

Borel approximation of coanalytic sets with Borel sections and the regularity properties for Σ_2^1 sets of reals

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In 2008, Fujita showed the following:

Theorem 1 (Fujita [5]). The following are equivalent:

1. If $A \subseteq \mathbb{R} \times \mathbb{R}$ is Π_1^1 and for any real x , A_x is Borel where $A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$, then there is a comeager Borel set $D \subseteq \mathbb{R}$ such that $A \cap (D \times \mathbb{R})$ is Borel, and
2. every Σ_2^1 set of reals has the Baire property.

We show that one can generalize the above theorem to a wide class of tree-type ccc forcings. More precisely:

Theorem 2. Let $\mathbb{P}$ be a strongly arboreal, Σ_1^1 , provably ccc forcing. Then the following are equivalent:

1. If $A \subseteq \mathbb{R} \times \mathbb{R}$ is Π_1^1 and for any real x , A_x is Borel, then there is a Borel set $D \subseteq \mathbb{R}$ such that D is of $\mathbb{P}$ -measure one and $A \cap (D \times \mathbb{R})$ is Borel, and
2. every Σ_2^1 set of reals is $\mathbb{P}$ -measurable.

We also show that this equivalence fails for non-ccc forcings. In fact, for Sacks forcing, the corresponding statement to the first fails in ZFC while the one for the second is consistent with ZFC.

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Throughout this paper, we assume the basic knowledge of descriptive set theory and forcing which can be obtained from e.g., [9], [8], and [2]. By *reals*, we mean elements of the Cantor space or those of the Baire space and we use $\mathbb{R}$ to denote the set of reals.

To prove Theorem 2, let us start with basic definitions. Let n be a natural number with $n \geq 1$. A partial order $\mathbb{P}$ is Σ_n^1 if the sets P , $\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ are Σ_n^1 , where $\mathbb{P} = (P, \leq_{\mathbb{P}})$ and $\perp_{\mathbb{P}}$ is the incompatibility relation in $\mathbb{P}$. A partial order $\mathbb{P}$ is *provably ccc* if there is a formula ϕ defining $\mathbb{P}$ and the statement “ ϕ defines a ccc partial order” is provable in ZFC. A partial order $\mathbb{P}$ is *arboreal* if its conditions are perfect trees on ω (or on 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as $\mathbb{P} = \{<^\omega \omega\}$. We need the following stronger notion:

Definition 3. A partial order $\mathbb{P}$ is *strongly arboreal* if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P}) (\forall t \in T) T_t \in \mathbb{P},$$

where $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$.

With strongly arboreal forcings, one can code generic objects by reals in the standard way: Let $\mathbb{P}$ be strongly arboreal and G be $\mathbb{P}$ -generic over V . Let $x_G = \bigcup \{\text{stem}(T) \mid T \in G\}$, where $\text{stem}(T)$ is the longest $t \in T$ such that $T_t = T$. Then x_G is a real and $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$, where $[T]$ is the set of all infinite paths through T . Hence $V[x_G] = V[G]$. We call such real x_G a *$\mathbb{P}$ -generic real over V* .

Almost all typical forcings related to regularity properties are strongly arboreal:

Example 4.

1. Cohen forcing $\mathbb{C}$: Let T_0 be $<^\omega \omega$. Consider the partial order $(\{(T_0)_s \mid s \in <^\omega \omega\}, \subseteq)$. Then this is strongly arboreal and equivalent to Cohen forcing.
2. Random forcing $\mathbb{B}$: Consider the set of all perfect trees T on 2 such that for any $t \in T$, $[T_t]$ has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.
3. Hechler forcing $\mathbb{D}$: For $(n, f) \in \mathbb{D}$, let

$$T_{(n,f)} = \left\{ t \in <^\omega \omega \mid \text{either } t \subseteq f \restriction n \text{ or } \left(t \supseteq f \restriction n \text{ and } (\forall m \in \text{dom}(t)) t(m) \geq f(m) \right) \right\}.$$

Then the partial order $(\{T_{(n,f)} \mid (n,f) \in \mathbb{D}\}, \subseteq)$ is strongly arboreal and equivalent to Hechler forcing.

