

ON NON-LINEAR SPECTRAL GAP FOR SYMMETRIC MARKOV CHAINS WITH COARSE RICCI CURVATURES

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ABSTRACT. In this note, we report the summary of [12] for the case that the target space is a complete separable $\text{CAT}(0)$ -space. We prove an upper estimate of spectral radius for (non-linear) transition operator P over L^p -maps in the framework of symmetric Markov chains on a Polish space with positive lower bound of n -step coarse Ricci curvatures without its proof. As consequences, strong L^p -Liouville property for P -harmonic maps, a global Poincaré inequality (spectral gaps) for energy functional over L^2 -maps (or functions), and spectral bounds of L^2 -generator of Markov chains are presented.

1. COARSE RICCI CURVATURE

Throughout this note, let (E, d) be a Polish space with complete distance d and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Denote by $\mathcal{P}^p(E)$, the family of probability measures on (E, d) with finite p -th moment. We consider a conservative Markov chain $\mathbf{X} = (\Omega, X_k, \theta_k, \mathcal{F}_k, \mathcal{F}_\infty, \mathbf{P}_x)_{x \in E}$ with state space (E, d) . Then the transition kernel $P(x, dy)$ (or $P_x(dy)$ in short) of $\mathbf{X}$ defined by $P(x, dy) := \mathbf{P}_x(X_1 \in dy)$, $x \in E$ satisfies

(P1) for each $x \in E$, $\mathcal{B}(E) \ni A \mapsto P(x, A)$ is a probability measure on $(E, \mathcal{B}(E))$.

(P2) for each $A \in \mathcal{B}(E)$, $E \ni x \mapsto P(x, A)$ is $\mathcal{B}(E)$ -measurable.

Conversely, for $P(x, dy)$ satisfying (P1) and (P2), we can construct a conservative Markov chain $\mathbf{X}$ such that $P(x, dy) = \mathbf{P}_x(X_1 \in dy)$, $x \in E$. We set $Pf(x) := \int_X f(y)P(x, dy) = \mathbf{E}_x[f(X_1)]$ for any non-negative or bounded $\mathcal{B}(E)$ -measurable function f on E . For $n \in \mathbb{N}$, if we set $P^n f(x) := P(P^{n-1}f)(x)$ inductively, then $P^n f(x) = \mathbf{E}_x[f(X_n)]$

Date: December 22, 2012.

2000 *Mathematics Subject Classification.* Primary 53C20; Secondary 53C21, 53C23.

Key words and phrases. $\text{CAT}(0)$ -space, barycenter, Jensen's inequality, optimal mass transportation, Wasserstein distance, symmetric Markov chain, coarse Ricci curvature, n -step coarse Ricci curvature, P -harmonic map, spectral radius, spectral gap, eigenvalue, strong L^p -Liouville property.

The second author is partially supported by a Grant-in-Aid for Scientific Research No. 22340036 from the Japan Society for the Promotion of Science.

and $P^n(x, A) := (P^n \mathbf{1}_A)(x) = P_x(X_n \in A)$. For any non-negative measure ν on $(E, \mathcal{B}(E))$ and $n \in \mathbb{N}$, we define a measure νP^n by $\nu P^n(A) := \langle \nu, P^n \mathbf{1}_A \rangle := \int_E P^n(x, A) \nu(dx) = P_\nu(X_n \in A)$, $A \in \mathcal{B}(E)$. Note that $\delta_x P^n = P_x^n$, $x \in E$.

We further assume the following condition to $\mathbf{X}$:

(P3) for each $x \in E$, $P_x \in \mathcal{P}^1(E)$.

For the given Markov chain $\mathbf{X}$ as above and a fixed $n \in \mathbb{N}$, a Markov chain $\mathbf{X}^n = (\Omega, X_k^n, \theta_k^n, \mathcal{F}_k^n, \mathcal{F}_\infty^n, P_x^n)_{x \in E}$ with state space (E, d) defined by the transition kernel $P^n(x, dy)$ is called an *n-step Markov chain*. Note that if $\mathbf{X}$ satisfies (P3), then $\mathbf{X}^n$ does so.

For $\mu, \nu \in \mathcal{P}^1(E)$, the L^1 -Wasserstein/Kantorovich-Rubinstein distance $d_{W_1}(\mu, \nu)$ is defined by

$$d_{W_1}(\mu, \nu) := \inf \left\{ \int_{E \times E} d(x, y) \pi(dxdy) \mid \pi \in \Pi(\mu, \nu) \right\},$$

where $\Pi(\mu, \nu) := \{ \pi \in \mathcal{P}(E \times E) \mid \pi(A \times E) = \mu(A), \pi(E \times B) = \nu(B) \text{ for any } A, B \in \mathcal{B}(E) \}$.

Definition 1.1 (Coarse Ricci Curvature, [22]). For a pair of distinct points $x, y \in E$, the *coarse Ricci curvature* $\kappa(x, y)$ of $\mathbf{X}$ along (xy) is defined to be

$$\kappa(x, y) := 1 - \frac{d_{W_1}(P_x, P_y)}{d(x, y)} (> -\infty), \quad (x, y) \in E \times E \setminus \text{diag}$$

and $\kappa := \inf \{ \kappa(x, y) \mid (x, y) \in E \times E \setminus \text{diag} \} \in [-\infty, 1]$ is said to be the *lower bound of the coarse Ricci curvature*. The *n-step coarse Ricci curvature* $\kappa_n(x, y)$ of $\mathbf{X}$ along (xy) is defined to be

$$\kappa_n(x, y) := 1 - \frac{d_{W_1}(P_x^n, P_y^n)}{d(x, y)} (\geq -\infty), \quad (x, y) \in E \times E \setminus \text{diag}$$

and $\kappa_n := \inf \{ \kappa_n(x, y) \mid (x, y) \in E \times E \setminus \text{diag} \} \in [-\infty, 1]$ is said to be the *lower bound of the n-step coarse Ricci curvature*.

Remark 1.2. We denote the family of Lipschitz functions on E by $\text{Lip}(E)$.

