

On the m -step solvable anabelian geometry of mixed-characteristic local fields

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1 Background

Anabelian geometry is a branch of arithmetic geometry that studies how the arithmetic information of a scheme X is embedded within its étale fundamental group. One could easily translate this description into the language of field arithmetic by setting

$$X = \operatorname{Spec} k$$

for a field k : *Anabelian geometry of fields* studies how the field-theoretic/arithmetic invariants (including the field itself) of k can be “recovered” from its absolute Galois group G_k as a profinite group. The celebrated theorem of Neukirch-Uchida, one of the earliest validations of anabelian geometry’s central philosophy, demonstrates that number fields can be recovered by their absolute Galois groups [11, Corollary 2], i.e., for two number fields $F_\circ$ and $F_\bullet$, it holds that

$$F_\circ \cong F_\bullet \iff G_{F_\circ} \cong G_{F_\bullet}.$$

This monumental result naturally led to investigations into the analogous statement in a local setting. That is, for two mixed-characteristic local fields $K_\circ$ and $K_\bullet$, it holds that

$$K_\circ \cong K_\bullet \stackrel{?}{\iff} G_{K_\circ} \cong G_{K_\bullet}.$$

It is known that this statement does *not* hold in general (cf., e.g., [12] for a counterexample). However, by imposing stronger constraints on the side of the absolute Galois groups, Mochizuki [6] proved that

$$K_\circ \cong K_\bullet \iff G_{K_\circ} \cong_{\text{filt}} G_{K_\bullet},$$

where $\cong_{\text{filt}}$ means that the two objects are isomorphic as *filtered* profinite groups. (Here, the absolute Galois groups and their quotients shall be regarded as filtered profinite groups by the *ramification groups in upper numbering*, cf. [9, Chap. IV, §3].)

On the other hand, the recent work of Saïdi-Tamagawa [7] has shown that number fields can be characterized by the “maximal 3-step solvable quotients” (cf. Definition 2.4) of their absolute Galois groups. That is, for number fields $F_\circ$ and $F_\bullet$,

$$F_\circ \cong F_\bullet \iff G_{F_\circ}^3 \cong G_{F_\bullet}^3,$$

where G^m denotes the maximal m -step solvable quotient of a profinite group G . An implication of this result is that these quotients, despite carrying less information than the full absolute Galois groups, still retain enough arithmetic information to determine the field structure.

2 Main results

In this paper, we mainly discuss the following result on mixed-characteristic local fields (analogous to the result of Saïdi-Tamagawa): For two mixed-characteristic local fields $K_\circ$ and $K_\bullet$,

$$K_\circ \cong K_\bullet \iff G_{K_\circ}^2 \cong_{\text{filt}} G_{K_\bullet}^2.$$

To do so, we begin by recalling Mochizuki's and Saïdi-Tamagawa's results in a *functorial* form.

Theorem 2.1 (Mochizuki [6, Theorem 4.2]). *Let $K_\circ$ and $K_\bullet$ be mixed-characteristic local fields. Then, for an isomorphism*

$$\alpha: G_{K_\circ} \xrightarrow{\cong} G_{K_\bullet}$$

of filtered profinite groups, there exists a unique isomorphism $\theta: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$ such that

$$\alpha(\sigma) = \theta \circ \sigma \circ \theta^{-1} \tag{1}$$

for every $\sigma \in G_{K_\circ}$. ◇

Theorem 2.1 implies that the isomorphism class of a given mixed-characteristic local field K can be determined *functorially* from the isomorphism class of its absolute Galois group G_K (as a filtered profinite group).

Corollary 2.2. *Let $K_\circ$ and $K_\bullet$ be mixed-characteristic local fields, and assume that there exists an isomorphism*

$$\alpha: G_{K_\circ} \xrightarrow{\cong} G_{K_\bullet}$$

of filtered profinite groups. Then there exists an isomorphism $f: K_\circ \xrightarrow{\cong} K_\bullet$. ◇

Proof. By Theorem 2.1, there exists an isomorphism $\theta: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$ such that (1) holds for all $\sigma \in G_{K_\circ}$. Hence

$$\theta(K_\circ) = \theta((K_\circ^{\text{alg}})^{G_{K_\circ}}) = (K_\bullet^{\text{alg}})^{G_{K_\bullet}} = K_\bullet$$

and we obtain the isomorphism $f = \theta|_{K_\circ}$. □

We restate the result of Saïdi-Tamagawa.