4. Mathias forcing $\mathbb{R}_M$: For a condition (s, A) in $\mathbb{R}_M$, let

$$T_{(s,A)} = \{t \in {}^{<\omega}\omega \mid t \text{ is strictly increasing and } s \subseteq \text{ran}(t) \subseteq s \cup A\}.$$

Then $\{T_{(s,A)} \mid (s, A) \in \mathbb{R}_M\}$ is a strongly arboreal forcing equivalent to Mathias forcing.

5. Sacks forcing $\mathbb{S}$, Silver forcing $\mathbb{V}$, Miller forcing $\mathbb{M}$, Laver forcing $\mathbb{L}$: These forcings can be naturally seen as strongly arboreal forcings.

The following is as expected:

Lemma 5. Let $\mathbb{P}$ be a strongly arboreal, Σ_1^1 , provably ccc forcing and M be an inner model of ZFC containing parameter defining $\mathbb{P}$ with a Σ_1^1 -formula. Then if x is $\mathbb{P}$ -generic over V , then x is $\mathbb{P}^M$ -generic over M .

Proof. Since $\mathbb{P}$ is Σ_1^1 , $\mathbb{P}^M = \mathbb{P} \cap M$. So it suffices to show that if $A \subseteq \mathbb{P}^M$ is a maximal antichain in M , so is in V . Let $A \subseteq \mathbb{P}^M$ be a maximal antichain in M . Since $\mathbb{P}$ is provably ccc, M thinks $\mathbb{P}^M$ is ccc. So A is countable in M and there is a real r coding A in M . Since $\mathbb{P}$ is Σ_1^1 , the statement “a real r codes a maximal antichain in $\mathbb{P}$ ” is Π_2^1 . So the real r also codes the maximal antichain A in V , as desired. ■

We now introduce a σ -ideal $I_{\mathbb{P}}$ on the reals expressing “smallness” for each strongly arboreal forcing $\mathbb{P}$.

Definition 6. Let $\mathbb{P}$ be a strongly arboreal forcing. A set of reals A is $\mathbb{P}$ -null if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. Let $N_{\mathbb{P}}$ denote the set of all $\mathbb{P}$ -null sets and $I_{\mathbb{P}}$ denote the σ -ideal generated by $\mathbb{P}$ -null sets, i.e., the set of all countable unions of $\mathbb{P}$ -null sets. A set of reals A is of $\mathbb{P}$ -measure one if $\mathbb{R} \setminus A$ is in $I_{\mathbb{P}}$.

Example 7.

1. Cohen forcing $\mathbb{C}$: $\mathbb{C}$ -null sets are the same as nowhere dense sets of reals and $I_{\mathbb{C}}$ is the ideal of meager sets of reals.

2. Random forcing $\mathbb{B}$: $\mathbb{B}$ -null sets are the same as Lebesgue null sets in the Baire space and $I_{\mathbb{B}}$ is the Lebesgue null ideal.

3. Hechler forcing $\mathbb{D}$: $\mathbb{D}$ -null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by $\{[s, f] \mid (s, f) \in \mathbb{D}\}$ where

$$[s, f] = \{x \in {}^\omega\omega \mid s \subseteq x \text{ and } (\forall n \geq \text{dom}(s)) x(n) \geq f(n)\}.$$

Hence $I_{\mathbb{D}}$ is the meager ideal in the dominating topology.

4. Mathias forcing $\mathbb{R}_M$: A set of reals A is $\mathbb{R}_M$ -null if and only if $\{\text{ran}(x) \mid x \in A \cap A_0\}$ is Ramsey null or meager in the Ellentuck topology, where A_0 is the set of strictly increasing infinite sequences of natural numbers. Hence $I_{\mathbb{R}_M} = N_{\mathbb{R}_M}$.

