X^{sl+l'} e_{a(i)} & \quad \text{for } 2 \leq i \leq s, 0 \leq l \leq a-1, 0 \leq l' \leq s-1, \\ Y^{tl+l'} e_{b(j)} & \quad \text{for } 1 \leq j \leq t, 0 \leq l \leq b-1, 0 \leq l' \leq t-1, \\ X^{si+l} Y^{tj+l'} & \quad \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 0 \leq l' \leq t-1, \\ X^{si} Y^{tj+l'} & \quad \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 0 \leq l' \leq t-1, \\ X^{si} Y^{tj} X^l & \quad \text{for } 0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l \leq s-1, \\ Y^{l'} X^{si} Y^{tj} & \quad \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l' \leq t-1, \\ Y^{l'} X^{si} Y^{tj} X^l & \quad \text{for } 0 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l \leq s-1, 1 \leq l' \leq t-1, \\ X^{si+l} Y^{tj} X^{l'} & \quad \text{for } 0 \leq i \leq a-1, 1 \leq j \leq b-1, 1 \leq l, l' \leq s-1, \\ Y^l X^{si} Y^{tj+l'} & \quad \text{for } 1 \leq i \leq a-1, 0 \leq j \leq b-1, 1 \leq l, l' \leq t-1. \end{aligned}$$

So we have $\dim_k A_q = ab(s+t-1)^2$.

3.1 Projective resolution of A_q

For $n \geq 0$, we define left A_q^e -modules, equivalently A_q -bimodules

$$\begin{aligned} P_{2n} &= \prod_{l=0}^{2n} A_q e_1 \otimes e_1 A_q \oplus \prod_{i=2}^s A_q e_{a(i)} \otimes e_{a(i)} A_q \oplus \prod_{j=2}^t A_q e_{b(j)} \otimes e_{b(j)} A_q, \\ P_{2n+1} &= \prod_{l=1}^{2n} A_q e_1 \otimes e_1 A_q \oplus \prod_{i=1}^s A_q e_{a(i+1)} \otimes e_{a(i)} A_q \oplus \prod_{j=1}^t A_q e_{b(j+1)} \otimes e_{b(j)} A_q, \end{aligned}$$

where $\prod_{l=1}^0 A_q e_1 \otimes e_1 A_q = 0$. The generators $e_1 \otimes e_1$, $e_{a(i)} \otimes e_{a(i)}$ and $e_{b(j)} \otimes e_{b(j)}$ of P_{2n} are labeled ε_l^{2n} for $0 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n}$ for $2 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively.

Similarly, we denote the generators $e_1 \otimes e_1$, $e_{a(i+1)} \otimes e_{a(i)}$ and $e_{b(j+1)} \otimes e_{b(j)}$ of P_{2n+1} by ε_l^{2n+1} for $1 \leq l \leq 2n$, $\varepsilon_{a(i)}^{2n+1}$ for $1 \leq i \leq s$, and $\varepsilon_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. In [15], we give the minimal projective bimodule resolution of A_q as follows.

Theorem 3.1 ([15, Theorem 1.1]). *The following sequence $\mathbb{P}$ is a minimal projective resolution of the left A_q^e -module A_q :*

$$\mathbb{P} : \cdots \rightarrow P_{2n+1} \xrightarrow{d_{2n+1}} P_{2n} \xrightarrow{d_{2n}} P_{2n-1} \rightarrow \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} A_q \rightarrow 0.$$

where $\pi: P_0 \rightarrow A_q$ is the multiplication map, and we define left A^e -homomorphisms d_{2n+1} and d_{2n+2} by

$d_{2n+1} :$

$$\begin{cases} \varepsilon_{b(j)}^{2n+1} \mapsto \varepsilon_{b(j+1)}^{2n} Y - Y \varepsilon_{b(j)}^{2n} \text{ for } 1 \leq j \leq t, \\ \varepsilon_{l''}^{2n+1} \mapsto \begin{cases} -\sum_{j=0}^{b-1} q^{j(al+1)} Y^{tj} \varepsilon_{2l+1}^{2n} Y^{t(b-1-j)} - X^s \varepsilon_{2l}^{2n} + q^{b(n-l)} \varepsilon_{2l}^{2n} X^s & \text{if } l'' = 2l + 1 \text{ for } 0 \leq l \leq n-1, \\ -q^{al'} Y^t \varepsilon_{2l'}^{2n} + \varepsilon_{2l'}^{2n} Y^t + \sum_{i=0}^{a-1} q^{i(b(n-l')+1)} X^{s(a-1-i)} \varepsilon_{2l'-1}^{2n} X^{si} & \text{if } l'' = 2l' \text{ for } 1 \leq l' \leq n, \end{cases} \\ \varepsilon_{a(i)}^{2n+1} \mapsto \varepsilon_{a(i+1)}^{2n} X - X \varepsilon_{a(i)}^{2n} \text{ for } 1 \leq i \leq s, \end{cases}$$

$d_{2n+2} :$

$$\begin{cases} \varepsilon_0^{2n+2} \mapsto \sum_{l=0}^{b-1} \sum_{l'=0}^{t-1} Y^{tl+l'} \varepsilon_{b(t-l')}^{2n+1} Y^{t(b-1-l)+t-l'-1}, \\ \varepsilon_{b(j)}^{2n+2} \mapsto \sum_{l=0}^{b-1} \sum_{l'=0}^{t-1} Y^{tl+l'} \varepsilon_{b(j-1-l')}^{2n+1} Y^{t(b-1-l)+t-l'-1} \text{ for } 2 \leq j \leq t, \\ \varepsilon_1^{2n+2} \mapsto q Y^t \varepsilon_1^{2n+1} - \varepsilon_1^{2n+1} Y^t - \sum_{j=1}^t X^s Y^{t-j} \varepsilon_{b(j)}^{2n+1} Y^{j-1} + q^{bn+1} \sum_{j=1}^t Y^{t-j} \varepsilon_{b(j)}^{2n+1} Y^{j-1} X^s, \\ \varepsilon_{l''}^{2n+2} \mapsto \begin{cases} \sum_{j=0}^{b-1} q^{alj} Y^{tj} \varepsilon_{2l}^{2n+1} Y^{t(b-1-j)} + \sum_{i=0}^{a-1} q^{ib(n-l+1)} X^{s(a-1-i)} \varepsilon_{2l-1}^{2n+1} X^{si} & \text{if } l'' = 2l \text{ for } 1 \leq l \leq n, \\ q^{al'+1} Y^t \varepsilon_{2l'+1}^{2n+1} - \varepsilon_{2l'+1}^{2n+1} Y^t - X^s \varepsilon_{2l'}^{2n+1} + q^{b(n-l')+1} \varepsilon_{2l'}^{2n+1} X^s & \text{if } l'' = 2l' + 1 \text{ for } 1 \leq l' \leq n-1, \end{cases} \\ \varepsilon_{2n+1}^{2n+2} \mapsto q^{an+1} \sum_{i=1}^s Y^t X^{s-i} \varepsilon_{a(i)}^{2n+1} X^{i-1} - \sum_{i=1}^s X^{s-i} \varepsilon_{a(i)}^{2n+1} X^{i-1} Y^t - X^s \varepsilon_{2n}^{2n+1} + q \varepsilon_{2n}^{2n+1} X^s, \\ \varepsilon_{a(i)}^{2n+2} \mapsto \sum_{l=0}^{a-1} \sum_{l'=0}^{s-1} X^{sl+l'} \varepsilon_{a(i-1-l')}^{2n+1} X^{s(a-1-l)+s-l'-1} \text{ for } 2 \leq i \leq s, \\ \varepsilon_{2n+2}^{2n+2} \mapsto \sum_{l=0}^{a-1} \sum_{l'=0}^{s-1} X^{sl+l'} \varepsilon_{a(s-l')}^{2n+1} X^{s(a-1-l)+s-l'-1}, \end{cases}$$

