

多種 Lotka-Volterra 非自励競争モデルの解の漸近的性質

Asymptotic property of solutions of nonautonomous Lotka-Volterra model for
 N -competing species

広島大学・理学研究科 谷口 公仁彦 (Kunihiko TANIGUCHI)

Department of Mathematics, Graduate School of Science, Hiroshima University

1 Introduction and Statements of the main results

In this paper we consider the system of differential equations

$$u'_i = u_i \left[a_i(t) - \sum_{j=1}^N b_{ij}(t) f_{ij}(u_i, u_j) \right], \quad i = 1, \dots, N, \quad N \geq 2, \quad (\text{GLV})$$

where the functions $a_i(t)$, $1 \leq i \leq N$, and $b_{ij}(t)$, $1 \leq i, j \leq N$, are assumed to be continuous and nonnegative on $\mathbb{R}$. Furthermore, let the functions $f_{ij}(x, y)$, $1 \leq i, j \leq N$, be continuously differentiable on $\mathbb{R}_+^2 = (0, \infty)^2$, and we impose the following conditions on f'_{ij} s:

$$\left\{ \begin{array}{l} f_{ii}(x, y), \quad 1 \leq i \leq N, \text{ is continuously differentiable on } [0, \infty) \times [0, \infty); \\ f_{ij}(x, y) > 0, \quad (x, y) \in \mathbb{R}_+^2, \quad 1 \leq i, j \leq N; \\ (D_1 f_{ii} + D_2 f_{ii})(x, x) > 0, \quad x \in \mathbb{R}_+, \quad 1 \leq i \leq N; \\ D_1 f_{ij}(x, y) \geq 0, \quad (x, y) \in \mathbb{R}_+^2, \quad 1 \leq i, j \leq N; \\ D_2 f_{ij}(x, y) \geq 0, \quad (x, y) \in \mathbb{R}_+^2, \quad 1 \leq i, j \leq N; \\ f_{ii}(0, 0) = 0, \quad 1 \leq i \leq N; \\ \lim_{x \rightarrow \infty} f_{ii}(x, x) = \infty, \quad 1 \leq i \leq N, \end{array} \right. \quad (1.1)$$

where D_i , $i = 1, 2$, denotes the differentiation with respect to the i -th variable.

System (GLV) is a generalization of the following nonautonomous N -dimensional Lotka-Volterra competition system which S. Ahmad and A. C. Lazer [2] considered:

$$u'_i = u_i \left[a_i(t) - \sum_{j=1}^N b_{ij}(t) u_j \right], \quad i = 1, \dots, N, \quad N \geq 2. \quad (\text{LV})$$

An prototype of system (LV), as well as (GLV), is the classical Lotka-Volterra competition model for two species:

$$\left\{ \begin{array}{l} u'_1 = u_1(a_1 - b_{11}u_1 - b_{12}u_2), \\ u'_2 = u_2(a_2 - b_{21}u_1 - b_{22}u_2), \end{array} \right. \quad (1.2)$$

where a_i , $i = 1, 2$, and b_{ij} , $i, j = 1, 2$, are positive constants. When the growth rates a_i , $i = 1, 2$, and the interaction coefficients b_{ij} , $i, j = 1, 2$, satisfy

$$a_1 - b_{12} \left(\frac{a_2}{b_{22}} \right) > 0, \quad a_2 - b_{21} \left(\frac{a_1}{b_{11}} \right) > 0, \quad (1.3)$$

there exists a unique equilibrium point $(u_1^*, u_2^*) \in \mathbb{R}_+^2$. It is known that, if (1.3) hold, then any solution $(u_1(t), u_2(t))$ of system (1.2) with $(u_1(t_0), u_2(t_0)) \in \mathbb{R}_+^2$ satisfies

$$u_1(t) \rightarrow u_1^* \quad \text{and} \quad u_2(t) \rightarrow u_2^* \quad \text{as } t \rightarrow \infty.$$

In [2]–[4] it is shown that analogous results still hold for the nonautonomous equation (LV), as seen below. In this paper we intend to generalize such results further.

We introduce notation. Put $c_M := \sup_{t \in \mathbb{R}} c(t)$ for bounded functions $c(t)$ on $\mathbb{R}$. For $i = 1, \dots, N$, we put

$$\tilde{f}_{ii}(x) = f_{ii}(x, x), \quad x \in \mathbb{R}_+.$$

By assumption (1.1) $\tilde{f}_{ii}$, $i = 1, \dots, N$, have the inverse function $\tilde{f}_{ii}^{-1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The assumptions employed in the paper will be selected from the following list:

$$(A1) \quad b_{ii}(t) > 0, \quad t \in \mathbb{R}, \quad 1 \leq i \leq N;$$

$$(A2) \quad \int_0^\infty b_{ii}(s) ds = \infty, \quad 1 \leq i \leq N;$$

$$(A3) \quad \left(\frac{a_i}{b_{ii}} \right)_M < \infty, \quad 1 \leq i \leq N;$$