Theorem 2.3 (Saïdi-Tamagawa [7, Theorem 1]). *Let $F_\circ$ and $F_\bullet$ be number fields, and assume that there exists an isomorphism*

$$A_3: G_{F_\circ}^3 \xrightarrow{\cong} G_{F_\bullet}^3$$

of profinite groups. Then there exists an isomorphism $h: F_\circ \xrightarrow{\cong} F_\bullet$. ◇

In Theorem 2.3 (resp. Corollary 2.2), there is no clear description of how A_3 (resp. α) and h (resp. f) are related to each other; we often call this kind of results *weak bi-anabelian*. Contrastingly, *strong bi-anabelian* (or simply *bi-anabelian*) results are those that provide a functorial description of the relationship between *the morphisms of the étale fundamental groups (absolute Galois groups)* and *the morphisms of the geometric objects (fields)*. For instance, Theorem 2.1 is a strong bi-anabelian result. To proceed to the strong bi-anabelian version of Saïdi-Tamagawa's result, we define the *maximal m -step solvable extension* of a field.

Definition 2.4. For a profinite group G , we denote by $\overline{[G, G]}$ the closed subgroup generated by the commutators of G , i.e., the elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$, where $\sigma, \tau \in G$.

(1) We inductively define the decreasing sequence

$$G = G^{[0]} \supseteq G^{[1]} \supseteq \dots \supseteq G^{[m]} \supseteq \dots$$

of closed normal subgroups of G , by $G^{[m+1]} = \overline{[G^{[m]}, G^{[m]}]}$. We say that a profinite group G is *m-step solvable* (resp. *abelian*, resp. *metabelian*) if $G^{[m]}$ (resp. $G^{[1]}$, resp. $G^{[2]}$) is trivial.

(2) We denote by G^m the quotient $G/G^{[m]}$, and call it the *maximal m-step solvable quotient* of G . We will often write G^{ab} instead of G^1 , and call it the *maximal abelian quotient* or *abelianization* of G .

(3) Let k be a field. For an integer $m \geq 0$, we denote by k^m/k the subfield of k^{sep} fixed by $G_k^{[m]}$, and call it the *maximal m-step solvable extension* of k , so that $G_k^m = \text{Gal}(k^m/k)$. We will often write k^{ab} instead of k^1 , and call it the *maximal abelian extension* of k .

◇

Theorem 2.5 (Saïdi-Tamagawa [7, Theorem 2]). *Let m be an integer ≥ 0 , $F_\circ$ and $F_\bullet$ number fields. Then, for an isomorphism*

$$A_{m+4}: G_{F_\circ}^{m+4} \rightarrow G_{F_\bullet}^{m+4}$$

of profinite groups, there exists an isomorphism $\Theta_{m+1}: F_\circ^{m+1} \rightarrow F_\bullet^{m+1}$ such that

$$A_{m+1}(\sigma) = \Theta_{m+1} \circ \sigma \circ \Theta_{m+1}^{-1}$$

for every $\sigma \in G_{F_\circ}^{m+1}$, where $A_{m+1}: G_{F_\circ}^{m+1} \rightarrow G_{F_\bullet}^{m+1}$ is the isomorphism induced by A_{m+4} . Moreover,

- *if $m \geq 1$, the isomorphism Θ_{m+1} is uniquely determined by A_{m+4} ;*
- *if $m = 0$, the isomorphism $\Theta_{m+1}|_{F_\circ}: F_\circ \rightarrow F_\bullet$ is uniquely determined by A_{m+4} .*

◇

Now, we are ready to proceed to the main results, which are:

- local analogues of Theorem 2.3 and Theorem 2.5;
- refinements of Theorem 2.1 for maximal m -step solvable quotients ($m \geq 2$).

Theorem 2.6 ([4, Theorem 1.2]). *Let $K_\circ$ and $K_\bullet$ be mixed-characteristic local fields, and assume that there exists an isomorphism*

$$\alpha_2: G_{K_\circ}^2 \xrightarrow{\cong} G_{K_\bullet}^2$$

of filtered profinite groups. Then there exists an isomorphism $f: K_\circ \xrightarrow{\cong} K_\bullet$.