for $n \geq 0$, where in the case $n = 0$, $\varepsilon_{l''}^1$ and $\varepsilon_{l''}^2$ vanish, and the image of ε_1^2 by d_2 is

$$\begin{aligned} & -\sum_{j=1}^t X^s Y^{t-j} \varepsilon_{b(j)}^1 Y^{j-1} + q \sum_{j=1}^t Y^{t-j} \varepsilon_{b(j)}^1 Y^{j-1} X^s \\ & + q \sum_{i=1}^s Y^t X^{s-i} \varepsilon_{a(i)}^1 X^{i-1} - \sum_{i=1}^s X^{s-i} \varepsilon_{a(i)}^1 X^{i-1} Y^t. \end{aligned}$$

3.2 Hochschild cohomology group of A_q

Next, we give a basis of the n -th Hochschild cohomology group $\mathrm{HH}^n(A_q) := \mathrm{Ext}_{A_q^e}^n(A_q, A_q)$ for $n \geq 0$, using the minimal projective A_q^e -resolution given in Theorem 3.1. Now we

consider the case $s, t \geq 2$. In the case $s = 1$ or $t = 1$, we can give a basis of $\mathrm{HH}^n(A_q)$ by the same method.

Now, we consider the vector space structure of $\mathrm{HH}^n(A_q)$ for all $n \geq 0$. By the definition of P_n , we have isomorphisms

$$\begin{aligned} u_{2n} : \mathrm{Hom}_{A_q^e}(P_{2n}, A_q) &\xrightarrow{\sim} \prod_{l=0}^{2n} e_1 A_q e_1 \oplus \prod_{i=2}^s e_{a(i)} A_q e_{a(i)} \oplus \prod_{j=2}^t e_{b(j)} A_q e_{b(j)}, \\ u_{2n+1} : \mathrm{Hom}_{A_q^e}(P_{2n+1}, A_q) &\xrightarrow{\sim} \prod_{l=1}^{2n} e_1 A_q e_1 \oplus \prod_{i=1}^s e_{a(i+1)} A_q e_{a(i)} \oplus \prod_{j=1}^t e_{b(j+1)} A_q e_{b(j)}, \end{aligned}$$

where $\prod_{l=1}^0 e_1 A_q e_1 = 0$. We denote the k -modules $\mathrm{Im} u_{2n}$ and $\mathrm{Im} u_{2n+1}$ by P_{2n}^* and P_{2n+1}^* respectively. We see that the dimension of P_n^* is given by

$$\dim_k P_n^* = nab + ab(s + t - 1).$$

The elements $e_1, e_{a(i)}$ and $e_{b(j)}$ of P_{2n}^* are labeled e_l^{2n} for $0 \leq l \leq 2n$, $e_{a(i)}^{2n}$ for $2 \leq i \leq s$ and $e_{b(j)}^{2n}$ for $2 \leq j \leq t$ respectively. Similarly, we denote the elements $e_1, e_{a(i)}$ and $e_{b(j)}$ of P_{2n+1}^* by e_l^{2n+1} for $1 \leq l \leq 2n$, $e_{a(i)}^{2n+1}$ for $1 \leq i \leq s$ and $e_{b(j)}^{2n+1}$ for $1 \leq j \leq t$ respectively. These labels correspond to that of generators of P_n . Using the maps $u_{2n}, u_{2n+1}, d_{2n+1}, d_{2n+2}$ for $n \geq 0$, we obtain the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{A_q^e}(P_0, A_q) & \xrightarrow{\bar{d}_1} & \mathrm{Hom}_{A_q^e}(P_1, A_q) & \xrightarrow{\bar{d}_2} & \mathrm{Hom}_{A_q^e}(P_2, A_q) \xrightarrow{\bar{d}_3} \cdots \\ & & \sim \downarrow u_0 & & \sim \downarrow u_1 & & \sim \downarrow u_2 \\ 0 & \longrightarrow & P_0^* & \xrightarrow{d_1^*} & P_1^* & \xrightarrow{d_2^*} & P_2^* \xrightarrow{d_3^*} \cdots, \end{array}$$

where we put $\bar{d}_n = \mathrm{Hom}_{A_q^e}(d_n, A_q)$, $d_n^* = u_n \bar{d}_n u_{n-1}^{-1}$ for $n \geq 1$. Hence we have the complex

$$\mathbb{P} : 0 \rightarrow P_0^* \xrightarrow{d_1^*} P_1^* \rightarrow \cdots \rightarrow P_{n-1}^* \xrightarrow{d_n^*} P_n^* \rightarrow \cdots.$$

See [16] for the homomorphism d_n^* . Now, we denote some elements of P_n^* as follows:

- For $n = 0$:

$$\begin{aligned} T_{l,l'} &:= X^{sl} Y^{tl'} e_0^0 + \sum_{j=2}^t Y^{j-1} X^{sl} Y^{t(l'-1)+t-j+1} e_{b(j)}^0 \\ &\quad + \sum_{i=2}^s X^{s(l-1)+i-1} Y^{tl'} X^{s-i+1} e_{a(i)}^0 \text{ for } 1 \leq l \leq a \text{ and } 1 \leq l' \leq b, \end{aligned}$$

