

# Differential forms and stratifications<sup>(1)\*</sup>

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## 前書き:

§0. 9月のセミナーで話す予定であった題材は次の二つである。

- 1° Division properties of diff. forms に関して, [3]への補足。
- 2° 我が国[2], [3]で行った議論の基礎をなすある種の stratification についての説明。

9月のセミナーでは, 講演の爲の原稿の準備不足の爲 1° まで終了してしましたが, ここでは勿論 2° についても議論をかえる。

❶ 1° について: まず若干 abstract な formulation より始める。最初に次の如き datum を考えよう。

(i) 集合  $X_1, X_2$ , (ii) 写像,  $\theta: X_1 \rightarrow X_2$ , (iii) '対象'  $\Sigma$  .

以下に於いては, 写像  $\theta$  をしばしば, 作用素 と呼ぶ事もある。 (i), (ii), (iii) に於ける datum  $\{X_1, X_2, \theta, \Sigma\}$  に対して次の写

(\*) この題は, 9月のセミナーの講演の題とは異なる。尚 9月の 'complex manifold' の seminar での筆者の同 title の講演参照。

像  $\div_{\mathbb{Z}}$  が定義されているとする。

$$(\star) \quad \div_{\mathbb{Z}} : \mathbb{X}_1 \vee \mathbb{X}_2 \longrightarrow \mathbb{Z}^+ = \{0, 1, 2, \dots\}.$$

$\mathbb{X}_1 \vee \mathbb{X}_2$  の元  $x_i$  ( $i=1, 2$ ) に対して,  $\div_{\mathbb{Z}}(x_i) \in \mathbb{Z}^+$  は division index w.r.t.  $\mathbb{Z}$  と呼ぶ。さて,  $(\mathbb{X}_1, \mathbb{X}_2, \mathcal{D}, \mathbb{Z}, \div_{\mathbb{Z}})$  は上記の如きものであるとして, 次の如き定義をする。

定義 Datum  $(\mathbb{X}_1, \mathbb{X}_2, \mathcal{D}, \mathbb{Z}, \div_{\mathbb{Z}})$  の division property を持つとは, 次の如き条件  $(\star)$  が満たされる事である。

$(\star)$ .  $\exists \Delta : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$  such that  $\lim_{n \rightarrow \infty} \Delta(n) = \infty$  が存在して,

$$\mathcal{D}(\mathbb{X}_1^{\Delta(n)}) \supset \mathbb{X}_2^{\Delta(n)} \vee \mathcal{D}(\mathbb{X}_2).$$

$$\text{ここで, } \mathbb{X}_i^{m'} \ (i=1, 2) = \{x_i \in \mathbb{X}_i : \div_{\mathbb{Z}} x_i \geq m'\}.$$

division property を有する datum の例

例 1. (Aztec-Ree の定理).  $\mathcal{O}$  は noetherian ring,  $\mathcal{A}, \mathcal{B}$  を  $\mathcal{O}$  の ideal とし,  $\mathcal{B}$  の basis  $(f) = (f_1, \dots, f_e)$  を固定する。この時,  $\mathbb{X}_1 = \mathbb{X}_2 = \mathcal{B}$ ,  $\mathbb{Z} = \mathbb{Z}$  とし更に  $\mathcal{D} = \mathcal{D}_{(f)}$  を次の如く定める。

$$\mathcal{D}_{(f)} : \mathcal{O}^e \ni (x_1, \dots, x_e) \longrightarrow \mathcal{O} \ni \mathcal{D}(x_1, \dots, x_e) = \sum_{j=1}^e x_j f_j.$$

更に  $\div_{\mathcal{O}}$  を次式により定める。

$$\text{div}_{\mathcal{O}_\alpha}(x_1, \dots, x_r) = \min v_\alpha(x_i), x_i \in \mathcal{O}_\alpha, \text{div}_\alpha(b) = v_\alpha(b), b \in \mathcal{O}_\alpha.$$

ここで  $v_\alpha(x_i), v_\alpha(b)$  は,  $x_i, b$  の  $\mathcal{O}_\alpha$  に関する order: ④

$$x_i \in \mathcal{O}_\alpha^{v_\alpha(x_i)}, \quad x_i \notin \mathcal{O}_\alpha^{v_\alpha(x_i)+1}, \quad b \text{ についても同様.}$$

この時,  $\Delta(n) = n \cdot K_0$  ( $K_0$  は  $\mathcal{O}_\alpha$  のみによる  $\mathbb{Q}$  整数) と ~~なる~~  
 べる。(これは, 勿論, Artin-Rees の定理 (ring theory における) ~~の~~ 言い  
 換えに他ならない.)