- (1) If $\kappa \in \mathbb{R}$, then $P^n f \in \text{Lip}(E)$ for any $f \in \text{Lip}(E)$ and $\text{Lip}(P^n f) \leq (1 - \kappa)^n \text{Lip}(f)$ by [22, Proposition 20], which implies that (P3) holds for all $\mathbf{X}^n$ provided (P3) holds for $\mathbf{X}$ and $\kappa \in \mathbb{R}$, in particular, $\kappa_n(x, y) > -\infty$ for all $n \in \mathbb{N}$ and $x \neq y$ under $\kappa \in \mathbb{R}$.
- (2) The *n-step coarse Ricci curvature* $\kappa_n(x, y)$ is nothing but the coarse Ricci curvature for $\mathbf{X}^n$ and $\kappa_1(x, y) = \kappa(x, y)$ for $(x, y) \in E \times E \setminus \text{diag}$. In general, we have

$$(1 - \kappa_{k+\ell}) \leq (1 - \kappa_k)(1 - \kappa_\ell), \quad k, \ell \in \mathbb{N},$$

which implies $\lim_{\ell \rightarrow \infty} (1 - \kappa_\ell)^{1/\ell} = \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{1/n} \in [0, +\infty]$. In particular, $\kappa \in \mathbb{R}$ implies $\kappa_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

The following example due to [4] shows that the lower bound 0 for the coarse Ricci curvature does not necessarily mean the same bound for the n -step coarse Ricci curvature.

Example 1.3 (Simple Random Walk on Cycle Graph, see [4]). Let $G = (V, E)$ be a cycle graph of size N , that is, G is an unweighted finite graph with vertices $V := \{x_i\}_{i=1}^N$ and edges $E := \{x_i x_{i+1}\}_{i=1}^N$ by regarding $x_{N+1} = x_1$ ($i \in \mathbb{N}$). The degree $d_x(G)$ for $x \in V$ is the number of edges starting from x is given by $d_x(G) = 2$ for this cycle graph G . The weight w_{xy} for $xy \in E$ is given by $w_{x_i x_{i+1}} = 1$ for $i = 1, 2, \dots, N$. Consider a symmetric Markov chain $\mathbf{X}$ defined by the transition kernel $P_{x_i}(dy) := \frac{1}{2}\delta_{x_{i-1}}(dy) + \frac{1}{2}\delta_{x_{i+1}}(dy)$. As for the simple random walk on $\mathbb{Z}^1$, the coarse Ricci curvature $\kappa(x, y)$ on $\mathbf{X}$ satisfies $\kappa(x, y) = 0$ for $(x, y) \in V \times V \setminus \text{diag}$ (by [8, Theorems 2,3,4 and 5], [4, Theorems 6 and 7]), hence the n -step coarse Ricci curvature $\kappa_n(x, y)$ satisfies $\kappa_n(x, y) \geq 0$ for $(x, y) \in V \times V \setminus \text{diag}$ by [22, Proposition 25]. $\mathbf{X}$ (hence the n -step Markov chain $\mathbf{X}^n$) is m -symmetric with respect to $m(dy) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}(dy)$. For simplicity, hereafter, we assume $N = 5$. 3-step Markov chain $\mathbf{X}^3$ is associated with the Cayley graph $G^3 := (V^3, E^3)$ defined by $V^3 := V$ and $E^3 := \{x_i x_j \mid i, j = 1, 2, 3, 4, 5 \text{ with } i \neq j\}$. G^3 is a weighted complete graph. The transition kernel $P_x^3(dy)$ of $\mathbf{X}^3$ is given by $P_{x_i}^3(dy) = \frac{1}{8}\delta_{x_{i-2}}(dy) + \frac{3}{8}\delta_{x_{i-1}}(dy) + \frac{3}{8}\delta_{x_{i+1}}(dy) + \frac{1}{8}\delta_{x_{i+2}}(dy)$. G^3 is a weighted graph with no loop and the degree $d_x(G^3)$ for $x \in V$ is given by $d_x(G^3) = 4$. The weight w_{xy} for $xy \in E^3$ is given by $w_{x_i x_i} = \frac{3}{2}$, $w_{x_{i-2} x_i} = w_{x_i x_{i+2}} = \frac{3}{4}$. Note here that our degree $d_x(G^3) = 4$ and the way for weighting on edges are different from those used in Section 6 of [4], but the conclusion is the same as we calculate below. The 3-step coarse Ricci curvature $\kappa_3(x, y)$ for $xy \in E^3$ can be estimated by use of [4, Theorems 6 and 7]:

$$\kappa_3(x_i, x_{i+1}) = \frac{3}{8}, \quad \frac{5}{8} \leq \kappa_3(x_i, x_{i+2}) \leq \frac{7}{8}.$$

Therefore, $\kappa_3(x, y) \geq \frac{3}{8}$ for all $(x, y) \in V \times V \setminus \text{diag}$.

Remark 1.4. For a continuous time parameter Markov process $\mathbf{M}$, the notion of *coarse Ricci curvature* $\kappa(x, y)$ for $\mathbf{M}$ is discussed in [22], [28]:

$$\kappa(x, y) := \lim_{t \rightarrow 0} \frac{1}{t} \left(1 - \frac{d_{W_1}(P_t(x, \cdot), P_t(y, \cdot))}{d(x, y)} \right) \quad \text{for } (x, y) \in E \times E \setminus \text{diag}.$$

We can also define the n -step coarse Ricci curvature $\kappa_n(x, y)$:

$$\kappa_n(x, y) := \lim_{t \rightarrow 0} \frac{1}{t} \left(1 - \frac{d_{W_1}(P_{nt}(x, \cdot), P_{nt}(y, \cdot))}{d(x, y)} \right) \quad \text{for } (x, y) \in E \times E \setminus \text{diag},$$

which is nothing but the coarse Ricci curvature for the time changed process $\mathbf{M}^n = (\Omega, X_{nt}, \mathbf{P}_x)$. Then, we easily see $\kappa_n(x, y) = n\kappa(x, y)$ for $(x, y) \in E \times E \setminus \text{diag}$, in particular, the positivity of lower bound

for curvature is equivalent to each other between both coarse Ricci curvatures. In our discrete setting, we have no such a relation.