5. Sacks forcing $\mathbb{S}$: In this case, $I_{\mathbb{S}} = N_{\mathbb{S}}$ by a standard fusion argument. The ideal $I_{\mathbb{S}}$ is called the Marczewski ideal and often denoted by s_0 .

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation $I_{\mathbb{P}} = N_{\mathbb{P}}$. In the case of ccc forcings, $I_{\mathbb{P}}$ is often different from $N_{\mathbb{P}}$ (e.g., Cohen forcing and Hechler forcing).

We now introduce $\mathbb{P}$ -measurability:

Definition 8. Let $\mathbb{P}$ be strongly arboreal. A set of reals A is $\mathbb{P}$ -measurable if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] \setminus A \in I_{\mathbb{P}}$.

As is expected, $\mathbb{P}$ -measurability coincides with a known regularity property for $\mathbb{P}$ when $\mathbb{P}$ is ccc:

Proposition 9. Let $\mathbb{P}$ be a strongly arboreal, ccc forcing and let A be a set of reals. Then A is $\mathbb{P}$ -measurable if and only if there is a Borel set B such that $A \Delta B \in I_{\mathbb{P}}$, where $A \Delta B$ is the symmetric difference between A and B .

Proof. See Proposition 2.9 in [6]. □

Proposition 9 does not hold for non-ccc forcings such as Sacks forcing.¹

But $\mathbb{P}$ -measurability is almost the same as the regularity properties for non-ccc forcings $\mathbb{P}$, e.g., for Mathias forcing, a set of reals A is $\mathbb{R}_M$ -measurable if and only if $\{\text{ran}(x) \mid x \in A \cap A_0\}$ is completely Ramsey (or has the

¹For example, assuming every Π_1^1 set has the perfect set property (i.e., either the set is countable or contains a perfect subset), there is no Σ_1^1 Bernstein set (i.e., a set where neither it nor its complement contains a perfect subset) but for a Σ_1^1 set of reals A , A is approximated by a Borel set modulo $I_{\mathbb{S}}$ if and only if A is Borel. This is because $I_{\mathbb{S}}$ restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals by the assumption that every Π_1^1 set has the perfect set property.

Baire property in the Ellentuck topology), where A_0 is the set of all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

Proposition 10 (Brendle, Löwe). Let Γ be a topologically reasonable point-class, i.e., it is a set of sets of reals closed under continuous preimages and any intersection between a set in Γ and a closed set of reals. Then every set in Γ is $\mathbb{S}$ -measurable if and only if there is no Bernstein set in Γ .²

Proof. See [3, Lemma 2.1]. □

As expected, every Σ_1^1 set of reals is $\mathbb{P}$ -measurable:

Theorem 11. Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then every Σ_1^1 set of reals is $\mathbb{P}$ -measurable.

Proof. It follows from the fact that every Σ_1^1 set of reals is universally Baire, that every universally Baire set of reals is $\mathbb{P}$ -Baire, and that every $\mathbb{P}$ -Baire set of reals is $\mathbb{P}$ -measurable. For the details, see [4] and Section 3 in [6]. □

We are now ready to state the theorem characterizing the regularity properties for Σ_2^1 sets of reals in terms of the existence of many generic reals over $L[r]$ for a real r , which we will use for the proof of Theorem 2:

Theorem 12. Let $\mathbb{P}$ be a strongly arboreal, Σ_1^1 , provably ccc forcing. Then the following are equivalent:

1. Every Σ_2^1 set of reals is $\mathbb{P}$ -measurable, and
2. for any real r , the set of $\mathbb{P}$ -generic reals over $L[r]$ is of $\mathbb{P}$ -measure one.

Proof. See Definition 2.11, Lemma 2.13 (3), Definition 2.15, Proposition 2.17 (3), and Theorem 4.4 in [6]. □

We are now ready to prove Theorem 2:

²In general, the property not being a Bernstein set does not imply $\mathbb{S}$ -measurability while the converse is true. By using the axiom of choice, one can construct a set of reals which is not $\mathbb{S}$ -measurable but is not a Bernstein set.