例 2. (division properties of diff. forms [2]) 今  $\mathcal{O}_1$  を原点  $0$  を含  
 む  $\mathbb{C}^n$  の領域,  $\mathcal{O}_2$  を  $\mathbb{C}^n$  を含む充分小さい 一つの (適当な) 領域 と  
 し, ~~更~~ <sup>に (4) =</sup>  $(f_1, \dots, f_r)$  を  $\mathcal{O}_1$  で定義された正則函数とする。この<sup>時</sup>  
 ④  $\mathcal{D}(\mathcal{O}_2) = \mathcal{I}_i$  を  $\mathcal{O}_2$  上の hol. diff. forms の全体とする。更に  $\mathcal{F} = (f_1, \dots, f_r)$   
 とする。亦  $\mathcal{D} = d$  を外微分作用素 (holomorphic) とする。  
 更に亦,  $\text{div}_{\mathcal{F}} = \text{div}_{(f)}$  を次の如く定める。

$$\text{div}_{(f)} \varphi_i, \varphi_i \in \mathcal{I}_i = \max_v \{ v \in \mathbb{Z}^+ : \varphi_i \equiv 0 \cdot (f_1^v, \dots, f_r^v) \cdot \mathcal{D}(\mathcal{O}_2) \}.$$

この時  $\Delta(n)$  を一次式  $\Delta(n) = C_1 n + C_2, C_1, C_2 \in \mathbb{Q}$  と  
 取れる。(2.3.3)。

例 1 は, ideal の completion の理論の基礎をなし, 例 2 は  $\hat{\mathcal{O}}^{(f)}$   
 $= \mathcal{O}_\alpha$  の (4) による完備化の理論で役割を果たす。

例 3.  $\Sigma_i = \Omega_{\text{alg}}(\mathbb{C}^n)$  ( $i=1,2$ ) を  $\mathbb{C}^n$  上の regular diff. forms 全体.  
 $V \in \mathbb{C}^n$  内の algebraic variety (affine variety/c) とする. 更に  $\mathcal{D} = d$ .  
 は外微分作用素とし,  $\Sigma = V = \text{affine variety in } \mathbb{C}^n$  とする. この時  
 $\mathcal{I}_V$  は  $V$  の ideal を表わし,  $\text{div}_{\Sigma \in V} \varphi$  ( $\varphi \in \Omega_{\text{alg}}(\mathbb{C}^n)$ ) は例 2 と類  
 似に  $\varphi \equiv 0 \pmod{\mathcal{I}_V^{v(\varphi)} \cdot \Omega}$ ,  $\varphi \notin 0 \pmod{\mathcal{I}_V^{v(\varphi)+1} \cdot \Omega}$ , (但し  $v(\varphi) = \text{div}_{\Sigma \in V} \varphi$ )  
 と定める. この時

$(\Sigma_1, \Sigma_2, \mathcal{D}, \Sigma, \text{div}_{\Sigma})$  が division property を持つ  $\Rightarrow H^*(V; \mathbb{C}) \cong 0$  ( $q \geq 1$ ).

例 3 は大域的な一々の結果である. [3] に於いては, かなり漠然とした形で述べたが, 要するに筆者の質問は, 次の通りである.

1° Division property を有する datum  $(\Sigma_1, \Sigma_2, \mathcal{D}, \Sigma, \text{div}_{\Sigma})$  の例を見出せ. 特に  $\mathcal{D}$  が楕円型微分作用素, or Cauchy-Riemann operator 等の ~~代数~~ 代数 (or 解析) 幾何にしばしば現われたものについて, 論ぜよ.

2° 例 1, 例 2 は, 直接的に variety  $V$  の ~~代数~~ 代数, or 幾何学的意味と結び付く. (例 2 に因しては [2], [3]. 例 1 は周知である.) <sup>この時,</sup> variety (alg. or analytic) の幾何学的性質に結び付き得てかつ (iii) division property を持つ 微分 (or より ~~広範囲の~~) 作用素は, 外微分作用素  $d$  の他にあるか?