Example 1.5 (Riemannian manifold). Let (M, g) be a d -dimensional complete smooth Riemannian manifold whose Ricci curvature is bound below by $\kappa > 0$. In view of Bonnet-Myers theorem, M is compact. Consider a Brownian motion $\mathbf{M} = (\Omega, X_t, \mathbf{P}_x)$ on M associated with the following Dirichlet energy form on $L^2(M; m)$;

$$\begin{cases} D(\mathcal{E}) &:= \{u \in L^2(M; m) \mid \int_M g(\nabla f, \nabla f) dm < \infty\} \\ \mathcal{E}(f, g) &:= \int_M g(\nabla f, \nabla g) dm, \quad f, g \in D(\mathcal{E}) \end{cases}$$

where m is the volume element of (M, g) . Let $P_t(x, dy)$ be the transition kernel of $\mathbf{M}$. Under the Ricci curvature lower bound, $\mathbf{M}$ is a conservative process, that is, $P_t(x, \cdot) \in \mathcal{P}(M)$ for any $t > 0$. Moreover, we see $P_t(x, \cdot) \in \mathcal{P}^1(M)$ for any $t > 0$. We set $P(x, dy) := P_1(x, dy)$ and consider an m -symmetric Markov chain $\mathbf{X}$ associated with $P(x, dy)$. It is proved in [30] that

$$d_{W_1}(P_t(x, \cdot), P_t(y, \cdot)) \leq e^{-\kappa t} d(x, y), \quad x, y \in M.$$

So the coarse Ricci curvature $\kappa_{\mathbf{M}}(x, y)$ of $\mathbf{M}$ has the lower estimate

$$\begin{aligned} \kappa_{\mathbf{M}}(x, y) &:= \lim_{t \rightarrow 0} \frac{1}{t} \left(1 - \frac{d_{W_1}(P_t(x, \cdot), P_t(y, \cdot))}{d(x, y)} \right) \\ &\geq \frac{d}{dt} (1 - e^{-\kappa t}) \Big|_{t=0} = \kappa > 0, \quad (x, y) \in M \times M \setminus \text{diag}. \end{aligned}$$

On the other hand, the n -step coarse Ricci curvature $\kappa_n(x, y)$ of $\mathbf{X}$ has the lower estimate

$$\kappa_n(x, y) = 1 - \frac{d_{W_1}(P_x^n, P_y^n)}{d(x, y)} \geq 1 - e^{-\kappa n} > 0, \quad (x, y) \in M \times M \setminus \text{diag}.$$

Note that the same conclusion also holds for a Markov process whose coarse Ricci curvature is bounded below by $\kappa > 0$.

2. CAT(0)-SPACES

In this section, we summarize the notions of CAT(0)-space and its properties.

Definition 2.1 (CAT(0)-space). A metric space (Y, d) is called the *CAT(0)-space* (*Hadamard space*, or *global NPC space*) if for any pair of points $\gamma_0, \gamma_1 \in Y$ and any $t \in [0, 1]$ there exists a point $\gamma_t \in Y$ such that for any $z \in Y$

$$(2.1) \quad d_Y^2(z, \gamma_t) \leq (1-t)d_Y^2(z, \gamma_0) + td_Y^2(z, \gamma_1) - t(1-t)d_Y^2(\gamma_0, \gamma_1).$$

By definition, $\gamma := (\gamma_t)_{t \in [0, 1]}$ is the minimal geodesic joining γ_0 and γ_1 . Any CAT(0)-space is simply connected. Hadamard manifolds, Euclidean Bruhat-Tits buildings (e.g. metric tree), spiders, booklets and Hilbert spaces are typical examples of CAT(0)-spaces (cf. [25]).

Let (Y, d_Y) be a CAT(0)-space. Then the distance function $d_Y : Y \times Y \rightarrow [0, \infty[$ is convex (Corollary 2.5 in [25]) and Jensen's inequality (Theorem 6.3 in [25]) can be applied to the convex function $Y \ni w \mapsto d_Y(w, z)$ for each $z \in Y$.

The inequality (2.1) yields the (strict) convexity of $Y \ni x \mapsto d_Y^2(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a CAT(0)-space is again a CAT(0)-space.

The unique existence of projection (or foot-point) to closed convex set of CAT(0)-space is proved in [14] in more general setting.

Lemma 2.2 (Projection Map to Convex Set, see [25]). *Let (Y, d_Y) be a complete CAT(0)-space. The following hold:*

- (1) *Let F be a closed convex subset of (Y, d_Y) . Then, for each $x \in Y$, there exists a unique element $\pi_F(x) \in F$ such that $d_Y(x, F) = d_Y(\pi_F(x), x)$ holds. We call $\pi_F : Y \rightarrow F$ the projection map to F .*
- (2) *Let F be as above. Then π_F satisfies*

$$(2.2) \quad d_Y^2(z, \pi_F(z)) + d_Y^2(\pi_F(z), w) \leq d_Y^2(z, w), \quad \text{for } z \in Y, w \in F,$$
in particular, $d_Y(\pi_F(z), w) \leq d_Y(z, w)$ for $z \in Y, w \in F$.

Let (Y, d_Y) be a metric space and $\mathcal{P}(Y)$ a family of Borel probability measures on Y . For $p \geq 1$, we set

$$\mathcal{P}^p(Y) := \left\{ \mu \in \mathcal{P}(Y) \mid \int_Y d_Y^p(x, y) \mu(dy) < \infty \text{ for any/some } x \in Y \right\}.$$

Each element $\mu \in \mathcal{P}^p(Y)$ is called a *probability measure with p -th moment*.

Definition 2.3 (Barycenter or Center of Mass, see [25]). For $\mu \in \mathcal{P}^2(Y)$, if $z \mapsto \int_Y d_Y^2(z, x) \mu(dx)$ has a minimizer $b(\mu) \in Y$, then we call $b(\mu)$ the *barycenter, or center of mass* of $\mu \in \mathcal{P}^2(Y)$. For $\mu \in \mathcal{P}^1(Y)$ and $w \in Y$, we consider the following function F_w :

$$(2.3) \quad F_w(z) := \int_Y (d_Y^2(z, x) - d_Y^2(w, x)) \mu(dx).$$

We easily see

$$|F_w(z)| \leq 2d_Y(z, w) \int_Y (d_Y(z, x) + d_Y(w, x)) \mu(dx) < \infty.$$

If $Y \ni z \mapsto F_w(z)$ admits a minimizer $b(\mu)$ independent of w in the sense that $F_w(z) \geq F_w(b(\mu))$ if and only if $F_v(z) \geq F_v(b(\mu))$ for all $z, w, v \in Y$, we call it *barycenter, or center of mass* of $\mu \in \mathcal{P}^1(Y)$. If the barycenter of $\mu \in \mathcal{P}^2(Y)$ exists, then it is a barycenter of $\mu \in \mathcal{P}^1(Y)$.

Assume that (Y, d_Y) is a geodesic space. For a subset F of Y , denote by $C(F)$ the closed convex hull of F . That is, $C(F)$ is the smallest closed convex subset of Y containing F .

If (Y, d_Y) is a complete CAT(0)-space, we can obtain the unique existence of barycenter of $\mu \in \mathcal{P}^1(Y)$ proved in [25].