Proof of Theorem 2. The argument is exactly the same as the one in Theorem 1 in [5]. For the sake of completeness, we will give the proof.

We first show the implication from 1. to 2. Let P be a Σ_2^1 set of reals. We will show that P is $\mathbb{P}$ -measurable. Since P is Σ_2^1 , there is a Π_1^1 set $A \subseteq \mathbb{R} \times \mathbb{R}$ such that $P = \{x \in \mathbb{R} \mid (\exists y) (x, y) \in A\}$. By Kondô's uniformization theorem, there is a Π_1^1 function $f: P \rightarrow \mathbb{R}$ uniformizing A . Then for any real x , $f_x = \{y \mid (x, y) \in f\} = \{f(x)\}$ is Borel, so by applying the assumption for f , there is a Borel set D of reals such that D is of $\mathbb{P}$ -measure one and $f \cap (D \times \mathbb{R})$ is Borel. Hence $P \cap D = \{x \mid (\exists y) (x, y) \in f \cap (D \times \mathbb{R})\}$ is Σ_1^1 and is $\mathbb{P}$ -measurable by Theorem 11. So by Proposition 9, there is a Borel set B such that $(P \cap D) \triangle B$ is in $I_{\mathbb{P}}$. Since D is of $\mathbb{P}$ -measure one, $P \triangle B$ is also in $I_{\mathbb{P}}$. Again by Proposition 9, P is $\mathbb{P}$ -measurable, as desired.

We now show the implication from 2. to 1. Let WO be the set of reals coding a well-order on ω . It is well-known that WO is a complete Π_1^1 set of reals. For an element w of WO, $|w|$ denotes the countable ordinal that w codes. We need the following notion and lemma for the proof:

Definition 13. Let r be a real. A set $X \subseteq \mathbb{R} \times \omega_1$ is $\Pi_2^1(r)$ in the codes if the set

$$\{(x, w) \in \mathbb{R} \times \mathbb{R} \mid w \in \text{WO and } (x, |w|) \in X\}$$

is $\Pi_2^1(r)$.

Lemma 14. Let r be a real and $X \subseteq \mathbb{R} \times \omega_1$ be $\Pi_2^1(r)$ in the codes. Suppose that for any real x there is a $\xi < \omega_1$ such that $(x, \xi) \in X$. Then there is a countable ordinal δ such that for any $\mathbb{P}$ -generic real x over $L[r]$, there is a $\xi < \delta$ such that $(x, \xi) \in X$.

Proof of Lemma 14. Since X is $\Pi_2^1(r)$ in the codes, pick a Π_2^1 -formula $\phi(x, w, v)$ such that

$$(\forall x, w) (\phi(x, w, r) \iff w \in \text{WO and } (x, |w|) \in X).$$

Let $\tilde{\phi}(x, \xi, r)$ be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in \text{WO}) |w| = \xi \rightarrow \phi(x, w, r).$$

Then $\tilde{\phi}$ is absolute among all the transitive proper class models of ZFC in which ξ is countable.

For each $\xi < \omega_1$, let

$$X_\xi = \{T \in \mathbb{P} \mid (T, \mathbf{1}_{\mathbb{P}_\xi}) \Vdash_{\mathbb{P} \times \mathbb{P}_\xi} \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})\}^{L[r]},$$

where $\mathbb{P}_\xi$ is $\text{Coll}(\omega, \xi)$ and $\dot{x}$ is a canonical $\mathbb{P}$ -name for a generic real.