2° に於いて, 幾何学的性質 とは,  $H^*(V; \mathbb{C})$  or  $H^*(V, \mathbb{R})$ ,  $\mathbb{R} = V$  上の

連接層の意味にこの論説では了解しておく。(勿論より広範囲の対象について論じられはすうと良いと思う) 微分作用素の議論には不案内であるから、慚慚は避けるが、20の問題は必ずしも肯定的でない感を有するが、問題1は、一定の興味を有するのではないかと思われる。

(II) ---本論--- Diff. forms and stratifications : この部分は[4]と照会して頂きたい。[4]及びここで述べべき事は、かなりの部分が、[2], [3]への準備であり、亦いくらかの部分は、[2], [3]の ~~parts~~ <sup>parts</sup> の拡張である。大雑把に言って [4] は、代数的部分であり、~~ここ~~ <sup>ここ</sup> で述べる事柄は(あまり適当でないかも知れないが) 幾何学的と呼んでおく。一般の variety  $V$  の研究に於いて、stratification の方法は基本的であり、 $V$  の cohomology 群の研究に於いても、---その cohomology 群の表示方法をどの様なものにしても--- stratification の方法と cohomology 群の研究の方法を結び付けねばとするのは、自然と思われる。特に我々が、用いている analytic de Rham cohomology group と stratification の ideas の幾つかの関係を論じねばとするのが [4]、及びここでの主題である。我々の基本的な立場を明らかにする為に [5] を挿入する。( [5] は、内にも掲載する予定であるが、この論説と ~~を~~ を独立に読める様にする為であって、寛容をお願いする次第である。) [5] で述べた様に stratifications の idea の幾つかは、与問題の

局所化' 及び '局所的結果の大域化' という解釈を許す様に思われる。ここで述べる事柄は [5] で述べている (1) に相当する部分である。既に我々の de Rham cohomology への approach に好都合な stratification の type を定義する。そして亦 <sup>theory</sup> germ (analy. variety) に対して、望ましい形での stratification の存在を記述する\*。ここで述べる stratification は [3] の議論での基礎になる。我々がここで ~~行っている~~ 行っている、ある与えられた variety に対して normalized series of prestratified spaces という概念を定義する事である。(この §0 の後に掲載する。英又 type を参照)。

normalized series of prestratified spaces は N. S. S. と略される事があり亦

一般に記号  $(R, \mathcal{F})$  で表わす。(定義の詳細は、英又 type の §: (Germann, 大文字)

Normalized series of prestratified spaces を参照されたい)。N. S. S. に関して簡単な説明を付する。代数あるいは解析 variety に於いて、

与えられた variety  $V$  に対し、適当な ~~射~~ 写像  $\pi: V \rightarrow W = \pi(V)$

(\*) タイプ作成の都合上、証明を記載するのは、間に合わなかった。

問題の reduction を載せる。残りの部分は、本質的に困難を含まない (他の場所で詳細を議論する。)

を取り,  $W$  を  $V$  より簡単なものに帰着させて,  $V$  の性質を調べるのは, standard  
 である. 特に  $\dim V = \dim W$  である場合, 射影元  $\pi: V \rightarrow W$  を適当に  
 選ぶ,  $W$  及び  $\pi$  の小生質より  $V$  の小生質を考察する方法を ramified covering

map の方法 と呼ぶ事になる. ramified covering map の方法は, algebraic

variety, complex analytic variety の考察のみならず, real analytic variety  
 の考察においても ~~基本的な意味~~ 重要である. (cf. S. Zojasiewicz [1], ...)

そして ramified covering map の方法 においては, Weierstrass の予備定理  
 及び E. Noether の正規化定理 (normalization theorem) が基本的な意味  
 を持つ事も周知である. 亦上記 ramified covering map の方法は多論争の  
 技術的な意味での長所を有するが同時に variety に対し 幾何学的に直観的  
な approach を許す (直観的) という特徴を有すると言えると思う. 我々の N.S.S.  
 の導入の一つの理由は, 次の様に言い表わされると思う.

(I) Ramified covering の方法を多かれ少かれ組織化する事.