Lemma 2.4 ([25], cf. [16],[21]). *Let (Y, d_Y) be a complete CAT(0)-space. Then $\mu \in \mathcal{P}^1(Y)$ admits a unique barycenter.*

For any metric space (Y, d_Y) , we easily see $b(\delta_x) = x$ for $x \in Y$.

The following proposition is proved in Proposition 5.5 in [25].

Theorem 2.5 (Jensen's Inequality, see [25, Theorem 6.3]). *Let (Y, d_Y) be a complete CAT(0)-space. Let φ be a lower semi-continuous convex function on Y and $\mu \in \mathcal{P}^1(Y)$. Suppose $\varphi \in L^1(Y; \mu)$. Then we have*

$$(2.4) \quad \varphi(b(\mu)) \leq \int_Y \varphi(x) \mu(dx).$$

Corollary 2.6 (Fundamental Contraction Property, see [25]). *Let (Y, d_Y) be a complete CAT(0)-space. Let $\mu, \nu \in \mathcal{P}^1(Y)$. Then*

$$d_Y(b(\mu), b(\nu)) \leq d_{W_1}(\mu, \nu),$$

where $d_{W_1}(\mu, \nu)$ is the L^1 -Wasserstein distance on $\mathcal{P}^1(Y)$ defined by

$$d_{W_1}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{Y \times Y} d_Y(x, y) \pi(dxdy).$$

Here $\Pi(\mu, \nu) := \{\pi \in \mathcal{P}(Y \times Y) \mid \pi(A \times Y) = \mu(A), \pi(Y \times B) = \nu(B) \text{ for } A, B \in \mathcal{B}(Y)\}$.

3. L^p -MAPS

Let $(E, \mathcal{E}, \mu)$ be a σ -finite measure space and $\mathcal{E}^\mu$ a completion of $\mathcal{E}$ with respect to μ . In what follows, we say *measurable* (resp. μ -*measurable*) for $\mathcal{E}$ -measurable (resp. $\mathcal{E}^\mu$ -measurable). For function $f : E \rightarrow [-\infty, \infty]$, we set $\|f\|_p := (\int_E |f(x)|^p \mu(dx))^{1/p}$, $\|f\|_\infty := \inf\{\lambda > 0 \mid |f(x)| \leq \lambda \text{ } \mu\text{-a.e. } x \in E\}$. For two $\overline{\mathbb{R}}$ -valued functions f, g , they are said to be μ -*equivalent* if $f = g$ μ -a.e.

For $p \in]0, \infty]$, $L^p(E; \mu)$ denotes the family of μ -equivalence class of functions with finite $\|\cdot\|_p$ -norm. Also $L^0(E; \mu)$ denotes the family of μ -equivalence class of functions having finite value μ -a.e. Fix a metric space (Y, d_Y) . For $p \in]0, \infty]$ and measurable maps $u, v : E \rightarrow Y$, the pseudo-distance $d_{L^p}(u, v)$ is defined by $d_{L^p}(u, v) := \|d_Y(u, v)\|_p$. More precisely, for $p \in]0, \infty[$ we set

$$d_{L^p}(u, v) := \left(\int_E d_Y^p(u(x), v(x)) \mu(dx) \right)^{1/p},$$

and for $p = \infty$, $d_\infty(u, v)$ is the μ -essentially supremum of $x \mapsto d_Y(u(x), v(x))$.

We say that u and v are μ -equivalent ($u \stackrel{\mu}{\sim} v$ in short) if

$$u(x) = v(x) \text{ } \mu\text{-a.e. } x \in E.$$

For a fixed measurable map $h : E \rightarrow Y$, we set

$$L_h^p(E, Y; \mu) := \{f \in \mathcal{E}/\mathcal{B}(Y) \mid d_Y(f, h) \in L^p(E; \mu)\} / \sim.$$

Such a map $h : E \rightarrow Y$ is called a *base map* of $L_h^p(E, Y; \mu)$. If $\mu(E) < \infty$ and the image of $h : E \rightarrow Y$ is bounded, $L_h^p(E, Y; \mu)$ is independent of the choice of such a base map h . In this case, we can assume $h \equiv o$ for some fixed point $o \in Y$.

Proposition 3.1 ([23, Proposition 3.3]). *Let (Y, d_Y) be a metric space and $h : E \rightarrow Y$ a measurable map. Take $p \in [1, \infty]$. Then we have the following:*

- (1) *If (Y, d_Y) is complete, then $(L_h^p(E, Y; \mu), d_{L^p})$ is so.*
- (2) *If (Y, d_Y) is a geodesic space and any point γ_t of the constant speed geodesic $\gamma : [0, 1] \rightarrow Y$ joining γ_0 to γ_1 is a continuous map with respect to (γ_0, γ_1) , then $(L_h^p(E, Y; \mu), d_{L^p})$ is also a geodesic space.*

In what follows, we assume $m(E) < \infty$. Let $L^p(E, Y; m)$ be the space of L^p -maps with bounded base maps, that is,

$$L^p(E, Y; m) := \left\{ u : E \rightarrow Y \mid u \text{ is } m\text{-measurable,} \right. \\ \left. \int_E d_Y^p(u, o) dm < \infty \text{ for some } o \in Y \right\} / \sim.$$

Definition 3.2 (Lipschitz Maps). Let (Y, d_Y) be a geodesic space and (E, d) a metric space. For a map $u : E \rightarrow Y$, we set $\text{Lip}(u) := \sup_{x \neq y} \frac{d_Y(u(x), u(y))}{d(x, y)}$ and

$$\text{Lip}(E, Y) := \{u : E \rightarrow Y \mid \text{Lip}(u) < \infty\}.$$

Lemma 3.3. *Let (Y, d_Y) be a geodesic space and (E, d) a metric space. Suppose that m has a p -th moment, that is, $\int_E d^p(x, x_0) m(dx) < \infty$ for some/any point $x_0 \in E$. Then $\text{Lip}(E, Y) \subset L^p(E, Y; m)$.*

Let $S(E, Y)$ be a space of finite valued maps from E to Y . Any element of $S(E, Y)$ is called a *step map* or a *simple map*. Since $m(E) < \infty$, $S(E, Y)$ (more precisely $S(E, Y)/\sim$) is a subset of $L^p(E, Y; m)$.