◇

Theorem 2.7 ([4, Theorem 1.3]). *Let m be an integer ≥ 0 , $K_\circ$ and $K_\bullet$ mixed-characteristic local fields. Then, for an isomorphism*

$$\alpha_{m+3}: G_{K_\circ}^{m+3} \xrightarrow{\cong} G_{K_\bullet}^{m+3}$$

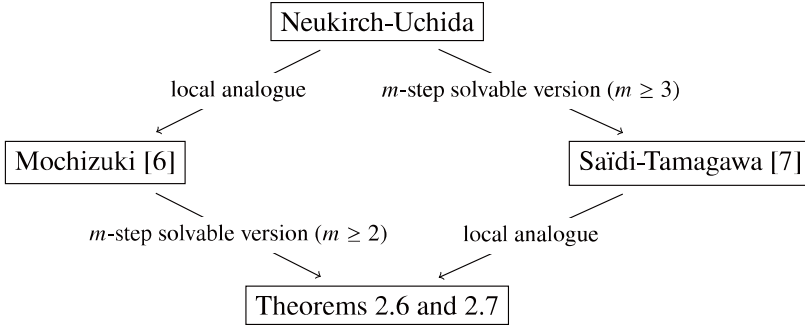
of filtered profinite groups, there exists an isomorphism $\theta_{m+1}: K_\circ^{m+1} \rightarrow K_\bullet^{m+1}$ such that

$$\alpha_{m+1}(\sigma) = \theta_{m+1} \circ \sigma \circ \theta_{m+1}^{-1}$$

for every $\sigma \in G_{K_\circ}^{m+1}$, where $\alpha_{m+1}: G_{K_\circ}^{m+1} \rightarrow G_{K_\bullet}^{m+1}$ is the isomorphism induced by α_{m+3} . Moreover,

- (i) *if $m \geq 1$, the isomorphism θ_{m+1} is uniquely determined by α_{m+3} ;*
- (ii) *if $m = 0$, the isomorphism $\theta_{m+1}|_{K_\circ}: K_\circ \rightarrow K_\bullet$ is uniquely determined by α_{m+3} .*

◇



Definition 2.8.

- (1) A group G is said to be *center-free* if the center $Z(G)$ of G is trivial, i.e.,

$$\{\sigma \in G \mid \sigma\tau = \tau\sigma \text{ for all } \tau \in G\} = \{1\}.$$

- (2) A topological group G is said to be *slim* if the centralizer $Z_G(H)$ of any open subgroup H of G is trivial in G , i.e.,

$$\{\sigma \in G \mid \sigma\tau = \tau\sigma \text{ for all } \tau \in H\} = \{1\}.$$

for all open subgroup $H \subseteq G$.

◇

Proposition 2.9. *Let K be a mixed-characteristic local field.*

- (1) *The absolute Galois group G_K is slim.*
- (2) *The maximal m -step solvable quotient G_K^m of G_K is center-free for all $m \geq 2$.*

◇

Proof. (1) cf. [3, Lemma 1.8]. (2) cf. [4, Proposition A.1], [7, Proposition 1.1 (ix)]. □

The “uniqueness” parts of Theorems 2.1 and 2.7 are consequences of the *slimness* of the absolute Galois group $G_{K_\circ}$ and the *center-freeness* of the maximal m -step solvable quotients $G_{K_\circ}^m$ ($m \geq 2$), respectively.

Uniqueness of the isomorphism θ in Theorem 2.1. Suppose that both isomorphisms

$$\theta_1, \theta_2: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$$

satisfy the condition (1). Then

$$\theta_2^{-1} \circ \theta_1 \in Z_G(G_{K_\circ}),$$

where G is the absolute Galois group of $\mathbf{Q}_{p_{K_\circ}}$. Since G is slim, $\theta_2^{-1} \circ \theta_1 = \text{id}_{K_\circ^{\text{alg}}}$.