さて定義の詳細は後にゆずるとして, N.S.S.  $(R, F)$  を構成する成分の  
 German 大文字

内  $\cup, \cap, \dots$  を取り上げる. ここで,  $\cup, \dots$  は概略次の  
 English 大文字斜体

如し。

(\*\*)  $\mathcal{U} = \{U_1, \dots, U_n\}$ ,  $\mathcal{V} = \{V_1, \dots, V_n\}$ ,  $\mathcal{S}_0 = \{S_0^1, \dots, S_0^n\}$ ,  
 $\mathcal{F} = \{F^1, \dots, F^n\}$ ,  $\mathcal{F}' = \{F'^1, \dots, F'^n\}$  の形で,  $U^i$  は  $\mathbb{R}^i = (x_1, \dots, x_i)$   
 の domain,  $V^i \subset U^i$  は proper subvariety of  $U^i$ ,  $S_0^i$  は  $\text{pair}(U^i, V^i)$   
 の prestratification ( $i=1, \dots, n$ ), 亦、 $F^i, F'^i$  は  $S_0^i$  の元  $S_{\lambda_j}^i$  に付随した  
 函数族  $f(S_{\lambda_j}^i), f'(S_{\lambda_j}^i)$  の寄せ集め:  $\mathcal{F}^i = \{f(S_{\lambda_j}^i); S_{\lambda_j}^i \in S_0^i\}$ ,  $\mathcal{F}'^i = \{f'(S_{\lambda_j}^i), f(S_{\lambda_j}^i)\}$   
 全体 (詳細は、後に付す英訳参照.)

ここで上記の様な列  $\mathcal{U}, \mathcal{V}, \mathcal{S}_0, \mathcal{F}, \mathcal{F}', \dots$  を考える理由は勿論命  
 題の理由である。

(II) ~~Varities の次元に関する帰納的~~ <sup>Varities の次元に関する帰納的</sup> 考察を, 我々の問題に就いて採り  
 採る事。

さて [2], [3] で述べた様に real analytic variety の二つの定量的  
 性質: (i) 与えられた subvariety に対しての Poly. growth properties (ii) 与えられた subvariety に対  
 しての division properties: の考察が我々には重要である。従って, 我々の  
 N. S. S. の定義に於いて, 次の事柄を考慮する事が望ましい。

(Ⅳ) Data  $\mathcal{U}, \mathcal{V}, \mathcal{A}_0, \mathcal{F}, \mathcal{F}', \dots$  が与えられた variety の定量的考察に対して有効である事。

さて上記の data  $\mathcal{U}, \mathcal{V}, \mathcal{A}_0, \mathcal{F}, \mathcal{F}', \dots$  の内 variety の定量的考察

に特に交りて来るのは 函数の寄せ集め  $\mathcal{F} = \{f_i(S_{\lambda_i}^i)_{i=1, \dots, n}\}$  である。  $f(S_{\lambda_i}^i) \in \mathcal{F}$

は,  $(\bar{\alpha} - \dim S_{\lambda_i}^i)$  個の monic polynomials  $f_1(S_{\lambda_i}^i), \dots, f_{\bar{\alpha} - \dim S_{\lambda_i}^i}(S_{\lambda_i}^i)$  よりなる。

(これは勿論 E. Noether の正規化定理の類似を狙ったものであり、筆者

の terminology 'normalized series of prestr' はこの事に基づく)

そして、(Ⅱ)で述べた 帰納的考察を  $\mathcal{F}$  に対して有効に行なう為の条件

として導入したのが Higher discriminant condition (cf. 英文外訳)

である。我々が導入した N.S.S. という概念は [2], [3] での基礎的な

意味も有する。[2], [3] でなした考察の各々の step と N.S.S. の定義に

用いた data あるいは、条件との関連を分析するのは、左程容易でなく<sup>ト</sup>余り

有意味とも思われない。然しいつかには、ramified covering map の

方法 が有する 技術的長所 あるいは 上記の方法 の underlying idea である。

ある 視覚的直観的な approach が [2], [3] での各段階に影響を有するといえると思われる。

(注)<sub>1</sub> 我々の N.S.S. の導入に於いて, Lojasiewicz [1] から刺激を受けた。

Lojasiewicz [1] からの影響の他は, N.S.S. の導入は かなり '経験的事実に基づく' という表現を取っておく。

(事実, N.S.S. の導入の 'motivation' を手短かに, 説明しようと試みたのであるが, ~~それは~~ 'theoretical' あるいは 'a priori' の説明では, 余りうまく出来そうでもないように思われる)。