We show that $\bigcup_{\xi < \omega_1} X_\xi$ is a dense subset of $\mathbb{P}^{L(\mathbb{R})}$ in $L[r]$. Let T be any element of $\mathbb{P}^{L[r]}$. Take a $\mathbb{P}$ -generic real x over $L[r]$ in V with $x \in [T]$. Then by the assumption, there is a $\xi < \omega_1$ such that $(x, \xi) \in X$. Take a function $g: \omega \rightarrow \xi$ generic over $L[r, x]$. Then $L[r, x, g] \models \tilde{\phi}(x, \xi, r)$. Hence there is a $T' \leq T$ and a condition p in $\mathbb{P}_\xi$ such that $L[r] \models "(T', p) \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})"$. Since $\mathbb{P}_\xi$ is homogeneous, it follows that $L[r] \models "(T', \mathbf{1}_{\mathbb{P}_\xi}) \Vdash \tilde{\phi}(\dot{x}, \check{\xi}, \check{r})"$, so $T' \leq T$ and $T \in \bigcup_{\xi < \omega_1} X_\xi$, as desired.

Since $\mathbb{P}$ is provably ccc, $L[r] \models "\mathbb{P} \text{ is ccc}"$, so there is a $\delta < \omega_1$ such that $\bigcup_{\xi < \delta} X_\xi$ is a predense subset of $\mathbb{P}$ in $L[r]$. We show that this δ is the desired countable ordinal. Take any $\mathbb{P}$ -generic real x over $L[r]$. Then since $L[r]$ thinks $\bigcup_{\xi < \delta} X_\xi$ is a predense subset of $\mathbb{P}$, the generic filter G_x meets $\bigcup_{\xi < \delta} X_\xi$ and hence there is a $\xi < \delta$ such that $G_x \cap X_\xi \neq \emptyset$. By the definition of X_ξ , for a function $g: \omega \rightarrow \xi$ generic over $L[r, x]$, $L[r, x, g] \models \tilde{\phi}(x, \xi, r)$, hence $\tilde{\phi}(x, \xi, r)$ holds also in V and $(x, \xi) \in X$, as desired. $\square$ (Lemma 14)

We now finish showing the implication from 2. to 1. Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be Π_1^1 such that for any real x , A_x is Borel. Let $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f^{-1}(\text{WO}) = A$. Take any real x . Since A_x is Borel, the set $f^{-1}(\{x\} \times A_x)$ is Σ_1^1 , hence by boundedness theorem, it is bounded in WO, i.e.,

$$(\forall x) (\exists \xi) (\forall y) \text{ if } (x, y) \in A, \text{ then } |f(x, y)| < \xi.$$

Set

$$X = \{(x, \xi) \mid f^{-1}(\{x\} \times A_x) \subseteq \text{WO}_\xi\},$$

where $\text{WO}_\xi = \{w \in \text{WO} \mid |w| < \xi\}$ for each $\xi < \omega_1$.

Then for any x there is a ξ with $(x, \xi) \in X$. It is also easy to see that X is $\Pi_2^1(r)$ in the codes for some real r . By Lemma 14, there is a $\delta < \omega_1$ such that for any $\mathbb{P}$ -generic real x over $L[r]$ there is a $\xi < \delta$ such that $(x, \xi) \in X$. Hence A is the same as the Borel set $f^{-1}(\text{WO}_\delta)$ on $G(L[r]) \times \mathbb{R}$, where $G(L[r])$ is the set of $\mathbb{P}$ -generic reals over $L[r]$. By 2. and Theorem 12, the set $G(L[r])$ is of $\mathbb{P}$ -measure one. Since $\mathbb{P}$ is ccc, $I_{\mathbb{P}}$ is Borel generated, so there is a Borel set $D \subseteq G(L[r])$ of $\mathbb{P}$ -measure one and $A \cap (D \times \mathbb{R})$ is Borel, as desired.

■ (Theorem 2)

After the RIMS set theory conference in 2008, Fujita asked if one could take δ in Lemma 14 below γ_2^1 if X is Π_2^1 (lightface) in the codes and if $\mathbb{P}$ is Cohen forcing, where γ_2^1 is the least countable ordinal that meets every set $A \subseteq \omega_1$ which is Π_2^1 (lightface) in the codes.³ We show that this is generally the case for each strongly arboreal, Σ_1^1 , provably ccc forcing $\mathbb{P}$:

Proposition 15. Let $\mathbb{P}$ be a strongly arboreal, Σ_1^1 (lightface), ccc forcing and $X \subseteq \mathbb{R} \times \omega_1$ be Π_2^1 (lightface) in the codes such that for any real x there is a $\xi < \omega_1$ with $(x, \xi) \in X$. Then there is a $\delta < \gamma_2^1$ such that for any $\mathbb{P}$ -generic real x over L , there is a $\xi < \delta$ with $(x, \xi) \in X$.