Uniqueness of the isomorphism θ_{m+1} in Theorem 2.7. Suppose that both isomorphisms

$$\theta_{m+1,1}, \theta_{m+1,2}: K_\circ^{m+1} \rightarrow K_\bullet^{m+1}$$

satisfy the condition. Then there exist isomorphisms

$$\theta_1, \theta_2: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$$

that extend $\theta_{m+1,1}, \theta_{m+1,2}$, respectively; we have

$$(\gamma := (\theta_2)^{-1} \circ \theta_1 \in \text{Gal}(K_\circ^{\text{alg}}/\mathbf{Q}_{p_{K_\circ}}))$$

and

$$\gamma|_{K_\circ^{m+1}} \circ \sigma \circ \gamma^{-1}|_{K_\circ^{m+1}} = (\theta_{m+1,2})^{-1} \circ \theta_{m+1,1} \circ \sigma \circ (\theta_{m+1,1})^{-1} \circ \theta_{m+1,2} = \sigma, \quad (2)$$

for all $\sigma \in G_{K_\circ}^{m+1}$. Hence we see that, for all $x \in K_\circ^\times$,

$$\gamma|_{K_\circ^{\text{ab}}} \circ \text{Art}_{K_\circ}(x) \circ \gamma^{-1}|_{K_\circ^{\text{ab}}} = \text{Art}_{K_\circ}(x),$$

and $\gamma(x) = x$ by local class field theory; it follows that $\gamma \in \text{Gal}(K_\circ^{\text{alg}}/K_\circ)$. Furthermore,

$$\gamma|_{K_\circ^{m+1}} \in Z(G_{K_\circ}^{m+1})$$

since (2) holds for all $\sigma \in G_{K_\circ}^{m+1}$; together with the center-freeness of $Z(G_{K_\circ}^{m+1})$ for $m \geq 1$, we conclude that $\gamma|_{K_\circ^{m+1}} = \text{id}_{K_\circ^{m+1}}$.

For the rest of this paper, we give a sketch of the proof of Theorem 2.6 and the “existence” part of Theorem 2.7 via the following steps:

- (1) We first explain how Mochizuki utilized p -adic Hodge theory to prove Theorem 2.1. (§4.1)
- (2) We then explain how Mochizuki’s method can be applied to prove Theorems 2.6 and 2.7 with appropriate modifications. (§4.2)

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3 Notation

Given a mixed-characteristic local field K , we write:

- $\mathcal{O}_K$ for the *ring of integers* of K ;
- $\mathfrak{p}_K$ for the (unique) *maximal ideal* of $\mathcal{O}_K$;
- U_K for the *unit group* $\mathcal{O}_K^\times$ of $\mathcal{O}_K$;
- $\mathbb{k}_K$ for the *residue field* $\mathcal{O}_K/\mathfrak{p}_K$ of K ;
- p_K for the *residue characteristic* of K , i.e., the characteristic of $\mathbb{k}_K$;
- a_K for the largest integer ≥ 0 such that K contains a $(p_K^{a_K})^{\text{th}}$ root of unity;
- d_K for the *absolute degree* $[K : \mathbf{Q}_{p_K}]$ of K ;
- e_K for the *absolute ramification index* of K , so that $p_K \mathcal{O}_K = \mathfrak{p}_K^{e_K}$;
- f_K for the *absolute inertia degree* $[\mathbb{k}_K : \mathbf{F}_{p_K}]$, so that $|\mathbb{k}_K| = p_K^{f_K}$, and $d_K = e_K f_K$;
- $\chi_K : G_K \rightarrow \mathbf{Z}_{p_K}^\times$ for the p_K -*adic cyclotomic character* of K ;
- $\text{Art}_K : K^\times \rightarrow G_K^{\text{ab}}$ for the *local reciprocity map* of K .

4 Sketch of the proof

4.1 Group-theoretic algorithms and p -adic Hodge theory

Mochizuki's proof of Theorem 2.1 relies on the following key steps that characterize a mixed-characteristic local field by its absolute Galois group, equipped with the ramification filtration.