- For n odd, $n \geq 1$:

$$\begin{aligned} U_{0,l,l'}^n &:= Y X^{sl} Y^{tl'} e_{b(1)}^n \text{ for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \\ U_{m,l,l'}^n &:= X^{sl} Y^{tl'} e_m^n \text{ for } 1 \leq m \leq n-1, 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \\ U_{n,l,l'}^n &:= X^{sl+1} Y^{tl'} e_{a(1)}^n \text{ for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \end{aligned}$$

- For n even, $n \geq 2$:

$$\begin{aligned}
W_{0,l,l'}^n &:= X^{sl} Y^{tl'} e_0^n + \sum_{j=2}^t Y^{j-1} X^{sl} Y^{t(l'-1)+t-j+1} e_{b(j)}^n \\
&\quad \text{for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b, \\
W_{0,l,b-1}^{n*} &:= b X^{sl} Y^{t(b-1)} e_0^n + b \sum_{j=2}^t Y^{j-1} X^{sl} Y^{t(b-2)+t-j+1} e_{b(j)}^n \\
&\quad + (q^{b(n/2-1)+1} - 1) X^{s(l+1)} e_1^n \quad \text{for } 0 \leq l \leq a-1, \\
W_{m,l,l'}^n &:= X^{sl} Y^{tl'} e_m^n \text{ for } 1 \leq m \leq n-1, 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b-1, \\
W_{n,l,l'}^n &:= X^{sl} Y^{tl'} e_n^n + \sum_{i=2}^s X^{s(l-1)+i-1} Y^{tl'} X^{s-i+1} e_{a(i)}^n \\
&\quad \text{for } 0 \leq l \leq a \text{ and } 0 \leq l' \leq b-1, \\
W_{n,a-1,l'}^{n*} &:= a X^{s(a-1)} Y^{tl'} e_n^n + a \sum_{i=2}^s X^{s(a-2)+i-1} Y^{tl'} X^{s-i+1} e_{a(i)}^n \\
&\quad + (q^{a(n/2-1)+1} - 1) Y^{t(l'+1)} e_{n-1}^n \quad \text{for } 0 \leq l' \leq b-1.
\end{aligned}$$

In the following results we use the complex $\mathbb{P}^*$ to compute a basis of the Hochschild cohomology group $\mathrm{HH}^n(A_q) = \mathrm{Ker} d_{n+1}^* / \mathrm{Im} d_n^*$ of our algebra A_q for $n \geq 0$. First, we consider the case where q is an r -th root of unity for integer $r \geq 1$. Now, we set $\bar{z}$ is the remainder when we divide z by r for any integer z . Then we have $0 \leq \bar{z} \leq r-1$.

Theorem 3.2 ([15, Proposition 3.3] and [16, Theorem 2.1]). *If q is an r -th root of unity for $r \geq 1$ and $s, t \geq 2$, then the following elements form a basis of $\mathrm{HH}^n(A_q)$.*

- (1) *Basis of $\mathrm{HH}^0(A_q)$:*

- (a) $1_A = e_0^0 + \sum_{j=2}^t e_{b(j)}^0 + \sum_{i=2}^s e_{a(i)}^0$,
- (b) $T_{l,l'}$ for $1 \leq l \leq a-1, 1 \leq l' \leq b-1$ if $\bar{l} = \bar{l}' = 0$,
- (c) $T_{l,b}$ for $1 \leq l \leq a-1$,
- (d) $T_{a,l'}$ for $1 \leq l' \leq b-1$,

- (2) *Basis of $\mathrm{HH}^{2n}(A_q)$ for $n \geq 1$:*

- (a) $W_{0,0,l'}^{2n}$ for $0 \leq l' \leq b-1$ if $\bar{l}' = \bar{b}n$,
- (b) $W_{0,l,l'}^{2n}$ for $1 \leq l \leq a-1, 1 \leq l' \leq b-1$ if $\bar{l} = 0, \bar{l}' = \bar{b}n$,
- (c) $W_{0,l,b}^{2n}$ for $1 \leq l \leq a-1$ if $\bar{l} = 0, \overline{b(n-1)} = 0, \mathrm{char} k | b$,
- (d) $W_{0,a-1,l'}^{2n}$ for $0 \leq l' \leq b-1$ if $\bar{a} = 1, \overline{b(n-1)} \neq 0$,
- (e) $\begin{cases} W_{0,l,b-1}^{2n*} \text{ for } 0 \leq l \leq a-1 \text{ if } \bar{l} = 0, \overline{b(n-1)+1} \neq 0, \mathrm{char} k \nmid b, \\ W_{1,l+1,0}^{2n} \text{ for } 0 \leq l \leq a-2 \text{ if } \bar{l} = 0, \overline{b(n-1)+1} \neq 0, \mathrm{char} k | b, \end{cases}$
- (f) $W_{1,l+1,1}^{2n}$ for $1 \leq l \leq a-2$ if $\bar{l} = 0, \overline{b(n-1)} \neq 0$,
- (g) $\begin{cases} W_{2l''-1,l,l'}^{2n} \text{ for } 1 \leq l \leq a-1, 1 \leq l' \leq b-1 \text{ if } \mathrm{char} k \nmid a, \mathrm{char} k \nmid b, \\ W_{2l''-1,l,l'}^{2n} \text{ for } 1 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \mathrm{char} k \nmid a, \mathrm{char} k | b, \\ W_{2l''-1,l,l'}^{2n} \text{ for } 0 \leq l \leq a-1, 1 \leq l' \leq b-1 \text{ if } \mathrm{char} k | a, \mathrm{char} k \nmid b \\ W_{2l''-1,l,l'}^{2n} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \mathrm{char} k | a, \mathrm{char} k | b, \end{cases}$
for $1 \leq l'' \leq n$ if $\bar{l} = \overline{a(l''-1)+1}, \bar{l}' = \overline{b(n-l'')+1}$,