Proof. Let $X \subseteq \mathbb{R} \times \omega_1$ be Π_2^1 in the codes such that for any real x there is a ξ with $(x, \xi) \in X$.

Let A be as follows:

$$A = \{\gamma < \omega_1 \mid (\forall x: \mathbb{P}\text{-generic over } L) (\exists \xi < \gamma) (x, \xi) \in X\}.$$

By Lemma 14, A is nonempty. Hence it suffices to show that A is Π_2^1 in the codes.

Since X is Π_2^1 in the codes, pick a Π_2^1 -formula ϕ such that

$$(\forall x, w) (\phi(x, w) \iff w \in \text{WO and } (x, |w|) \in X).$$

Let $\tilde{\phi}$ be the following:

$$\tilde{\phi}(x, \xi) \iff (\forall w \in \text{WO}) |w| = \xi \rightarrow \phi(x, w).$$

Then

$$\begin{aligned} A &= \{\gamma < \omega_1 \mid (\forall x: \mathbb{P}\text{-generic over } L) (\exists \xi < \gamma) \tilde{\phi}(x, \xi)\} \\ &= \{\gamma < \omega_1 \mid L \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)" \}, \end{aligned}$$

where $\mathbb{P}_\gamma$ is $\text{Coll}(\omega, \gamma)$ and $\dot{x}$ is a canonical $\mathbb{P}$ -name for a generic real.

Claim 16. For $\gamma < \omega_1$,

$$L \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)" \iff V \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \tilde{\phi}(\dot{x}, \xi)"$$

³ γ_2^1 is also the least ordinal such that every Π_1^1 (lightface) Borel set is Π_α^0 (boldface) for some $\alpha < \omega_1$. For the details, see [7].

Proof of Claim 16. The direction from left to right follows from the fact that if (x, g) is $\mathbb{P} \times \mathbb{P}_\gamma$ -generic over V , then so is over L by Lemma 5.

For right to left, suppose $L \models "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \check{\phi}(\dot{x}, \xi)"$ fails. Then there is a $(p, q) \in \mathbb{P} \times \mathbb{P}_\gamma$ in L such that $L \models "(p, q) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\forall \xi < \check{\gamma}) \neg \check{\phi}(\dot{x}, \xi)"$. Take a $\mathbb{P} \times \mathbb{P}_\gamma$ -generic (x, g) over V with $x \in [p]$ and $g \supseteq q$. By the assumption, there exists a $\xi < \gamma$ such that $V[x, g] \models \phi(x, \xi)$. But (x, g) is also $\mathbb{P} \times \mathbb{P}_\gamma$ -generic over L and $L[x, g] \models \check{\phi}(x, \xi)$, contradicting $L \models "(p, q) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\forall \xi < \check{\gamma}) \neg \check{\phi}(\dot{x}, \xi)"$. $\square$ (Claim 16)

Therefore,

$$A = \{\gamma < \omega_1 \mid "(1_{\mathbb{P}}, 1_{\mathbb{P}_\gamma}) \Vdash_{\mathbb{P} \times \mathbb{P}_\gamma} (\exists \xi < \check{\gamma}) \check{\phi}(\dot{x}, \xi)"\}.$$