Recovering the cyclotomic character. The first step involves recovering the p_K -adic cyclotomic character

$$\chi_K : G_K \rightarrow \mathbf{Z}_{p_K}^\times$$

of K from the profinite group G_K , for a mixed-characteristic local field K . There are several known *group-theoretic algorithms* (cf. [3]) that recover important invariants of K , such as the residue characteristic p_K , the absolute ramification index e_K , the absolute inertia degree f_K , the absolute degree d_K and even the cyclotomic character χ_K . This means that, for example, we have a “procedure” p (defined in language that only depends on the topological group structure) which takes a profinite group G as an input, and returns an integer $p(G)$, such that

$$p(G_K) = p_K$$

for any mixed-characteristic local field K . Likewise, we have a group-theoretic algorithm χ , which takes a profinite group G and returns a character

$$\chi(G) : G \rightarrow \mathbf{Z}_{p(G)}^\times,$$

such that $\chi(G_K) = \chi_K$ for any mixed-characteristic local field K [6, Proposition 1.1]. (Note that we have not used the ramification filtration yet.)

Recovering the additive structure. The second key step involves reconstructing the topological G_K -module $K^{\text{alg},+}$ (the additive group of K^{alg}) equipped with the p_K -adic topology (and hence its p_K -adic completion $\mathcal{C}(K^{\text{alg}})^+ (= \mathbf{C}_{p_K}^+)$) from the *filtered* profinite group G_K . As in the previous step, this is achieved by establishing group-theoretic algorithms that recover the topological G_K -module $K^{\text{alg},+}$ and $\mathcal{C}(K^{\text{alg}})^+$ from G_K : We first establish a group-theoretic algorithm $k^{\text{alg},+}$, which takes a profinite group G and returns a G -module $k^{\text{alg},+}(G)$ such that

$$k^{\text{alg},+}(G_K) \cong K^{\text{alg},+}$$

as G_K -modules. We then establish $\mathcal{O}_{k^{\text{alg}}}^+$, which takes a filtered profinite group G and returns a sub- G -module $\mathcal{O}_{k^{\text{alg}}}^+(G)$ of $k^{\text{alg},+}(G)$, such that $\mathcal{O}_{k^{\text{alg}}}^+(G)$ is mapped onto the additive group of the integer ring of K^{alg} . (This is the reason why the ramification filtration is crucial in this step.) This yields another group-theoretic algorithm: Let $\mathcal{E}_{k^{\text{alg}}}^+(G)$ be the $p(G) \cdot \mathcal{O}_{k^{\text{alg}}}^+(G)$ -adic completion of $k^{\text{alg},+}(G)$ for a filtered profinite group G . Then

$$\mathcal{E}_{k^{\text{alg}}}^+(G_K) \cong \mathcal{C}(K^{\text{alg}})^+$$

as topological G_K -modules [6, Proposition 2.2].

Hodge-Tate representations. Let G be a profinite group, and (ρ, V) an ℓ -adic representation of G for a prime number ℓ . For an ℓ -adic character $\chi: G \rightarrow \mathbf{Z}_\ell^\times$, we shall write $\mathbf{Z}_\ell(\chi)$ for the $\mathbf{Z}_\ell$ -representation $(\chi, \mathbf{Z}_\ell^+)$ of G , and $V(\chi)$ for the ℓ -adic representation $V \otimes_{\mathbf{Q}_\ell} (\mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell(\chi))$ of G .

For a p_K -adic representation (ρ, V) of G_K and an integer i , we define the i^{th} *Hodge-Tate number* $d_{\text{HT},K}^i(\rho, V)$ of (ρ, V) as the dimension of the K -vector space

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V(-i) \right)^{G_K},$$

where $V(-i) = (\rho(-i), V(-i))$ denotes the $(-i)^{\text{th}}$ Tate twist $V(\chi_K^{-i})$ of V . It is known that

$$\sum_{i \in \mathbf{Z}} d_{\text{HT},K}^i(\rho, V) \leq \dim_{\mathbf{Q}_{p_K}}(V)$$

holds [2, §5.1], and we say that (ρ, V) is *Hodge-Tate* when the equality holds.