- (h)
$$\begin{cases} W_{2l'',l,l'}^{2n} \text{ for } 0 \leq l \leq a-2, 0 \leq l' \leq b-2 \text{ if } \text{char } k \nmid a, \text{char } k \nmid b, \\ W_{2l'',l,l'}^{2n} \text{ for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \text{char } k \nmid a, \text{char } k|b, \\ W_{2l'',l,l'}^{2n} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-2 \text{ if } \text{char } k|a, \text{char } k \nmid b, \\ W_{2l'',l,l'}^{2n} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \text{char } k|a, \text{char } k|b, \\ \text{for } 1 \leq l'' \leq n-1 \text{ if } \bar{l} = \overline{al''}, \bar{l}' = \overline{b(n-l'')}, \end{cases}$$
- (i) $W_{2n-1,1,l'+1}^{2n}$ for $1 \leq l' \leq b-2$ if $\bar{l}' = 0, \overline{a(n-1)} \neq 0$,
- (j)
$$\begin{cases} W_{2n,a-1,l'}^{2n*} \text{ for } 0 \leq l' \leq b-1 \text{ if } \bar{l}' = 0, \overline{a(n-1)+1} \neq 0, \text{char } k \nmid a, \\ W_{2n-1,0,l'+1}^{2n} \text{ for } 0 \leq l' \leq b-2 \text{ if } \bar{l}' = 0, \overline{a(n-1)+1} \neq 0, \text{char } k|a, \end{cases}$$
- (k) $W_{2n,l,0}^{2n}$ for $0 \leq l \leq a-1$ if $\bar{l} = \overline{an}$,
- (l) $W_{2n,l,l'}^{2n}$ for $1 \leq l \leq a-1, 1 \leq l' \leq b-1$ if $\bar{l}' = 0, \bar{l} = \overline{an}$,
- (m) $W_{2n,a,l'}^{2n}$ for $1 \leq l' \leq b-1$ if $\bar{l}' = 0, \overline{a(n-1)} = 0, \text{char } k|a$,
- (n) $W_{2n,l,b-1}^{2n}$ for $0 \leq l \leq a-1$ if $\bar{b} = 1, \overline{an-l} \neq 0$
- (o) *Additionally in the case of $q = -1$:*
- i. $W_{0,l,b}^{2n}$ for $0 \leq l \leq a-1$ if $\overline{an} = 0, \bar{b} = 0, \bar{l} = 1$,
 - ii. $W_{1,l,0}^{2n}$ for $1 \leq l \leq a-1$ if $\bar{b} = 0, \bar{l} = 0$,
 - iii. $W_{2l''-1,0,0}^{2n}$ for $1 \leq l'' \leq n$ if $\bar{a} = \bar{b} = 0$,
 - iv. $W_{2l'',a-1,b-1}^{2n}$ for $1 \leq l'' \leq n-1$ if $\bar{a} = \bar{b} = 0$,
 - v. $W_{2n-1,0,l'}^{2n}$ for $1 \leq l' \leq b-1$ if $\bar{a} = 0, \bar{l}' = 0$,
 - vi. $W_{2n,a,l'}^{2n}$ for $0 \leq l' \leq b-1$ if $\bar{a} = 0, \overline{bn} = 0, \bar{l}' = 1$,

(3) *Basis of $\text{HH}^{2n+1}(A_q)$ for $n \geq 0$:*

- (a) $U_{0,l,l'}^{2n+1}$ for $0 \leq l \leq a-1, 1 \leq l' \leq b-1$ if $\bar{l}' = \overline{bn}, \bar{l} = 0$,
- (b) $U_{0,a-1,l'}^{2n+1}$ for $1 \leq l' \leq b-1$ if $\bar{a} = 1, \overline{bn-l'} \neq 0$,
- (c) $U_{0,l,b-1}^{2n+1}$ for $0 \leq l \leq a-1$ if $\bar{l} = 0, \overline{b(n-1)+1} \neq 0$,
- (d)
$$\begin{cases} U_{0,0,0}^{2n+1} \text{ if } \overline{bn} = 0, \text{char } k \nmid b, \\ U_{0,l,0}^{2n+1} \text{ for } 0 \leq l \leq a-1 \text{ if } \bar{l} = 0, \overline{bn} = 0, \text{char } k|b, \end{cases}$$
- (e) $U_{1,l+1,0}^{2n+1}$ for $0 \leq l \leq a-2$ if $\bar{l} = 0, \overline{bn} \neq 0$,
- (f)
$$\begin{cases} U_{2l'',l,l'}^{2n+1} \text{ for } 0 \leq l \leq a-2, 1 \leq l' \leq b-1 \text{ if } \text{char } k \nmid a, \text{char } k \nmid b, \\ U_{2l'',l,l'}^{2n+1} \text{ for } 0 \leq l \leq a-1, 1 \leq l' \leq b-1 \text{ if } \text{char } k|a, \text{char } k \nmid b, \\ U_{2l'',l,l'}^{2n+1} \text{ for } 0 \leq l \leq a-2, 0 \leq l' \leq b-1 \text{ if } \text{char } k \nmid a, \text{char } k|b, \\ U_{2l'',l,l'}^{2n+1} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \text{char } k|a, \text{char } k|b, \\ \text{for } 1 \leq l'' \leq n \text{ if } \bar{l} = \overline{al''}, \bar{l}' = \overline{b(n-l'')+1}, \end{cases}$$
- (g)
$$\begin{cases} U_{2l''+1,l,l'}^{2n+1} \text{ for } 1 \leq l \leq a-1, 0 \leq l' \leq b-2 \text{ if } \text{char } k \nmid a, \text{char } k \nmid b, \\ U_{2l''+1,l,l'}^{2n+1} \text{ for } 1 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \text{char } k \nmid a, \text{char } k|b, \\ U_{2l''+1,l,l'}^{2n+1} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-2 \text{ if } \text{char } k|a, \text{char } k \nmid b, \\ U_{2l''+1,l,l'}^{2n+1} \text{ for } 0 \leq l \leq a-1, 0 \leq l' \leq b-1 \text{ if } \text{char } k|a, \text{char } k|b, \\ \text{for } 0 \leq l'' \leq n-1 \text{ if } \bar{l} = \overline{al''+1}, \bar{l}' = \overline{b(n-l'')}, \end{cases}$$