(注)<sub>2</sub> 我々は real analytic variety を対象としている。 ~~最も基本的な~~ 応用は

complex analytic case に対して行なうのであるが, [2], [3] で述べた事実は

次の様な意味で実解析的と思われる。

(★) Complex analytic variety  $V$  は 勿論自然な方法で real analytic variety の構造を有する。  $V$  は自然な real analytic variety の構造を与えていた。

ものを  $V_{re}$  と表わす。この時 [2], [3] で述べた事実は、  
 complex analytic variety  $V$  に対しては、 $V_{re}$  に付随された概念を  
 用いずに 定式化 出来る。他方 [2], [3] で述べた事実の 証明~~は~~  
 は複素 variety に対して行なうよりも、実 variety に対して行なう方  
 が容易であり同時に (恐らくは) 自然でもある。

実 variety を考察する種々の方法は、既に多くあると思う。注 2  
 を挿入した時、実 variety の意味を有される方に、若干の注意を  
~~挿入~~ いき起し、亦 意見を知らたいと思った為である。

§1. 以後の論説の説明： まず最初に、[5] との関連を  
 明らかにし亦同時に筆者の考えをはっきりさせる為に [4] を掲げ  
 する。( [4] は学士院紀要に投稿中で、本質的には announcement である  
 が [4] に述べられた事実を check する事は左程難しくない )。  
 次に 二つの §§ : § Normalized series of prestratification

及び § Normalized series attached to germs を掲載している

前<sup>者</sup>が §4, 後者が §6 となっているのは, 目下作成中の原稿の都合上  
で少々不体裁であるが, 実態を正確に示す次第である. 亦 §6 は完  
結したものでなく, 問題の reduction step 迄で止めているが, 残りは  
本質的に困難を含まぬ. 亦 §5 は, §6 の準備として local analytic  
variety の germ に対して elementary arguments をする. §5 に含まれる  
事実は多かれ少かれ周知と思われるので, 省略させて頂いた. この論議  
の大半  
を読者限りでは §5 は必要でない. 勿論 §5 と §6 の残りは既に  
存在するので, どこかにまとめて発表もしたいと思っている.

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~~14-11-79~~  
Nov/79

De Rham cohomologies and stratifications .  
Complex analytic de Rham cohomology III.

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The importance of the idea of stratifying varieties in the study of algebraic and analytic varieties is well known. The investigation of stratification of varieties would involve basically the following steps\*:

(1) To stratify varieties so that each stratum as well as the relations among the strata ,e.g., incidence relation ,..., are of simple (or typical) forms.

(2) To obtain results of desired nature for each stratum or each series of strata,etc. with respect to a fixed stratification for given varieties.

(3) To piece together results from the step (2) in order to obtain results of a desired sort for given varieties and subvarieties,....

The steps  $\{(1),(2)\}$  and (3) might reasonably be called , respectively, localization steps (for given global problems)

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(\*) See R.Thom [8] , H.Whitney [9]. The author learned the theories of stratifications in connection with his proposed approach to Complex analytic de Rham cohomology. (Cf. [4],[5].)

and globalization steps ( to be applied to local results).

In [5],[7] we investigated certain quantitative properties of real analytic varieties. Results of [5] are used in our study of the complex analytic de Rham cohomology. Our investigations in [7] are carried out using steps (1), (2) and (3). Exact sequences of Mayer- Vietoris type are used repeatedly in our globalization steps. The basis of our arguments used in the globalization steps is algebraic in nature.

The main purpose of the present note is to introduce the notion of cochain complex with incidence relations (C.C.I.) for a prestratified space. (See n.1. and n.2. below.) The arguments used in the study of C.C.I. are generalizations, as well as abstractions, of those in [7]. When C.C.I.'s are related to de Rham cohomologies of certain types, the arguments applicable to C.C.I.'s in general clarify relations between 'local' and 'global' data in the de Rham cohomologies in question. Actually the author's hope in introducing the notion of C.C.I. is to clarify relations between 'local' and 'global' data in de Rham cohomologies of various types.

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(\*) The terms 'local' and 'global' in this note should be understood in the sense explained at the beginning of this note.

The contents of this note are preliminary in nature. However, the arguments applicable to C.C.I. in general are indispensable in [5] and have certain theoretically pleasant aspects.

n.1. Prestratifications. Let  $X$  be a topological space. By a prestratification  $^*(\mathcal{S})$  of  $X$  we mean a collection  $(\mathcal{S}) = \{S_\lambda\}_{\lambda \in \Lambda}$  of subsets  $S_\lambda$ 's of  $X$  satisfying the following conditions.