$$(A4) \quad \inf_{t \in \mathbb{R}} \frac{a_i(t) - \sum_{j \neq i} b_{ij}(t)(a_j/b_{jj})_M}{b_{ii}(t)} > 0, \quad 1 \leq i \leq N;$$

$$(A5) \quad \inf_{t \in \mathbb{R}} \frac{a_i(t) - \sum_{j \neq i} b_{ij}(t)f_{ij}(\tilde{f}_{ii}^{-1}((a_i/b_{ii})_M), \tilde{f}_{jj}^{-1}((a_j/b_{jj})_M))}{b_{ii}(t)} > 0, \quad 1 \leq i \leq N;$$

$$(A6) \quad f_{ij}(x, y) \leq \tilde{f}_{jj}(y), \quad (x, y) \in \mathbb{R}_+^2, \quad 1 \leq i, j \leq N;$$

$$(A7) \quad \text{for any } s > 1 \text{ sufficiently close to } 1;$$

$$f_{ij}(\tilde{f}_{ii}^{-1}(sx), \tilde{f}_{jj}^{-1}(sy)) \leq sf_{ij}(\tilde{f}_{ii}^{-1}(x), \tilde{f}_{jj}^{-1}(y)), \quad (x, y) \in \mathbb{R}_+^2, \quad 1 \leq i, j \leq N.$$

REMARK 1.1. As in the case of (LV) and (1.2), if $f_{ij}(x, y)$, $1 \leq i, j \leq N$, are independent of x , (A6) is satisfied. For (LV) we can take $f_{ij}(x, y) = y$, $1 \leq i, j \leq N$, which satisfy (A6) and (A7).

REMARK 1.2. Let

$$f_{ij}(x, y) = \begin{cases} \frac{x^{\alpha_{ij}}}{1 + x^{\alpha_{ij}}} y^{\beta_{ij}}, & i \neq j, \\ x^{\alpha_{ij}} y^{\beta_{ij}}, & i = j, \end{cases}$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}_+$. If for $i \neq j$, $\beta_{ij} = \alpha_{jj} + \beta_{jj}$, then the functions f_{ij} , $1 \leq i, j \leq N$, satisfy (A6).

REMARK 1.3. Let

$$f_{ij}(x, y) = x^{\alpha_{ij}} y^{\beta_{ij}}, \quad (x, y) \in \mathbb{R}_+^2, \quad 1 \leq i, j \leq N,$$

where $\alpha_{ij}, \beta_{ij} \in \mathbb{R}_+$. If $\alpha_{ij} + \beta_{ij} \leq \min\{\alpha_{ii} + \beta_{ii}, \alpha_{jj} + \beta_{jj}\}$, then the functions f_{ij} , $1 \leq i, j \leq N$, satisfy (A7).

S. Ahmad and A. C. Lazer [2] supposed that the functions $a_i(t)$, $1 \leq i \leq N$ and $b_{ij}(t)$, $1 \leq i, j \leq N$, satisfy conditions (A1)–(A3) and (A4). Under these conditions they have shown the following [2]:

(I) If $u = (u_1, \dots, u_N)$ is a solution of (LV) with $u_i(t_0) > 0$, $1 \leq i \leq N$, $t_0 \in \mathbb{R}$, then

$$0 < \inf_{t \geq t_0} u_i(t) \leq \sup_{t \geq t_0} u_i(t) < \infty, \quad \text{for } 1 \leq i \leq N.$$

(II) If A is a compact subset of $\mathbb{R}_+^N$, then the Lebesgue measure of the set $\{u(t) \mid u \text{ is a solution of (LV) satisfying } u(t_0) \in A\}$ tends to 0 as $t \rightarrow \infty$.

Our main aim is to show that (I) and (II) are still valid for (GLV). To state the results we introduce the symbol: For compact subset A of $\mathbb{R}_+^N$ and $t_0 \in \mathbb{R}$ we set

$$u(t, t_0, A) = \{u(t) \mid u \text{ is a solution of (GLV) satisfying } u(t_0) \in A\}.$$

By $\mu(\cdot)$ we denote the Lebesgue measure of measurable sets in $\mathbb{R}_+^N$. We can show the following:

THEOREM 1.4. *Let conditions (A1)–(A3), (A4), and (A6) hold. Let A be a compact subset of $\mathbb{R}_+^N$ and let $t_0 \in \mathbb{R}$. Then,*

$$\mu(u(t, t_0, A)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

THEOREM 1.5. *Let conditions (A1)–(A3), (A5), and (A7) hold. Let A be a compact subset of $\mathbb{R}_+^N$ and let $t_0 \in \mathbb{R}$. Then,*

$$\mu(u(t, t_0, A)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

We give examples of systems (GLV) for which above conditions hold.