By the previous steps, it is clear that the Hodge-Tate numbers $d_{\text{HT},K}^i(\rho, V)$ of a given p_K -adic representation (ρ, V) of G_K can be recovered group-theoretically from the *filtered* profinite group G_K and its action on V , i.e., if $K_\circ$ and $K_\bullet$ are mixed-characteristic local fields and $G_{K_\circ} \rightarrow G_{K_\bullet}$ is an isomorphism of filtered profinite groups, then the i^{th} Hodge-Tate numbers of the representation

$$G_{K_\bullet} \rightarrow \text{Aut}_{\mathbf{Q}_{p_K}}(V)$$

and the representation defined by the composition

$$G_{K_\circ} \rightarrow G_{K_\bullet} \rightarrow \text{Aut}_{\mathbf{Q}_{p_K}}(V)$$

coincide for all $i \in \mathbf{Z}$.

Uniformizing representations. Let (ρ, V) be a p_K -adic representation of G_K , and E/K a finite extension such that $E/\mathbf{Q}_{p_K}$ is Galois. Suppose that V is an E -vector space of dimension 1 and the G_K -action on V is E -linear, so that $\rho: G_K \rightarrow \text{Aut}_{\mathbf{Q}_{p_K}}(V)$ factors through $\rho: G_K^{\text{ab}} \rightarrow E^\times$. We say that a representation (ρ, V) of this type is *uniformizing* if there exist an open subgroup $I \subseteq U_K$ and a field homomorphism $\iota: K \rightarrow E$ such that

$$(\rho \circ \text{Art}_K)|_I = \iota^\times|_I,$$

where $\iota^\times: K^\times \rightarrow E^\times$ is the group homomorphism induced by ι . For example, $V_E = E^+$ can be regarded as a uniformizing representation by local class field theory: We define the G_K -action on V_E via the composition

$$\rho_E: G_K^{\text{ab}} \rightarrow G_E^{\text{ab}} (\cong G_E^{\text{ab}}(0) \oplus \text{Gal}(E^{\text{un}}/E)) \twoheadrightarrow G_E^{\text{ab}}(0) \rightarrow U_E$$

of continuous homomorphisms, where the first (resp. second, resp. third) arrow is the transfer map (resp. the natural surjection, resp. the isomorphism Art_E^{-1}).

Proposition 4.1. *Let K be a mixed-characteristic local field, and let E/K be a finite extension such that $E/\mathbf{Q}_{p_K}$ is Galois. Suppose that (ρ, V) is an E -linear representation of G_K , of E -dimension 1. Then (ρ, V) is a uniformizing representation if and only if*

$$d_{\text{HT},K}^i(\rho, V) = \begin{cases} [E:K]([K:\mathbf{Q}_{p_K}] - 1) & i = 0 \\ [E:K] & i = 1 \end{cases}.$$

◇

Proof. cf. [8, Chap. III, A5].

□

In other words, the uniformizing representations are characterized by their Hodge-Tate numbers; if

- $K_\circ$ and $K_\bullet$ are mixed-characteristic local fields;
- $\alpha: G_{K_\circ} \rightarrow G_{K_\bullet}$ is an isomorphism of *filtered* profinite groups;
- $E/\mathbf{Q}_p$ is a finite Galois extension containing both $K_\circ$ and $K_\bullet$, where $p = p_{K_\circ} = p_{K_\bullet}$;
- $(\rho_\circ, V)$ is the (uniformizing) representation (ρ_E, V_E) of $G_{K_\circ}$,

then $(\rho_\bullet := \rho_\circ \circ \alpha_1^{-1}, V)$ is also uniformizing representation of $G_{K_\bullet}$, where $\alpha_1: G_{K_\circ}^{\text{ab}} \rightarrow G_{K_\bullet}^{\text{ab}}$ is the isomorphism induced by α . Hence there exist an open subgroup $I_\circ$ (resp. $I_\bullet$) of $U_{K_\circ}$ (resp. $U_{K_\bullet}$) and a field homomorphism $\iota_\circ: K_\circ \rightarrow E$ (resp. $\iota_\bullet: K_\bullet \rightarrow E$) such that $\alpha_1(I_\circ) = I_\bullet$ and the diagram