- (h) $U_{2n,0,l'+1}^{2n+1}$ for $0 \leq l' \leq b-2$ if $\bar{l}' = 0, \bar{a}\bar{n} \neq 0$,
- (i) $U_{2n+1,l,l'}^{2n+1}$ for $1 \leq l \leq a-1, 0 \leq l' \leq b-1$ if $\bar{l}' = 0, \bar{l} = \bar{a}\bar{n}$,
- (j) $U_{2n+1,a-1,l'}^{2n+1}$ for $0 \leq l' \leq b-1$ if $\bar{l}' = 0, \overline{a(n-1)+1} \neq 0$,
- (k) $U_{2n+1,l,b-1}^{2n+1}$ for $1 \leq l \leq a-1$ if $\bar{b} = 1, \overline{an-l} \neq 0$,
- (l) $\begin{cases} U_{2n+1,0,0}^{2n+1} & \text{if } \bar{a}\bar{n} = 0, \text{char } k \nmid a, \\ U_{2n+1,0,l'}^{2n+1} & \text{for } 0 \leq l' \leq b-1 \text{ if } \bar{l}' = 0, \bar{a}\bar{n} = 0, \text{char } k \mid a, \end{cases}$
- (m) Additionally in the case of $q = -1$:
 - i. $U_{0,l,0}^{2n+1}$ for $0 \leq l \leq a-1$ if $\overline{a(n-1)} = 0, \bar{b} = 0, \bar{l} = 1$,
 - ii. $U_{1,l,b-1}^{2n+1}$ for $1 \leq l \leq a-1$ if $\bar{b} = 0, \bar{l} = 0$,
 - iii. $U_{2l''-1,0,b-1}^{2n+1}$ for $1 \leq l'' \leq n$ if $\bar{a} = \bar{b} = 0$,
 - iv. $U_{2l'',a-1,0}^{2n+1}$ for $1 \leq l'' \leq n$ if $\bar{a} = \bar{b} = 0$,
 - v. $U_{2n,a-1,l'}^{2n+1}$ for $1 \leq l' \leq b-1$ if $\bar{a} = 0, \bar{l}' = 0$,
 - vi. $U_{2n+1,0,l'}^{2n+1}$ for $0 \leq l' \leq b-1$ if $\bar{a} = 0, \overline{b(n-1)} = 0, \bar{l}' = 1$.

In the case $q = 1$, q is a first root of unity. Then $\bar{z} = 0$ for any integer z . Hence if $q = 1$ then a basis of $\text{HH}^n(A_q)$ is formed by the elements of (1), (2)(a),(b),(c),(g),(h),(k),(l),(m) and (3)(a), (d), (f), (g), (i), (l).

Next, in the case where q is not a root of unity, we give a basis of $\text{HH}^n(A_q)$ for $n \geq 0$.

Theorem 3.3 ([16, Theorem 2.2]). *If q is not a root of unity and $s, t \geq 2$, then the following elements form a basis of $\text{HH}^n(A_q)$.*

- (1) Basis of $\text{HH}^0(A_q)$:
 - (a) $1_{A_q} = e_0^0 + \sum_{j=2}^t e_{b(j)}^0 + \sum_{i=2}^s e_{a(i)}^0$,
 - (b) $T_{l,b}$ for $1 \leq l \leq a-1$,
 - (c) $T_{a,l'}$ for $1 \leq l' \leq b-1$,
- (2) Basis of $\text{HH}^{2n+1}(A_q)$ for $n \geq 0$:
 - (a) $U_{0,l+1,b-1}^1$ for $0 \leq l \leq a-2$ if $n = 0$,
 - (b) $U_{0,0,b-1}^{2n+1}$,
 - (c) $U_{0,0,0}^1$ if $n = 0$,
 - (d) $U_{1,a-1,l'+1}^1$ for $0 \leq l \leq b-3$ if $n = 0$,
 - (e) $U_{2n+1,a-1,0}^{2n+1}$,
 - (f) $U_{1,0,0}^1$ if $n = 0$,
- (3) Basis of $\text{HH}^{2n+2}(A_q)$ for $n \geq 0$:
 - (a) $W_{1,1,1}^2$ if $n = 0$,
 - (b) $W_{0,0,b-1}^{2n+2*}$,
 - (c) $W_{2n+2,a-1,0}^{2n+2*}$.

3.3 Hochschild cohomology ring of A_q

In this section, we determine the Hochschild cohomology ring of A_q modulo nilpotence. Now we recall the Yoneda product in $\mathrm{HH}^*(A)$ (see [8]). For homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$ represented by $\eta: P_m \rightarrow A$ and $\theta: P_n \rightarrow A$ respectively, the Yoneda product $\eta\theta \in \mathrm{HH}^{m+n}(A)$ is given as follows: There exists a commutative diagram of A -bimodules

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_{m+n} & \xrightarrow{d_{m+n}} & \cdots & \xrightarrow{d_{n+2}} & P_{n+1} \xrightarrow{d_{n+1}} P_n \\ & & \downarrow \sigma_m & & & \downarrow \sigma_1 & \downarrow \sigma_0 \searrow \theta \\ \cdots & \longrightarrow & P_m & \xrightarrow{d_m} & \cdots & \xrightarrow{d_2} & P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} A \longrightarrow 0, \end{array}$$

where σ_i are liftings of θ . Here we see that such liftings always exist but are not unique. Then we have $\eta\theta = \eta\sigma_m \in \mathrm{HH}^{m+n}(A)$. We note that $\eta\theta$ is independent of the choice of representation η, θ and liftings σ_i ($0 \leq i \leq m$). See [16, Proposition 3.1] for the liftings of the basis of $\mathrm{HH}^n(A_q)$ ($n \geq 0$). In the case where q is a root of unity, by the liftings given in [16, Proposition 3.1], we see that $\mathrm{HH}^{n+2r}(A_q)$ is generated by the elements in $\mathrm{HH}^n(A_q)$ and that in $\mathrm{HH}^{2r}(A_q)$ for $n \geq 2r$. By corresponding Yoneda product of the basis elements of $\mathrm{HH}^*(A_q) := \bigoplus_{n \geq 0} \mathrm{HH}^n(A_q)$ given in Theorem 3.2, we now have the generators of $\mathrm{HH}^*(A_q)$. In this paper, we consider the case where $s, t \geq 2$ and $\bar{a}, \bar{b} \neq 0$. In the other cases, we have similar results to the following theorem and corollary. See [16] for the other cases.

Theorem 3.4 ([16, Theorem 3.2]). *In the case where $\bar{a}, \bar{b} \neq 0$, $\mathrm{HH}^*(A_q)$ is generated as an algebra by the following generators:*

- (1) *The generators of $\mathrm{HH}^*(A_q)$ in degree 0:*

$$1_{A_q}, T_{r,r}, T_{l_1,r}, T_{r,l'_1}, T_{l_2,b}, T_{a,l'_2} \\ \text{for } 1 \leq l_1, l_2 \leq a-1, 1 \leq l'_1, l'_2 \leq b-1 \text{ if } \bar{l}_1 = \bar{l}'_1 = 0.$$

- (2) *The generators of $\mathrm{HH}^*(A_q)$ in degree 1:*

- $U_{0,0,l'}, U_{1,l,0}$ for $0 \leq l \leq a-1, 0 \leq l' \leq b-1$ if $\bar{l} = \bar{l}' = 0$,
- $\begin{cases} U_{0,a-1,l'}^1 & \text{for } 1 \leq l' \leq b-1 \text{ if } \bar{a} = 1, \bar{l}' \neq 0, \\ U_{1,a-1,l'}^1 & \text{for } 0 \leq l' \leq b-1 \text{ if } \bar{a} \neq 1, \bar{l}' = 0, \end{cases}$
- $\begin{cases} U_{1,l,b-1}^1 & \text{for } 1 \leq l' \leq a-1 \text{ if } \bar{l} \neq 0, \bar{b} = 1, \\ U_{0,l,b-1}^1 & \text{for } 1 \leq l \leq a-1 \text{ if } \bar{l} = 0, \bar{b} \neq 1, \end{cases}$
- $U_{1,0,l'}^1$ for $1 \leq l' \leq b-1$ if $\bar{l}' = 0$, $\text{char } k | a$,
- $U_{0,l,0}^1$ for $1 \leq l \leq a-1$ if $\bar{l} = 0$, $\text{char } k | b$.