(1.1)<sub>1</sub>  $X$  is the disjoint union of  $S_\lambda$ 's in  $(\mathcal{S})$ :  $X = \bigcup_{\lambda \in \Lambda} S_\lambda$ ,  $S_\lambda \in (\mathcal{S})$ .

(1.1)<sub>2</sub> For each stratum  $S_\lambda \in (\mathcal{S})$ , the dimension:  $\dim S_\lambda \in \mathbb{Z}^+$  is given.

(1.1)<sub>3</sub> Frontier condition: For each stratum  $S_\lambda \in (\mathcal{S})$ ,  $\text{fron}(S_\lambda) = \overline{S_\lambda} - S_\lambda$  is the disjoint union of lower dimensional strata of  $(\mathcal{S})$ .

To substantially simplify notations in later arguments we assume that

(1.1)<sub>4</sub>  $(\mathcal{S})$  is a finite set. \*\*

A pair  $(X, (\mathcal{S}))$  consisting of a topological space and its prestratification  $(\mathcal{S})$  will be called a prestratified space.

Let  $(X, (\mathcal{S}))$  be a prestratified space. For  $\mathcal{T} \subset (\mathcal{S})$ , let  $|\mathcal{T}|$  denote the

(\*) For definitions of stratifications and prestratifications, see J.Mather [2], R.Thom[8]. Our definition of prestratifications is, for technical reasons, not the same as in [2],[8].

(\*\*) For the case where  $(\mathcal{S})$  is locally finite, see [6]

support of  $(T) : |T| = \bigcup S_\lambda$ ,  $S_\lambda \in (T)$ . Moreover, for  $(T) \subset (S)$ ,  $\overline{(T)}|_{(T)}$  denotes the closure of  $(T)$  in  $(T)$ . We list certain notations used below. For  $(T) \subset (S)$ , define  $(T)_C, (T)_m$  by

$$(T)_C = \{(T) \subset (T) : \overline{(T)}|_{(T)} = (T) \text{ or closed} \}, \quad \overline{(T)}|_{(T)}$$

$$(T)_m = \{S_\lambda \in (T) : l(S_\lambda) \leq m\}.$$

Moreover, let  $(T)_0$  denote the collection of series of strata in  $(T)$ :

$$(T)_0 = \{S_{\lambda_1}, \dots, S_{\lambda_t} : S_{\lambda_1} < \dots < S_{\lambda_t} : S_{\lambda_j} \in (T) (j = 1, \dots, t)\}.$$

In the above  $S_{\lambda_1} < S_{\lambda_2}$  means that  $S_{\lambda_1} \subset \text{fron}(S_{\lambda_2})$ , .... For  $(T) \subset (S)$  and a series  $S_{\lambda_1}, \dots, S_{\lambda_t}$ ,  $S_{\lambda_j} \in (T)$ , let  $(T)_m(S_{\lambda_1}, \dots, S_{\lambda_t})$  denote the intersection  $(T)_m \cap \{S \in (T) : S_{\lambda_j} < S\}$ . For  $(T) \subset (S)$  define  $(T)_{OC}$  by

$$(T)_{OC} = \{(T)_m(S_{\lambda_1}, \dots, S_{\lambda_t}) : (T) \subset (T), S_{\lambda_j} \in (T), j = 1, \dots, t\}.$$

Then one easily derives the following fact.

(1.2) If  $(T) \in (S)_C$ , then  $(T)_m \in (S)_C$  and  $(T)_C, (T)_{OC} \subset (S)_C$ . Moreover, if  $(T)_1, (T)_2 \in (S)_C$  satisfy the relation  $(T)_1 \vee (T)_2$ , then  $(T)_1 \cup (T)_2 \in (S)_C$ .