Let ψ be the following:

$$\psi(w) \iff w \in \text{WO and } (1_{\mathbb{P}}, 1_{\mathbb{P}_{|w|}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{|w|}} "(\exists n \in \omega) \phi(\dot{x}, w \restriction n)",$$

where $w \restriction n$ is the real coding the well-order $\leq_w$ below n , i.e. $\leq_{w \restriction n} = \{(l, m) \mid l \leq_w m <_w n\}$. Then

$$(\forall w)(\psi(w) \iff w \in \text{WO and } |w| \in A).$$

Hence it suffices to show that ψ is equivalent to a Π_2^1 -formula. Since $\mathbb{P}_{|w|}$ is ccc in $V^{\mathbb{P}}$, $\mathbb{P} \times \mathbb{P}_{|w|}$ is also ccc. Moreover, it is easy to see that $\mathbb{P} \times \mathbb{P}_{|w|}$ is $\Sigma_1^1(w)$ uniformly in $w \in \text{WO}$. Hence, by the same argument as in Theorem 2.7 (1) in Bagaria and Bosch [1], since $(\exists n \in \omega) \phi(x, w \restriction n)$ is Π_2^1 in x and w , so is $(1_{\mathbb{P}}, 1_{\mathbb{P}_{|w|}}) \Vdash_{\mathbb{P} \times \mathbb{P}_{|w|}} "(\exists n \in \omega) \phi(\dot{x}, w \restriction n)"$ in w . Therefore, ψ is equivalent to a Π_2^1 -formula. $\blacksquare$ (Proposition 15)

As announced in the beginning of this paper, we now show that the first item in Theorem 2 fails in ZFC for $\mathbb{P} = \mathbb{S}$ (Sacks forcing):

Proposition 17. There is a Π_1^1 set $A \subseteq \mathbb{R} \times \mathbb{R}$ such that for every x , A_x is Borel and there is no set D of $\mathbb{S}$ -measure one such that $A \cap (D \times \mathbb{R})$ is Borel.

Proof. Let A be the following:

$$A = \{(x, y) \mid x, y \in \text{WO and } |x| = |y|\}.$$

It is easy to see that A is Π_1^1 and A_x is Borel for every x .

To derive a contradiction, let D be a set of $\mathbb{S}$ -measure one such that $A \cap (D \times \mathbb{R})$ is Borel. Let B be the projection of $A \cap (D \times \mathbb{R})$ to the first coordinate. Then B is analytic and by boundedness lemma, there is a $\delta < \omega_1$ such that the length of any element of B is less than δ .

But this means that the set $C = \{y \mid |y| = \delta\}$ is disjoint from B . Since C is a subset of the projection of A to the first coordinate, it is disjoint from D and it clearly contains a perfect set, contradicting the choice of D . ■

It is also notable that Lemma 14 can consistently fail for Sacks forcing:

Proposition 18. Let s be a Sacks real over L . Then in $L[s]$, there is an $X \subseteq \mathbb{R} \times \omega_1$ which is Π_2^1 in the codes such that for every real x , there is a $\xi < \omega_1$ with $(x, \xi) \in X$ and that there is no $\delta < \omega_1$ such that for any Sacks real x over L , there is a $\xi < \delta$ with $(x, \xi) \in X$.

Proof. We work in $L[s]$. Let X be the following:

$$X = \{(x, \xi) \mid x \in \text{WO and } |x| = \xi\} \cup \{(x, 0) \mid x \notin \text{WO}\}.$$

It is easy to see that A is Π_2^1 in the codes and that for any every x there is an ordinal ξ with $(x, \xi) \in A$.

To derive a contradiction, suppose there is a $\delta < \omega_1$ such that for any Sacks real x over L , there is a $\xi < \delta$ with $(x, \xi) \in A$. It is easy to find a non-constructible surjection from ω to δ . Code that real as a relation on ω and make it a real in WO. Call it x . Then $(x, \delta) \in A$. But since x is non-constructible, x is also a Sacks real over L , contradicting the choice of δ . ■

Finally note that the second item of Theorem 2 for Sacks forcing is consistent with ZFC: In fact, it is equivalent to the statement that for any real r there is a real x which is not in $L[r]$,⁴ which is easily seen to be consistent with ZFC.

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⁴For the proof, see [3, Theorem 7.1].

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