- (3) *The generators of $\mathrm{HH}^*(A_q)$ in degree 2:*

- $W_{0,0,l'_1}^2, W_{2,l_1,0}^2, W_{0,l_2,\bar{b}}^2, W_{2,\bar{a},l'_2}^2$ for $1 \leq l_1, l_2 \leq a-1, 1 \leq l'_1, l'_2 \leq b-1$ if $\bar{l}_1 = \bar{a}, \bar{l}'_1 = \bar{b}, \bar{l}_2 = \bar{l}'_2 = 0$,
- $\begin{cases} W_{0,l,b-1}^{2*} & \text{for } 0 \leq l \leq a-1 \text{ if } \bar{l} = 0, \text{char } k \nmid b, \\ W_{1,l+1,0}^2 & \text{for } 0 \leq l \leq a-2 \text{ if } \bar{l} = 0, \text{char } k | b, \end{cases}$

- $\begin{cases} W_{2,a-1,l'}^{2*} \text{ for } 0 \leq l' \leq b-1 \text{ if } \bar{l}' = 0, \text{char } k \nmid a, \\ W_{1,0,l'+1}^2 \text{ for } 0 \leq l' \leq b-2 \text{ if } \bar{l}' = 0, \text{char } k|a, \end{cases}$
- $W_{0,a-1,l'}^2 \text{ for } 0 \leq l' \leq b-1 \text{ if } \bar{a} = 1, \bar{l}' \neq \bar{b},$
- $W_{2,l,b-1}^2 \text{ for } 0 \leq l \leq a-1 \text{ if } \bar{b} = 1, \bar{l} \neq \bar{a}.$

(4) The generators of $\text{HH}^*(A_q)$ in degree $2n$ for $2 \leq n \leq r$:

- $\begin{cases} W_{0,0,\bar{b}n}^{2n} \text{ if } \min\{\bar{b}n'|1 \leq n' \leq n-1\} > \bar{b}n, \\ W_{0,l,\bar{b}n}^{2n} \text{ for } 1 \leq l \leq a-1 \text{ if } \bar{l} = 0, \bar{b}n \neq 0, \min\{\bar{b}n'|1 \leq n' \leq n-1\} > \bar{b}n, \\ W_{0,a-1,\bar{l}'_1+\bar{l}'_2-r}^{2n} \\ \text{if } \bar{a} = 1, \bar{l}'_1 + \bar{l}'_2 \geq r+1, \bar{l}'_1 = \min\{\bar{b}n'|1 \leq n' \leq n-1\}, \\ \bar{l}'_2 \neq \overline{b(n-n')} \text{ where } n' \text{ is integer such that } 1 \leq n' \leq n-1 \text{ and} \\ \bar{b}n' \leq \bar{b}n'' \text{ for any } 1 \leq n'' \leq n-1, \end{cases}$
- $\begin{cases} W_{0,l,b-1}^{2n*} \text{ for } 0 \leq l \leq a-1 \text{ if } \bar{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char } k \nmid b, \\ W_{1,l+1,0}^{2n} \text{ for } 0 \leq l \leq a-2 \text{ if } \bar{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char } k|b, \end{cases}$
- $W_{1,l+1,1}^{2n} \text{ for } 1 \leq l \leq a-2 \text{ if } \bar{l} = 0, \overline{b(n-1)} \neq 0,$
- $\begin{cases} W_{1,l,0}^{2n} \text{ for } 1 \leq l \leq a-1 \text{ if } \bar{l} = 1, \overline{b(n-1)+1} = 0, \text{char } k|b, \\ W_{2n-1,0,l'}^{2n} \text{ for } 1 \leq l' \leq b-1 \text{ if } \overline{a(n-1)+1} = 0, \bar{l}' = 1, \text{char } k|a, \\ W_{2l''-1,0,0}^{2n} \text{ for } 1 \leq l'' \leq n \\ \text{if } \overline{a(l''-1)+1} = \overline{b(n-l'')+1} = 0, \text{char } k|a, \text{char } k|b, \end{cases}$
- $W_{2n-1,1,l'+1}^{2n} \text{ for } 1 \leq l' \leq b-2 \text{ if } \bar{l}' = 0, \overline{a(n-1)} \neq 0,$
- $\begin{cases} W_{2n,a-1,l'}^{2n*} \text{ for } 0 \leq l' \leq b-1 \text{ if } \bar{l}' = 0, \overline{a(n-1)+1} \neq 0, \text{char } k \nmid a, \\ W_{2n-1,0,l'+1}^{2n} \text{ for } 0 \leq l' \leq b-2 \text{ if } \bar{l}' = 0, \overline{a(n-1)+1} \neq 0, \text{char } k|a, \end{cases}$
- $\begin{cases} W_{2n,\bar{a}n,0}^{2n} \text{ if } \min\{\bar{a}n'|1 \leq n' \leq n-1\} > \bar{a}n, \\ W_{2n,\bar{a}n,l'}^{2n} \\ \text{for } 1 \leq l' \leq b-1 \text{ if } \bar{l}' = 0, \bar{a}n \neq 0, \min\{\bar{a}n'|1 \leq n' \leq n-1\} > \bar{a}n, \\ W_{2n,\bar{l}'_1+\bar{l}'_2-r,b-1}^{2n} \\ \text{if } \bar{b} = 1, \bar{l}'_1 + \bar{l}'_2 \geq r+1, \bar{l}'_1 = \min\{\bar{a}n'|1 \leq n' \leq n-1\}, \\ \bar{l}'_2 \neq \overline{a(n-n')} \text{ where } n' \text{ is integer such that } 1 \leq n' \leq n-1 \text{ and} \\ \bar{a}n' \leq \bar{a}n'' \text{ for any } 1 \leq n'' \leq n-1. \end{cases}$