Theorem 3.4. *Suppose that (E, d) is a Polish space and (Y, d_Y) is a separable geodesic space. Take $p \in [1, \infty[$. Then any element of $L^p(E, Y; m)$ can be L^p -approximated by elements in $S(E, Y)$. In particular, if $E = \text{supp}[m]$, then $(L^p(E, Y; m), d_{L^p})$ is a separable metric space. Moreover, if m has a finite p -th moment, then $L^p(E, Y; m)$ can be L^p -approximated by elements in $\text{Lip}(E, Y)$, if further $E = \text{supp}[m]$, then $\text{Lip}(E, Y)$ is a dense subset of $L^p(E, Y; m)$.*

In what follows, (E, d) denotes a Polish space with complete distance d .

Definition 3.5 ($P^\ell u$ for Borel Map u). Let $\mathbf{X}$ be a conservative Markov chain on (E, d) . Suppose that (Y, d_Y) is a complete CAT(0)-space and a $\mathcal{B}(E)/\mathcal{B}(Y)$ -measurable map $u : E \rightarrow Y$ satisfies $u_\# P_x^\ell \in \mathcal{P}^1(Y)$ for $\ell \in \mathbb{N}$. Then we set

$$P^\ell u(x) := b(u_\# P_x^\ell).$$

Here $u_\# P_x^\ell$ is a push-forward measure of $P(x, \cdot)$ by u ; $u_\# P_x^\ell(A) := P^\ell(x, u^{-1}(A))$, $A \in \mathcal{B}(Y)$.

Remark 3.6. Note that any $u \in S(E, Y) \cup \text{Lip}(E, Y)$ satisfies $u_\# P_x \in \mathcal{P}^1(Y)$. Indeed, for $u \in S(E, Y)$, u is a constant on each Borel set A_i , where $\{A_i\}_{i=1}^l$ is a finite family of disjoint Borel sets satisfying $E = \bigcup_{i=1}^l A_i$, hence $\int_E d_Y(z_0, z) u_\# P_x(dz) = \sum_{i=1}^l \|d_Y(z_0, u)\|_{\infty, A_i} P_x(A_i) < \infty$. For $u \in \text{Lip}(E, Y)$, we have

$$\begin{aligned} \int_E d_Y(z_0, z) u_\# P_x(dz) &= \int_E d_Y(z_0, u(y)) P_x(dy) \\ &\leq d_Y(z_0, u(y_0)) + \text{Lip}(u) \int_E d(y_0, y) P_x(dy) < \infty. \end{aligned}$$

Lemma 3.7 (Lemma 6.4 in [23]). Let $\mathbf{X}$ be a conservative Markov chain on (E, d) . Suppose that (Y, d_Y) is a complete separable CAT(0)-space. Then, for any Borel map $u : E \rightarrow Y$ satisfying $u_\# P_x \in \mathcal{P}^1(Y)$ for all $x \in E$, $Pu : E \rightarrow Y$ is $\mathcal{B}(E)/\mathcal{B}(Y)$ -measurable.

Definition 3.8 (Pu for L^p -map u). Fix $p \geq 1$. Let $\mathbf{X}$ be an m -symmetric conservative Markov chain on (E, d) . Suppose that (Y, d_Y) is a complete CAT(0)-space and $u \in L^p(E, Y; m)$, we can define $Pu \in L^p(E, Y; m)$ in the following way: Let $\{u_k\} \subset S(E, Y)$ be an L^p -approximating sequence to u . Applying the Jensen's inequality to the convex function d_Y^p on $Y \times Y$ and the m -symmetry, we have the following inequality for any maps $v, w \in S(E, Y)$.

$$\begin{aligned} (3.1) \quad d_{L^p}^p(Pv, Pw) &= \int_E d_Y^p(Pv(x), Pw(x)) m(dx) \\ &\leq \int_E P d_Y^p(v, w) dm \leq d_{L^p}^p(v, w). \end{aligned}$$

These mean that $\{Pu_k\}$ forms an L^p -Cauchy sequence. We set $Pu := \lim_k Pu_k \in L^p(E, Y; m)$. The well-definedness of Pu is clear from (3.1) and this is valid for any $v, w \in L^p(E, Y; m)$.

Definition 3.9 (P -harmonic Map, [16],[15]). A (lower or upper) bounded Borel function $f : E \rightarrow \mathbb{R}$ is said to be P -subharmonic if $f \leq Pf$ on E and it is said to be P -harmonic if both f and $-f$ are P -subharmonic. A Borel map $u : E \rightarrow Y$ is said to be P -harmonic if $u = Pu$ on E holds under that $u_\# P_x \in \mathcal{P}^1(Y)$ for all $x \in E$.

Lemma 3.10. *Let $\mathbf{X}$ be a Markov chain on (E, d) . Fix $n \in \mathbb{N}$ and assume $\kappa \in \mathbb{R}$. Suppose that (Y, d_Y) is a complete CAT(0)-space. Then for $u \in \text{Lip}(E, Y)$ and $\ell \in \mathbb{N}$, we have $P^{n\ell}u \in \text{Lip}(E, Y)$ and*

$$\text{Lip}(P^{n\ell}u) \leq (1 - \kappa_n)^\ell \text{Lip}(u),$$

in particular,

$$\text{Lip}(P^\ell u) \leq (1 - \kappa)^\ell \text{Lip}(u).$$

Corollary 3.11 (Strong Liouville Property for Lipschitz Maps). *Let $\mathbf{X}$ be a Markov chain on (E, d) . Assume that $\kappa \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Suppose that (Y, d_Y) is a complete CAT(0)-space. Then any P^n -harmonic map $u \in \text{Lip}(E, Y)$ is a constant map.*

Definition 3.12 (Variance). Fix $p \geq 1$, $\mu \in \mathcal{P}(E)$, a metric space (Y, d_Y) and $u \in L^p(E, Y; \mu)$. The p -variance $\text{Var}_\mu^p(u)$ of u is defined by

$$\text{Var}_\mu^p(u) := \inf_{y \in Y} \int_E d_Y^p(u(x), y) \mu(dx) (< \infty).$$

The quasi p -variance $\overline{\text{Var}}_\mu^p(u)$ is defined by

$$\overline{\text{Var}}_\mu^p(u) := \frac{1}{2} \int_E \int_E d_Y^p(u(y), u(x)) \mu(dx) \mu(dy) (< \infty).$$