$$\begin{array}{ccccccc}
 & & K_\circ^\times & & & & \\
 & \nearrow \subseteq & & \searrow \iota_\circ^\times & & & \\
 I_\circ & \xrightarrow{\subseteq} & U_{K_\circ} & \xrightarrow{\text{Art}_{K_\circ}} & G_{K_\circ}^{\text{ab}} & \xrightarrow{\rho_\circ} & E^\times \\
 \downarrow \cong, \alpha_1|_{I_\circ} & & \downarrow \cong, \alpha_1|_{U_{K_\circ}} & & \downarrow \cong, \alpha_1 & & \parallel \\
 I_\bullet & \xrightarrow{\subseteq} & U_{K_\bullet} & \xrightarrow{\text{Art}_{K_\bullet}} & G_{K_\bullet}^{\text{ab}} & \xrightarrow{\rho_\bullet} & E^\times \\
 & \searrow \subseteq & & \nearrow \iota_\bullet^\times & & & \\
 & & K_\bullet^\times & & & &
 \end{array}$$

commutes. In particular, we have $\iota_\bullet|_{I_\bullet} \circ \alpha_1|_{I_\bullet} = \iota_\circ|_{I_\circ}$. Since the sub- $\mathbf{Q}_p$ -vector space generated by $I_\circ$ (resp. $I_\bullet$) in E equals $K_\circ$ (resp. $K_\bullet$) (cf. [6, Lemma 4.1]), it follows that we have the field isomorphism

$$f: K_\circ \xrightarrow{\cong, \iota_\circ} \iota_\circ(K_\circ) = \iota_\bullet(K_\bullet) \xrightarrow{\cong, \iota_\bullet^{-1}} K_\bullet.$$

Finally, we obtain the desired isomorphism θ by passage to the limit.

4.2 The m -step solvable anabelian geometry

The proofs of Theorems 2.6 and 2.7 follow similar steps but require additional machinery and refinement: We can observe from the previous subsection that it suffices to prove the Hodge-Tate numbers $d_{\text{HT}, K}^i(\rho, V)$ of a given *abelian* p_K -adic representation (ρ, V) of G_K can be recovered group-theoretically from the *filtered* profinite group G_K^2 (instead of G_K) and its action on V , so we can repeat the restoration process of §4.2, starting from the isomorphism

$$\alpha_2: G_{K_\circ}^2 \rightarrow G_{K_\bullet}^2.$$

of filtered profinite groups. (From the exactly same discussion as the previous subsection, we obtain the isomorphism $f: K_\circ \rightarrow K_\bullet$, and by passage to the limit again, the desired isomorphism θ_{m+1} (with the 2-step-loss of solvability) [4, §6]. These amount to the proofs of Theorem 2.6 and the “existence” part of Theorem 2.7, respectively.)

Recovering the cyclotomic character from G_K^2 . The first step is to observe that:

- the residue characteristic p_K , the absolute ramification index e_K , the absolute inertia degree f_K , and the absolute degree d_K can be recovered group-theoretically from G_K^{ab} [4, Proposition 3.1];
- the cyclotomic character χ_K can be recovered group-theoretically from G_K^2 [4, Proposition 3.2].

For example, let G be a profinite group isomorphic to G_K^{ab} for some mixed-characteristic local field K . Then it follows from local class field theory that we have

$$G \cong (\mathbf{Z}/(p_K^{f_K} - 1)\mathbf{Z})^+ \oplus (\mathbf{Z}/p_K^{a_K}\mathbf{Z})^+ \oplus (\mathbf{Z}_{p_K}^+)^{\oplus d_K} \oplus \widehat{\mathbf{Z}}^+$$

as profinite groups. (Here, $\widehat{\mathbf{Z}}$ is the profinite completion of the ring $\mathbf{Z}$ of rational integers.) We denote by $p(G)$ the uniquely determined prime number ℓ such that

$$\log_\ell |G_{\text{tor}}/\ell \cdot G_{\text{tor}}| \geq 2,$$

where G_{tor} is the torsion subgroup of G and $G_{\text{tor}} := G/G_{\text{tor}}$. Then we have

$$p(G_K^{\text{ab}}) = p_K,$$

and see that the invariant p_K can be recovered group-theoretically from G_K^{ab} , instead of the full absolute Galois group G_K .

Recovering the additive structure of K^m . The second step involves reconstructing the topological G_K^{m+1} -module $\mathcal{C}_{K^m}^+$, the p_K -adic completion of $K^{m,+}$, from the *filtered* profinite group G_K^{m+1} .