(5) The generators of $\text{HH}^*(A_q)$ in degree $2n+1$ for $1 \leq n \leq r-1$:

- $U_{0,l,\bar{b}n}^{2n+1} \text{ for } 0 \leq l \leq a-1 \text{ if } \bar{l} = 0, \min\{\bar{b}n'|1 \leq n' \leq n-1\} \geq \bar{b}n,$
- $U_{0,l,b-1}^{2n+1} \text{ for } 0 \leq l \leq a-1 \text{ if } \bar{l} = 0, \overline{b(n-1)+1} \neq 0, \text{char } k|b,$
- $U_{1,l+1,0}^{2n+1} \text{ for } 0 \leq l \leq a-2 \text{ if } \bar{l} = 0, \bar{b}n \neq 0,$
- $\begin{cases} U_{2,l,0}^{2n+1} \text{ for } 0 \leq l \leq a-2 \text{ if } \bar{l} = \bar{a}, \overline{b(n-1)+1} = 0, \\ U_{2l'',\bar{a}l''',0}^{2n+1} \text{ for } 2 \leq l'' \leq n, 0 \leq \bar{a}l''' \leq \begin{cases} a-2 \text{ if } \text{char } k \nmid a, \\ a-1 \text{ if } \text{char } k|a, \end{cases} \\ \text{if } \overline{b(n-l'')+1} = 0, \min\{\bar{a}l'''|1 \leq l''' \leq l''-1\} > \bar{a}l''', \end{cases} \quad \text{if } \text{char } k|b,$

- $\begin{cases} U_{2l''+1,0,\overline{b(n-l'')}}^{2n+1} \text{ for } 0 \leq l'' \leq n-1, 0 \leq \overline{b(n-l'')} \leq \begin{cases} b-2 & \text{if char } k \nmid b, \\ b-1 & \text{if char } k|b, \end{cases} \\ \text{if } \overline{al''+1} = 0, \min\{\overline{b(n'-l'')|l''+1} \leq n' \leq n-1\} > \overline{b(n-l'')}, \\ U_{2n-1,0,l'}^{2n+1} \text{ for } 0 \leq l' \leq b-2 \text{ if } \overline{a(n-1)+1} = 0, \overline{l'} = \overline{b}, \\ \text{if char } k|a, \end{cases}$
- $U_{2n,0,l'+1}^{2n+1}$ for $0 \leq l' \leq b-2$ if $\overline{l'} = 0, \overline{an} \neq 0$,
- $U_{2n+1,\overline{an},l'}^{2n+1}$ for $0 \leq l' \leq b-1$ if $\overline{l'} = 0, \min\{\overline{an'|1} \leq n' \leq n-1\} \geq \overline{an}$,
- $U_{2n+1,a-1,l'}^{2n+1}$ for $0 \leq l' \leq b-1$ if $\overline{l'} = 0, \overline{a(n-1)+1} \neq 0, \text{char } k|a$.

(6) The generators of $\text{HH}^*(A_q)$ in degree $2r+2n+1$ for $0 \leq n \leq r-2$:

- $U_{2l'',\overline{al''},0}^{2r+2n+1}$ for $n+1 \leq l'' \leq r, 0 \leq \overline{al''} \leq \begin{cases} a-2 & \text{if char } k \nmid a, \\ a-1 & \text{if char } k|a, \end{cases}$
if $\min\{\overline{al'''}|1 \leq l''' \leq l''-1\} > \overline{al''}, \overline{b(n-l'')}+1 = 0, \text{char } k|b$,
- $U_{2l''+1,0,\overline{b(n-l'')}}^{2r+2n+1}$ for $1 \leq l'' \leq r-1, 0 \leq \overline{b(n-l'')} \leq \begin{cases} b-2 & \text{if char } k \nmid b, \\ b-1 & \text{if char } k|b, \end{cases}$
if $\overline{al''+1} = 0, \min\{\overline{b(n'-l'')|l''+1} \leq n' \leq r+n-1\} > \overline{b(n-l'')}, \text{char } k|a$.

It follows from the Theorem 3.4 that 1_{A_q} , $W_{0,0,0}^{2r}$ and $W_{2r,0,0}^{2r}$ are not nilpotent and the other generators are nilpotent. Thus we have the following corollary.

Corollary 3.5. *If $s, t \geq 2$ and $\overline{a}, \overline{b} \neq 0$, then the quotient of the Hochschild cohomology ring of A_q modulo nilpotence is isomorphic to the polynomial ring of two variables in all characteristic:*

$$\text{HH}^*(A_q)/\mathcal{N} \cong k[W_{0,0,0}^{2r}, W_{2r,0,0}^{2r}].$$

Finally, we consider the ring structure of $\text{HH}^*(A_q)$ in the case where q is not a root of unity. It follows from the liftings given in [16] that all basis elements except 1_{A_q} of $\text{HH}^n(A_q)$ are nilpotent elements for $n \geq 0$. Thus we have the following results.

Theorem 3.6. *If q is not a root of unity then $\text{HH}^*(A_q)$ is not a finitely generated k -algebra.*

Corollary 3.7. *If q is not a root of unity then $\text{HH}^*(A_q)/\mathcal{N} \cong k$.*

In general, our algebra A_q is not self-injective, monomial or Koszul. Moreover A_q does not have a stratifying ideal. Therefore A_q is new example of a class of algebras for which the Hochschild cohomology ring modulo nilpotence is finitely generated as a k -algebra. For example, in the case where $s = 2, t = 1$ and $a = b = 2$, our algebra A_q is not self-injective, monomial or Koszul. Moreover A_q does not have a stratifying ideal.

3.4 Finiteness conditions for A_q

Finally, we show that A_q satisfies the finiteness conditions in the case where q is a root of unity.

Now we consider the case where q is an r -th root of unity, $s, t \geq 2$ and $\overline{a}, \overline{b} \neq 0$. In the other case, we see that A_q satisfies the finiteness conditions by the same method. The Yoneda algebra or Ext algebra of A_q is given by $E(A_q) = \bigoplus_{n \geq 0} \text{Ext}_{A_q}^n(A_q/\mathfrak{r}, A_q/\mathfrak{r})$

with the Yoneda product. We use the notation $E(A_q)^n = \text{Ext}_{A_q}^n(A_q/\mathfrak{r}, A_q/\mathfrak{r})$ for the n -th graded component of $E(A_q)$. Then it is easy to see that $E(A_q)^n \simeq \coprod_{l=0}^n ke_l^n \oplus \coprod_{j=2}^t ke_{b(j)}^n \oplus \coprod_{i=2}^s ke_{a(i)}^n$.