We easily see $\text{Var}_\mu^p(u) \leq 2\overline{\text{Var}}_\mu^p(u)$. When $p = 2$, we write $\text{Var}_\mu(u) := \text{Var}_\mu^2(u)$ and $\overline{\text{Var}}_\mu(u) := \overline{\text{Var}}_\mu^2(u)$, and call them simply *variance*, *quasi variance*, respectively. Let (Y, d_Y) be a complete CAT(0)-space. If $u \in L^2(E, Y; \mu)$, then $\text{Var}_\mu(u) = \int_E d_Y^2(u(x), b(u_\# \mu)) \mu(dx)$ holds. For $u \in L^2(E, Y; \mu)$, we have $\text{Var}_\mu(u) \leq \overline{\text{Var}}_\mu(u)$. If (Y, d_Y) is a Hilbert space H , then we have $\text{Var}_\mu(u) = \overline{\text{Var}}_\mu(u)$. In this case we can define the *covariance* $\text{Cov}_\mu(f, g)$ for $f, g \in L^2(E, H; \mu)$ by

$$\begin{aligned} \text{Cov}_\mu(f, g) &:= \int_E \langle f(x) - \langle \mu, f \rangle, g(x) - \langle \mu, g \rangle \rangle_H \mu(dx) \\ &= \langle \mu, \langle f, g \rangle_H \rangle - \langle \langle \mu, f \rangle, \langle \mu, g \rangle \rangle_H \\ &= \frac{1}{2} \int_E \int_E \langle f(y) - f(x), g(y) - g(x) \rangle_H \mu(dx) \mu(dy), \end{aligned}$$

where $\langle \mu, f \rangle := \int_H f(x) \mu(dx) \in H$ is the barycenter of $f_\# \mu \in \mathcal{P}^2(Y)$.

Definition 3.13 (Energy of Maps). Take $m \in \mathcal{P}(E)$ and let $\mathbf{X}$ be an m -symmetric Markov chain and (Y, d_Y) is a metric space. For $u \in L^p(E, Y; m)$,

$$E^p(u) := \frac{1}{2} \int_E \int_E d_Y^p(u(y), u(x)) P(x, dy) m(dx)$$

is said to be p -energy of u with respect to $\mathbf{X}$ and

$$E_*^p(u) := \frac{1}{2} \int_E d_Y^p(Pu(x), u(x)) m(dx) = \frac{1}{2} d_{L^p}^p(Pu, u)$$

is said to be *quasi p -energy* of u with respect to $\mathbf{X}$ for $p \geq 1$ when (Y, d_Y) is a complete separable CAT(0)-space.

When $p = 2$, we simply say *energy* (resp. *quasi 2-energy*) and write $E(u) := E^2(u)$ (resp. $E_*(u) := E_*^2(u)$). Since

$$\text{Var}_{P_x}^p(u) \leq \int_E d_Y^p(u(y), u(x)) P(x, dy),$$

we see

$$(3.2) \quad \frac{1}{2} \int_E \text{Var}_{P_x}^p(u) m(dx) \leq E^p(u).$$

We use

$$\begin{cases} D(E^p) &:= \{u \in L^p(E, Y; m) \mid E^p(u) < \infty\} \\ E^p(u) &:= \frac{1}{2} \int_E \int_E d_Y^p(u(y), u(x)) P(x, dy) m(dx), \quad u \in D(E^p). \end{cases}$$

When (Y, d_Y) is a Hilbert space H , we use the symbol $\mathcal{E}$ instead of E for the (2-)energy on $L^2(E, H; m)$ and set

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{E \times E} \langle f(y) - f(x), g(y) - g(x) \rangle_H P_x(dy) m(dx)$$

for $f, g \in D(\mathcal{E})$. We see $\mathcal{E}(f) = \mathcal{E}(f, f)$ for $f \in L^2(E, H; m)$.

Proposition 3.14. *Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) and (Y, d_Y) is a metric space. Fix $p \in [1, \infty[$. For measurable maps $u, v : E \rightarrow Y$, the following inequalities hold:*

$$(3.3) \quad E^p(u)^{\frac{1}{p}} \leq E^p(v)^{\frac{1}{p}} + 2^{1-\frac{1}{p}} d_{L^p}(u, v),$$

$$(3.4) \quad \text{Var}_m^p(u)^{\frac{1}{p}} \leq \text{Var}_m^p(v)^{\frac{1}{p}} + d_{L^p}(u, v),$$

$$(3.5) \quad \overline{\text{Var}}_m^p(u)^{\frac{1}{p}} \leq \overline{\text{Var}}_m^p(v)^{\frac{1}{p}} + 2^{1-\frac{1}{p}} d_{L^p}(u, v).$$

Corollary 3.15. *Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) . Suppose that (Y, d_Y) is a complete separable CAT(0)-space. For $p \geq 1$ and $u \in L^p(E, Y; m)$, the following inequalities hold:*

$$(3.6) \quad \text{Var}_m^p(u)^{\frac{1}{p}} \leq \text{Var}_m^p(Pu)^{\frac{1}{p}} + 2^{\frac{1}{p}} E_*^p(u)^{\frac{1}{p}},$$

$$(3.7) \quad \overline{\text{Var}}_m^p(u)^{\frac{1}{p}} \leq \overline{\text{Var}}_m^p(Pu)^{\frac{1}{p}} + 2 E_*^p(u)^{\frac{1}{p}}.$$

If $u \in L^2(E, Y; m)$, we have

$$(3.8) \quad E_*^2(u) \leq 4E^2(u).$$

Corollary 3.16 (Lower Semi Continuity of Energy). *Let $\mathbf{X}$ be an m -symmetric Markov chain with $m \in \mathcal{P}(E)$ and (Y, d_Y) a metric space. Take $p \geq 1$ and let $(E^p, D(E^p))$ be the p -energy on $L^p(E, Y; m)$ associated with $\mathbf{X}$. We set $E^p(u) := \infty$ for $u \in L^p(E, Y; m) \setminus D(E^p)$. Then E^p is a $[0, \infty]$ -valued lower semi continuous functional on $L^p(E, Y; m)$.*

Remark 3.17. When $p = 2$, $Y = \mathbb{R}$, the lower semi continuity of energy is equivalent to the completeness of $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$ defined by $\|f\|_{\mathcal{E}_1} := \sqrt{\mathcal{E}_1(f, f)}$. Here $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + (f, g)_m$, $f, g \in D(\mathcal{E})$.

Lemma 3.18 (Contraction Property). *Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) with $m \in \mathcal{P}(E)$. Fix $p \geq 1$. Let (Y, d_Y) be a complete separable CAT(0)-space. Then, for any $u \in L^p(E, Y; m)$, we have*

$$(3.9) \quad \text{Var}_m^p(Pu) \leq \text{Var}_m^p(u),$$

$$(3.10) \quad \overline{\text{Var}}_m^p(Pu) \leq \overline{\text{Var}}_m^p(u).$$

4. MAIN RESULTS

In this section, we fix $p \geq 1$ and assume $m \in \mathcal{P}(E)$ and $\text{supp}[m] = E$.