EXAMPLE 1.6. We consider system (GLV) for two species

$$\begin{aligned} u_1' &= u_1 \left[(\cos t + 7) - (\sin t + 7) \cdot u_1^2 - (\sin t + 1) \cdot \left(\frac{u_1^3}{1 + u_1^3} \cdot u_2^2 \right) \right], \\ u_2' &= u_2 \left[(\cos t + 9) - (\sin t + 2) \cdot \left(\frac{u_2^4}{1 + u_2^4} \cdot u_1^3 \right) - (\sin t + 9) \cdot u_2^3 \right]. \end{aligned}$$

Obviously (A6) holds. We have

$$\begin{aligned} a_1(t) - b_{12}(t) \left(\frac{a_2}{b_{22}} \right)_M &> \cos t + 7 - (\sin t + 1) \cdot \frac{10}{8} > 2, \\ a_2(t) - b_{21}(t) \left(\frac{a_1}{b_{11}} \right)_M &> \cos t + 9 - (\sin t + 2) \cdot \frac{8}{6} > 2. \end{aligned}$$

So conditions (A1)–(A3) and (A4) hold. Of course condition (1.1) hold.

EXAMPLE 1.7. We consider system (GLV) for two-species

$$\begin{aligned} u_1' &= u_1 [(\cos t + 7) - (\sin t + 7) \cdot u_1^4 - (\sin t + 1) \cdot u_1 u_2^2], \\ u_2' &= u_2 [(\cos t + 9) - (\sin t + 2) \cdot u_2^2 u_1^2 - (\sin t + 9) \cdot u_2^6]. \end{aligned}$$

Obviously (A7) holds. We have

$$\begin{aligned} a_1(t) - b_{12}(t) f_{12} \left(\tilde{f}_{11}^{-1} \left(\left(\frac{a_1}{b_{11}} \right)_M \right), \tilde{f}_{22}^{-1} \left(\left(\frac{a_2}{b_{22}} \right)_M \right) \right) \\ &> \cos t + 7 - (\sin t + 1) \cdot \left(\frac{8}{6} \right)^{1/4} \cdot \left(\frac{10}{8} \right)^{2/6} > 2, \\ a_2(t) - b_{21}(t) f_{21} \left(\tilde{f}_{22}^{-1} \left(\left(\frac{a_2}{b_{22}} \right)_M \right), \tilde{f}_{11}^{-1} \left(\left(\frac{a_1}{b_{11}} \right)_M \right) \right) \\ &> \cos t + 9 - (\sin t + 2) \cdot \left(\frac{10}{8} \right)^{2/6} \cdot \left(\frac{4}{3} \right)^{2/4} > 2. \end{aligned}$$

So conditions (A1)–(A3), (A5) hold. Of course condition (1.1) hold.

2 The sketch of the proof of the main results

In this section we give the sketch of the proof of the main results. As a first step, we note that every solutions u of (GLV) with $u(t_0) \in \mathbb{R}_+^N$ remains here as long as it exists. To see this we rewrite system (GLV) in the form

$$u'_i(t) = p_i(t)u_i(t), \quad i = 1, 2, \dots, N,$$

where the functions $p_i(t)$, $1 \leq i \leq N$, are given by

$$p_i(t) = a_i(t) - \sum_{j=1}^N b_{ij}(t)f_{ij}(u_i(t), u_j(t)).$$

Since p_i , $1 \leq i \leq N$, is continuous on the domain of u , for t in the domain of u we obtain

$$u_i(t) = u_i(t_0) \exp \int_{t_0}^t p_i(s)ds > 0.$$

Hence $u(t) \in \mathbb{R}_+^N$. Next we rewrite system (GLV) in the form

$$u' = g(u, t),$$

where $u(t) = (u_1(t), \dots, u_N(t)) \in \mathbb{R}^N$, and $g(u, t) = (g_1(u, t), \dots, g_N(u, t))$ is given by

$$g_i(x, t) = x_i \left[a_i(t) - \sum_{j=1}^N b_{ij}(t)f_{ij}(x_i, x_j) \right], \quad 1 \leq i \leq N,$$

for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. Since the functions a_i , $1 \leq i \leq N$, and b_{ij} , $1 \leq i, j \leq N$, are continuous on $\mathbb{R}$ and the functions f_{ij} , $1 \leq i, j \leq N$, are continuously differentiable on $\mathbb{R}_+^2$, for every $\xi = (\xi_i) \in \mathbb{R}_+^N$ and $\tau \in \mathbb{R}$, there exists a unique solution $u(t)$ of (GLV) with $u(\tau) = \xi$. We denote it by $u(t, \tau, \xi) = (u_i(t, \tau, \xi))$. Recall that we have introduced the notation:

$$u(t, t_0, A) = \{u(t, t_0, \xi) \mid \xi \in A\}$$

for $A \subset \mathbb{R}_+^N$. Furthermore, since the functions $g_i(x, t)$, $1 \leq i \leq N$, are continuously differentiable with respect to the components of $x \in \mathbb{R}^N$, $u(t, \tau, \xi)$ are continuously differentiable with respect to the components of $\xi \in \mathbb{R}^N$. Therefore we can introduce the following notations. We denote by $D_\xi(u(t, \tau, \xi))$ the $N \times N$ matrix with (i, j) th entry equal to $\partial u_i(t, \tau, \xi) / \partial \xi_j$:

$$D_\xi u(t, \tau, \xi) = \left[\frac{\partial u_i(t, \tau, \xi)}{\partial \xi_j} \right],$$

where $\xi \in \mathbb{R}_+^N$. Similarly we define $N \times N$ matrix $D_x g(x, t)$ by

$$D_x g(x, t) = \left[\frac{\partial g_i(x, t)}{\partial x_j} \right],$$

where $x \in \mathbb{R}_+^N$.

Now for $t \geq t_0$ and $\xi_0 \in \mathbb{R}_+^N$, we set $u_0(t) = u(t, t_0, \xi_0)$. Then it is well known [6] that

$$X'(t) = A(t)X(t), \quad X(t_0) = I,$$

where

$$X(t) = D_\xi u(t, t_0, \xi_0), \quad A(t) = D_x g(u_0(t), t),$$

and I is the $N \times N$ identity matrix. Furthermore we know that

$$\det X(t) = \exp \int_{t_0}^t \text{tr} A(s)ds.$$

Therefore, we have

$$\det D_{\xi} u(t, t_0, \xi_0) = \exp \int_{t_0}^t \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds.$$

Hence it follows from the change of variables formula that

$$\begin{aligned} \mu(u(t, t_0, A)) &= \int_{u(t, t_0, A)} dx = \int_A \det D_{\xi} u(t, t_0, \xi_0) d\xi_0 = \int_A \exp \int_{t_0}^t \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds d\xi_0 \\ &\leq \int_A \exp \left[\sum_{i=1}^N \log \frac{u_i(t)}{u_i(t_0)} - \int_{t_0}^t \sum_{i=1}^N b_{ii}(s) u_i(s) \tilde{f}'_{ii}(u_i(s)) ds \right] d\xi_0. \end{aligned}$$

Therefore, by (A2) and (1.1), in order to prove Theorems 1.4 and 1.5, it is sufficient to prove the following claim:

Claim (see Taniguchi [1, Lemmas 3.1 and 4.1]). *If either conditions (A1)–(A3), (A4), and (A6) or conditions (A1)–(A3), (A5) and (A7) hold, there exists some numbers $M_A, \delta_A > 0$ and $t_A \geq t_0$ such that for $t \geq t_A, i = 1, \dots, N$, and $\xi_0 \in A$,*

$$\delta_A \leq u_i(t, t_0, \xi_0) \leq M_A. \quad (2.1)$$

In fact, by (1.1), (2.1), we have

$$\int_{t_0}^t \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds \leq \sum_{i=1}^N \log \frac{M_A}{\delta_A} - \delta_A \delta'_A \int_{t_0}^t \sum_{i=1}^N b_{ii}(s) ds = N \log \frac{M_A}{\delta_A} - \delta_A \delta'_A \int_{t_0}^t \sum_{i=1}^N b_{ii}(s) ds,$$

where $\delta'_A := \min\{\tilde{f}_{ii}(\delta_A) \mid 1 \leq i \leq N\}$. Therefore, by (A2), we have

$$\int_{t_0}^t \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}(u_0(s), s) ds \rightarrow -\infty \quad \text{as } t \rightarrow \infty$$

uniformly with respect to $\xi_0 \in A$; that is

$$\mu(u(t, t_0, A)) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This completes the proof.

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REFERENCES

- [1] K. Taniguchi, Asymptotic property of solutions of nonautonomous Lotka-Volterra model for N -competing species, submitted.
- [2] S. Ahmad, A. C. Lazer, On a property of nonautonomous Lotka-Volterra competition model, *Nonlinear Analysis* 37 (1999) 603-611.
- [3] S. Ahmad, A. C. Lazer, On the nonautonomous N -competing species problem, *Appl. Anal.* 57 (1995) 309-323.
- [4] R. Redheffer, Nonautonomous Lotka-Volterra system I, *J. Differential Equations* 127 (1996) 519-540.
- [5] J. Hofbauer, K. Sigmund, *Evolutionary Games and Population Dynamics*, Cambridge University Press, New York, 1998.
- [6] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, P. C. Heath and company, Boston, 1965.