Let $\varphi: \text{HH}^*(A_q) \rightarrow E(A_q)$ be a homomorphism of graded rings given by $\varphi(\eta) = \eta \otimes_{A_q} A_q/\mathfrak{r}$. Then it is easy to see that the image of φ is precisely the graded ring $k[x, y]$ where $x := e_0^{2r} + \sum_{j=2}^t e_{b(j)}^{2r}$ and $y := e_{2r}^{2r} + \sum_{i=2}^s e_{a(i)}^{2r}$ in degree $2r$.

Proposition 3.8. *$E(A_q)$ is a finitely generated left $k[x, y]$ -module with generators:*

$$\begin{aligned} & e_l^{2n}, e_{b(j)}^{2n}, e_{a(i)}^{2n} \text{ for } 0 \leq l \leq 2n, 2 \leq j \leq t \text{ and } 2 \leq i \leq s \\ & \qquad \qquad \qquad \text{in degree } 2n \text{ for } 0 \leq n \leq r-1, \\ & e_l^{2n+1}, e_{b(j)}^{2n+1}, e_{a(i)}^{2n+1} \text{ for } 1 \leq l \leq 2n, 1 \leq j \leq t \text{ and } 1 \leq i \leq s \\ & \qquad \qquad \qquad \text{in degree } 2n+1 \text{ for } 0 \leq n \leq r-1, \\ & e_l^{2r} \text{ for } 1 \leq l \leq 2r-1 \quad \text{in degree } 2r, \\ & e_l^{2r+2n+1} \text{ for } 2n+1 \leq l \leq 2r \quad \text{in degree } 2r+2n+1 \text{ for } 0 \leq n \leq r-1, \\ & e_l^{2r+2n+2} \text{ for } 2n+1 \leq l \leq 2r-1 \quad \text{in degree } 2r+2n+2 \text{ for } 0 \leq n \leq r-2. \end{aligned}$$

Now we consider the conditions **(Fg1)** and **(Fg2)**. The element $W_{0,0,0}^{2r} \in \text{HH}^{2r}(A_q)$ is a pre-image of x and the element $W_{2r,0,0}^{2r} \in \text{HH}^{2r}(A_q)$ is a pre-image of y . Let H be the graded subalgebra of $\text{HH}^*(A_q)$ generated by $\text{HH}^0(A_q)$, $W_{0,0,0}^{2r}$ and $W_{2r,0,0}^{2r}$, so that H is a pre-image of $k[x, y]$ in $\text{HH}^*(A_q)$. Then we have the following immediate consequence of Proposition 3.8.

Theorem 3.9. *The conditions **(Fg1)** and **(Fg2)** hold for the algebra A_q with respect to the subring H of $\text{HH}^*(A_q)$.*

By [2], Theorem 3.6 and 3.9, we have the necessary and sufficient conditions for A_q to satisfy the finiteness conditions.

Theorem 3.10. *A_q satisfies the finiteness conditions if and only if q is a root of unity.*

References

- [1] P. A. Bergh, K. Erdmann, *Homology and cohomology of quantum complete intersections*, Algebra Number Theory **2** (2008), 501–522.
- [2] P. A. Bergh, S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra **320** (2008), 3327–3338.
- [3] R.-O. Buchweitz, E. L. Green, D. Madsen, Ø. Solberg, *Finite Hochschild cohomology without finite global dimension*, Math. Res. Lett. **12** (2005), 805–816.
- [4] Erdmann, K., Holloway, M., Snashall, N., Solberg, Ø., R. Taillefer, *Support varieties for selfinjective algebras*, K-Theory **33** (2004), 67–87.
- [5] K. Erdmann, Ø. Solberg, *Radical cube zero weakly symmetric algebras and support varieties*, J. Pure Appl. Algebra **215** (2011), 185–200.
- [6] L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239.

- [7] Friedlander, E. M. and Suslin, A., *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), 209–270.
- [8] T. Furuya, K. Sanada, *Hochschild cohomology of an algebra associated with a circular quiver*, Comm. Algebra **34** (2006), 2019–2037.
- [9] T. Furuya, N. Snashall, *Support varieties for modules over stacked monomial algebras*, Preprint arXiv:math/0903.5170 to appear in Comm. Algebra.
- [10] E. L. Green, N. Snashall, *Projective bimodule resolutions of an algebra and vanishing of the second Hochschild cohomology group*, Forum. Math. **16** (2004), 17–36.
- [11] E. L. Green, N. Snashall, Ø. Solberg, *The Hochschild cohomology ring of a self-injective algebra of finite representation type*, Proc. Amer. Math. Soc. **131** (2003), 3387–3393.
- [12] E. L. Green, N. Snashall, Ø. Solberg, *The Hochschild cohomology ring modulo nilpotence of a monomial algebra*, J. Algebra Appl. **5** (2006), 153–192.
- [13] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Springer Lecture Notes **1404** (1989), 108–126.
- [14] S. Koenig, H. Nagase, *Hochschild cohomology and stratifying ideals*, J. Pure Appl. Algebra **213** (2009), 886–891.
- [15] D. Obara, *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation*, Preprint to appear in Comm. Algebra.
- [16] D. Obara, *Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation II*, Preprint to appear in Comm. Algebra.
- [17] S. Schroll, N. Snashall, *Hochschild cohomology and support varieties for tame Hecke algebras*, Preprint arXiv:math./0907.3435 to appear in Quart. J Math..
- [18] N. Snashall, *Support varieties and the Hochschild cohomology ring modulo nilpotence*, In Proceedings of the 41st Symposium on Ring Theory and Representation Theory, Shizuoka, Japan, Sept. 5–7, 2008; Fujita, H., Ed.; Symp. Ring Theory Represent. Theory Organ. Comm., Tsukuba, 2009, pp. 68–82.
- [19] N. Snashall, Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. **88** (2004), 705–732.
- [20] N. Snashall, R. Taillefer, *The Hochschild cohomology ring of a class of special biserial algebras*, J. Algebra Appl. **9** (2010), 73–122.
- [21] Ø. Solberg, *Support varieties for modules and complexes*, Trends in representation theory of algebras and related topics, Amer. Math. Soc., pp. 239–270.
- [22] B. B. Venkov, *Cohomology algebras for some classifying spaces*, Dokl. Akad. Nauk SSSR **127** (1959), 943–944.
- [23] F. Xu, *Hochschild and ordinary cohomology rings of small categories*, Adv. Math. **219** (2008), 1872–1893.