Theorem 4.1 (Non-linear Spectral Radius of P on $L^p(E, Y; m)/\{\text{const}\}$). *Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) with $m \in \mathcal{P}^p(E)$ and assume $\kappa \in \mathbb{R}$. Let (Y, d_Y) be a complete separable CAT(0)-space. Then, we have*

$$(4.1) \quad \lim_{\ell \rightarrow \infty} \left(\sup_{u \in L^p(E, Y; m)} \frac{\text{Var}_m^p(P^\ell u)}{\text{Var}_m^p(u)} \right)^{\frac{1}{p\ell}} \leq \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1,$$

$$(4.2) \quad \lim_{\ell \rightarrow \infty} \left(\sup_{u \in L^p(E, Y; m)} \frac{\overline{\text{Var}}_m^p(P^\ell u)}{\overline{\text{Var}}_m^p(u)} \right)^{\frac{1}{p\ell}} \leq \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

Corollary 4.2 (Linear Spectral Radius of P on $L^2(E, H; m)/\{\text{const}\}$). *Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) and H a real separable Hilbert space. Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Then, we have*

$$(4.3) \quad \lim_{\ell \rightarrow \infty} \left(\sup_{f \in L^2(E, H; m)} \frac{\text{Var}_m(P^\ell f)}{\text{Var}_m(f)} \right)^{\frac{1}{2\ell}} \leq \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

Consequently, P is an $\inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1$ -contraction operator on $L^2(E, H; m)/\{\text{const}\}$. In particular, for $f \in L^2(E, H; m)/\{\text{const}\}$, the following hold:

$$(4.4) \quad \text{Var}_m(Pf) \leq \left(\inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{2}{n}} \wedge 1 \right) \text{Var}_m(f),$$

$$(4.5) \quad |\text{Cov}_m(Pf, f)| \leq \left(\inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 \right) \text{Var}_m(f).$$

The main part of the following theorem is a slight generalization of [22, Corollary 31], and its proof is similar as in [22] based on Corollary 4.2 above.

Theorem 4.3 (Poincaré Inequality, cf. Corollary 31 in [22]). Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) and H a real separable Hilbert space. Then, for $f \in L^2(E, H; m)$

$$(4.6) \quad (1 - \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{2}{n}} \wedge 1) \text{Var}_m(f) \leq \int_E \text{Var}_{P_x}(f) m(dx),$$

$$(4.7) \quad 1 - \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 \leq \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1 + \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

In particular, if $\kappa_n > 0$ for some $n \in \mathbb{N}$, we have a global Poincaré inequality:

$$\begin{aligned} 0 < 1 - \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} &\leq \inf_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \\ &\leq \sup_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1 + \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} < 2. \end{aligned}$$

Moreover, if $\mathbf{X}$ is an even step Markov chain obtained from an m -symmetric Markov chain, then

$$(4.8) \quad \sup_{f \in L^2(E, H; m)} \frac{\mathcal{E}(f)}{\text{Var}_m(f)} \leq 1.$$

Corollary 4.4 (Estimates of Eigenvalues). Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) with $m \in \mathcal{P}^2(E)$ and assume that $\kappa \in \mathbb{R}$ and the embedding $D(\mathcal{E}) \subset L^2(E; m)$ is compact. Then any non-zero eigenvalue λ of the L^2 -operator $-\Delta = I - P$ on $L^2(E; m)$ satisfies

$$(4.9) \quad 1 - \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 \leq \lambda \leq 1 + \inf_{n \in \mathbb{N}} (1 - \kappa_n)^{\frac{1}{n}} \wedge 1.$$

Moreover, if $\mathbf{X}$ is an even step Markov chain obtained from an m -symmetric Markov chain, then any eigenvalue λ satisfies $\lambda \leq 1$.

Corollary 4.5 (Recurrence). Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) . Suppose that there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Then $\mathbf{X}$ is recurrent, that is, for any non-trivial $f \in L^1_+(E; m)$, we have $Gf = \infty$ m -a.e. Here $Gf := \sum_{i=0}^{\infty} P^i f$.

Remark 4.6. (1) When $\mathbf{X}$ is an m -symmetric random walk on a finite undirected weighted connected graph $G = (V, E)$ with $m(\{x\}) := d_x(G)$, the degree of G at $x \in V$, Bauer-Jost-Liu [4] proved (4.9) for any $n \in \mathbb{N}$.

(2) If we assume the existence of non-constant Lipschitz eigenfunction of $-\Delta := I - P$, then we can directly prove the estimate for the associated real eigenvalue λ ;

$$(4.10) \quad 1 - (1 - \kappa_n)^{\frac{1}{n}} \leq \lambda \leq 1 + (1 - \kappa_n)^{\frac{1}{n}}$$

under $\kappa_n(x, y) \geq \kappa_n$ ($\kappa_n \in \mathbb{R}$) for $(x, y) \in E \times E \setminus \text{diag}$ without assuming the m -symmetricity of $\mathbf{X}$. If $\kappa_n > 0$, (4.10) is equivalent to (4.9). We show (4.10) as mentioned above. Let f

be a non-constant Lipschitz eigenfunction and assume that λ is a real eigenvalue of f with respect to $-\Delta$. Then, we have $(I - P)f = \lambda f$, equivalently, $P^k f = (1 - \lambda)^k f$ for any $k \in \mathbb{N}$. By scaling, we may assume that the Lipschitz constant of f is 1. Kantorovich-Rubinstein duality formula yields

$$\begin{aligned} d(x, y)(1 - \kappa_n) &\geq d_{W_1}(P_x^n, P_y^n) \geq P^n f(x) - P^n f(y) \\ &= (1 - \lambda)^n (f(x) - f(y)) \end{aligned}$$

for $(x, y) \in E \times E$, which implies $(1 - \kappa_n) \geq |1 - \lambda|^n$, that is, we obtain (4.10).