Again, this is achieved by establishing a group-theoretic algorithm that recovers the topological G_K^{m+1} -module $\mathcal{E}_{K^m}^+$ from G_K^{m+1} [4, Proposition 4.3].

Hodge-Tate representations, revisited. By the preceding steps, it is clear that the following invariant

$$\dim_K \left(\mathcal{E}_{K^{\text{ab}}} \otimes_{\mathbf{Q}_{p_K}} V(-i) \right)^{G_K}$$

of a given p_K -adic representation (ρ, V) of G_K can be recovered group-theoretically from the *filtered* profinite group G_K^2 and its action on V , where $\mathcal{E}_{K^{\text{ab}}}$ denotes the p_K -adic completion of K^{ab} . We complete the proof by showing that this invariant is equal to the i^{th} Hodge-Tate number $d_{\text{HT},K}^i(\rho, V)$ if (ρ, V) is an *abelian* representation (and hence we can recover the Hodge-Tate numbers $d_{\text{HT},K}^i(\rho, V)$ of a given abelian p_K -adic representation (ρ, V)).

Lemma 4.2 ([4, Lemma 5.1]). *Let (ρ, V) be a p_K -adic representation of G_K . Then we have*

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V \right)^{G_K} = \left(\mathcal{E}((K^{\text{alg}})^{\text{Ker}(\rho)}) \otimes_{\mathbf{Q}_{p_K}} V \right)^{G_K}. \quad (3)$$

◇

Proof. We choose a basis $v_1, \dots, v_n$ of V . For each $\sigma \in G_K$, we shall write $(a_{ij}(\sigma)) \in \mathbf{GL}_n(\mathbf{Q}_{p_K})$ for the matrix of the linear transformation $\rho_\sigma := \rho(\sigma)$ with respect to the basis $v_1, \dots, v_n$, so that

$$(\rho_\sigma(v_1) \cdots \rho_\sigma(v_n)) = (v_1 \cdots v_n)(a_{ij}(\sigma)).$$

Suppose that $c_1, \dots, c_n \in \mathbf{C}_{p_K}$, and that $c_1 \otimes v_1 + \cdots + c_n \otimes v_n$ belongs to the left hand side of (3). Then for all $\sigma \in G_K$,

$$\begin{aligned} c_1 \otimes v_1 + \cdots + c_n \otimes v_n &= \sigma(c_1) \otimes \rho_\sigma(v_1) + \cdots + \sigma(c_n) \otimes \rho_\sigma(v_n) \\ &= \left(\sum_{j=1}^n \sigma(c_j) a_{1j}(\sigma) \right) \otimes v_1 + \cdots + \left(\sum_{j=1}^n \sigma(c_j) a_{nj}(\sigma) \right) \otimes v_n, \end{aligned}$$

and hence

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = (a_{ij}(\sigma)) \begin{pmatrix} \sigma(c_1) \\ \vdots \\ \sigma(c_n) \end{pmatrix}.$$

In particular, we have

$$c_1 \otimes v_1 + \cdots + c_n \otimes v_n \in \mathbf{C}_{p_K}^{\text{Ker}(\rho)} \otimes_{\mathbf{Q}_{p_K}} V = \mathcal{E}((K^{\text{alg}})^{\text{Ker}(\rho)}) \otimes_{\mathbf{Q}_{p_K}} V,$$

since it holds that $\sigma(c_1) = c_1, \dots, \sigma(c_n) = c_n$ for all $\sigma \in \text{Ker}(\rho)$. (Note that, for any closed subgroup H of G_K ,

$$\mathbf{C}_{p_K}^H = \mathcal{E}((K^{\text{alg}})^H)$$

by the theorem of Ax-Sen-Tate—cf. [10], [1], [2, Chap. 3].) □

Corollary 4.3. *For an abelian p_K -adic representation (ρ, V) of G_K , we have*

$$d_{\text{HT},K}^i(\rho, V) = \dim_K \left(\mathcal{E}_{K^{\text{ab}}} \otimes_{\mathbf{Q}_{p_K}} V(-i) \right)^{G_K},$$

for all $i \in \mathbf{Z}$.

◇

Proof. Since $\rho(-i)$ annihilates the $G_K^{[1]}$, the equality holds by the above lemma. $\square$

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