Theorem 4.7 (Strong L^p -Liouville Property). *Assume $m \in \mathcal{P}^p(E)$. Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) . Suppose that $\kappa \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Let (Y, d_Y) be a complete separable CAT(0)-space. Suppose that $u \in L^p(E, Y; m)$ satisfies $Pu = u$ m -a.e. on E . Then u is a constant map m -a.e. In particular, if $u \in \text{Lip}(E, Y)$ is P -harmonic, then u is a constant map.*

Corollary 4.8 (Ergodicity). *Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) . Suppose that $\kappa \in \mathbb{R}$ and there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Then $\mathbf{X}$ is ergodic, that is, for any P -invariant Borel set A , $m(A) = 0$ or $m(A^c) = 0$.*

Theorem 4.9 (Poincaré Inequality). *Assume $m \in \mathcal{P}^2(E)$ and $\kappa \in \mathbb{R}$. Let $\mathbf{X}$ be an m -symmetric Markov chain on (E, d) . Suppose that there exists $n \in \mathbb{N}$ such that $\kappa_n > 0$. Let (Y, d_Y) be a complete separable CAT(0)-space. Then for any $\varepsilon \in]0, 1 - (1 - \kappa_n)^{\frac{1}{n}}[$, there exists $\ell_0 \in \mathbb{N}$ depending on $\varepsilon, \kappa_n, (E, d, m, \mathbf{X})$ and (Y, d_Y) such that*

$$\inf_{u \in L^2(E, Y; m)} \frac{E(u)}{\text{Var}_m(u)} \geq \frac{(1 - (1 - \kappa_n)^{\frac{1}{n}} \wedge 1 - \varepsilon)^2}{8\ell_0^2} > 0.$$

Remark 4.10. (1) For the random walk on an undirected weighted finite graph $G = (V, E)$ with $N := |V|$, Bauer-Jost-Liu [4] proved the equivalence among the following:

- (i) G is non-bipartite.
- (ii) $\lambda_{N-1} < 2$.
- (iii) There exists $n \in \mathbb{N}$ such that $\kappa_n > 0$.

Since G is connected, we have $\lambda_1 > 0$. Here λ_1 (resp. λ_{N-1}) is the smallest non-zero (resp. maximum) eigenvalue of the Laplace operator on G . Under the equivalent conditions (i)–(iii), we have a positivity of non-linear spectral gap as in Theorem 4.9. Remark that the positivity of non-linear spectral gap on the finite connected weighted graph G (having no loop and no multi-edges) with graph distance is already proved by Izeki-Kondo-Nayatani [7] for CAT(0)-space target.

- (2) Our Theorem 4.9 covers the case for the random walk derived from the Brownian motion on Riemannian manifolds with positive Ricci curvature as in Example 1.5.

Acknowledgment. The authors thank Professor Kazumasa Kuwada for his valuable comments.

REFERENCES

1. L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Second edition. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008.
2. K. Ball, E. A. Carlen and E. H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent. Math. **115** (1994), no. 3, 463–482.
3. F. Bauer and J. Jost, *Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian*, <http://arxiv.org/abs/0910.3118v3>, to appear in Comm. Anal. Geom. 2011.
4. F. Bauer, J. Jost and S. Liu, *Ollivier's Ricci curvature and the spectrum of the normalized graph Laplace operator*, preprint, 2011.
5. M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **319**, Springer-Verlag, Berlin, 1999.
6. F.-Z. Gong, Y. Liu and Z.-Y. Wen, *Some notes on Ricci-Ollivier curvature*, preprint 2012, to appear in Osaka J. Math.
7. H. Izeke, T. Kondo and S. Nayatani, private communication, (2012).
8. J. Jost and S. Liu, *Ollivier's Ricci curvature, local clustering and curvature dimension inequality on graphs*, preprint, 2011.
9. W. S. Kendall, *Probability, convexity, and harmonic maps with small image I: Uniqueness and fine existence*, Proc. London Math. Soc., (3) **61** (1990), no. 2, 371–406.
10. W. S. Kendall, *From stochastic parallel transport to harmonic maps*, New directions in Dirichlet forms, 49–115, AMS/IP Stud. Adv. Math., **8**, Amer. Math. Soc., Providence, RI, 1998.
11. Y. Kitabeppu, *Lower bound of coarse Ricci curvature on metric measure spaces and eigenvalues of Laplacian*, preprint 2012.
12. E. Kokubo and K. Kuwae, *On spectral bounds for symmetric Markov processes with coarse Ricci curvatures*, preprint, 2012.
13. K. Kuwae, *Jensen's inequality over $CAT(\kappa)$ -space with small diameter*, Proceedings of Potential Theory and Stochastics, Albac Romania, 173–182, Theta Ser. Adv. Math., **14**, Theta, Bucharest, 2009.
14. K. Kuwae, *Jensen's inequality on convex spaces*, preprint (2012).
15. K. Kuwae, *Variational convergence over convex spaces*, (2012), in preparation.
16. K. Kuwae and K.-Th. Sturm, *On a Liouville type theorem for harmonic maps to convex spaces via Markov chains*, Proceedings of German-Japanese symposium in Kyoto 2006, 177–192, RIMS Kôkyûroku Bessatsu B6, 2008.
17. Y. Lin and S.-T. Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. **17** (2010), no. 2, 343–356.
18. Y. Lin, L. Lu and S.-T. Yau, *Ricci curvature on graphs*, Tohoku Math. J. **63** (2011), no. 4, 605–627.
19. J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. **169** (2009), no. 3, 903–991.
20. S.-I. Ohta, *Convexities of metric spaces*, Geom. Dedicata **125**, (2007), no. 1, 225–250.

21. S.-I. Ohta, *Extending Lipschitz and Hölder maps between metric spaces*, Positivity **13** (2009), no. 2, 407–425.
22. Y. Ollivier, *Ricci curvature of Markov chains on metric spaces*, J. Func. Anal. **256** (2009), no. 3, 810–864.
23. K.-Th. Sturm, *Nonlinear Markov operators associated with symmetric Markov kernels and energy minimizing maps between singular spaces*, Calc. Var. Partial Differential Equations **12** (2001), no. 4, 317–357.
24. K.-Th. Sturm, *Nonlinear Markov operators, discrete heat flow, and harmonic maps between singular spaces*, Potential Anal. **16** (2002), no. 4, 305–340.
25. K.-Th. Sturm, *Probability measures on metric spaces of nonpositive curvature. Heat kernels and analysis on manifolds, graphs, and metric spaces* (Paris, 2002), 357–390, Contemp. Math., **338**, Amer. Math. Soc., Providence, RI, 2003.
26. K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131.
27. K.-T. Sturm, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177.
28. L. Veysseire, *Coarse Ricci curvature for continuous-time Markov processes*, preprint, 2012.
29. C. Villani, *Optimal transport, old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009.
30. M.-K. von Renesse and K.-Th. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math. **58** (2005), no. 7, 923–940.

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