Here  $(T)_1 \vee (T)_2$  if, for any  $S_{\lambda_i} \in (T)_1 (i = 1, 2)$ ,  $S_{\lambda_i} \not\subset S_{\lambda_j}, S_{\lambda_i} \not\supset S_{\lambda_j}$ .

n.2. Cochain complex with incidence relation (C.C.I):

Let  $(R)$  be a noetherian ring, and let  $(X, (S))$  be a prestratified space. Moreover, let  $(C((S)))$  be a collection

(\*) For  $S \in (S)$ , the length  $l(S)$  can be defined in an obvious manner (see [6]).

of assignments  $\{\mathcal{C}'(\mathbb{S}), \mathcal{E}(\mathbb{S}), \mathcal{E}'(\mathbb{S})\}$  of the following forms:  $\mathcal{C}'(\mathbb{S})$ :  $\mathbb{T} \in \mathbb{S}_C \rightarrow \mathcal{C}'(\mathbb{T}), \mathcal{E}(\mathbb{S}): \mathbb{U} \in \mathbb{S}_0 \rightarrow \mathcal{E}(\mathbb{U})$  and  $\mathcal{E}'(\mathbb{S}): \mathbb{U} \in \mathbb{S}_{0C} \rightarrow \mathcal{E}'(\mathbb{U})$ . Here  $\mathcal{C}'(\mathbb{T}), \mathcal{E}(\mathbb{U})$  and  $\mathcal{E}'(\mathbb{U})$  are  $\mathbb{R}$ -cochain complexes. The collection  $\mathcal{C}(\mathbb{S}) = \{\mathcal{C}'(\mathbb{S}), \mathcal{E}(\mathbb{S}), \mathcal{E}'(\mathbb{S})\}$  as above is called a cochain complex with incidence relations attached to  $(X, \mathbb{S})$  (C.C.I. attached to  $(X, \mathbb{S})$ ) if  $\mathcal{C}(\mathbb{S})$  is equipped with isomorphisms  $1(\mathbb{S}), 1(\mathbb{T}_1, \mathbb{T}_2), 1(\mathbb{U}_1)$  and homomorphisms  $h_k(\mathbb{T}_m), h_k(\mathbb{U}_m)$  of the following forms.

$$(1.3)_1 \quad 1(\mathbb{S}): 0 \rightarrow \mathcal{C}'(\mathbb{S}) \rightarrow \mathcal{E}(\mathbb{S}) \rightarrow 0, \mathbb{S} \in \mathbb{S}.$$

$$(1.3)_2 \quad 1(\mathbb{T}_1, \mathbb{T}_2): 0 \rightarrow \mathcal{C}'(\mathbb{T}_1 \cup \mathbb{T}_2) \rightarrow \mathcal{C}'(\mathbb{T}_1) \oplus \mathcal{C}'(\mathbb{T}_2) \rightarrow 0, \mathbb{T}_1, \mathbb{T}_2 \in \mathbb{S}_C$$

and  $\mathbb{T}_1 \vee \mathbb{T}_2$ .

$$(1.3)_2' \quad 1(\mathbb{U}_1): 0 \rightarrow \mathcal{E}'(\mathbb{U}_1) \rightarrow \bigoplus_{\lambda_{t+1}} \mathcal{E}'(\mathbb{S}_{\lambda_1}, \dots, \lambda_{t+1}) \rightarrow 0, \text{ where } (\mathbb{U}_1 = \mathbb{T}_1(\mathbb{S}_{\lambda_1}, \dots, \lambda_t) \text{ with } \mathbb{T}_1 \in \mathbb{S}_C \text{ and } \mathbb{S}_{\lambda_j} \in \mathbb{T}_1, \text{ Moreover, } \mathbb{S}_{\lambda_{t+1}} \in \mathbb{U}_1.$$

$$(1.3)_3 \quad 0 \rightarrow \mathcal{C}'(\mathbb{T}_{m+1}) \xrightarrow{h_1(\mathbb{T}_m)} \mathcal{C}'(\mathbb{T}_m) \oplus \bigoplus_{\lambda_{m+1}} \mathcal{C}'(\mathbb{S}_{\lambda_{m+1}}) \xrightarrow{h_2(\mathbb{T}_m)} \bigoplus_{\lambda} \mathcal{E}'(\mathbb{T}_m(\mathbb{S}_{\lambda_{m+1}})) \rightarrow 0, \text{ where } \mathbb{T} \in \mathbb{S}_C \text{ and } \mathbb{S}_{\lambda_{m+1}} \in \mathbb{T}_{m+1} - \mathbb{T}_m.$$

$$(1.3)_3' \quad 0 \rightarrow \mathcal{C}'(\mathbb{U}_{m+1}) \xrightarrow{h_1(\mathbb{U}_m)} \mathcal{E}'(\mathbb{U}_m) \oplus \bigoplus_{\lambda_{t+1}} \mathcal{E}'(\mathbb{S}_{\lambda_1}, \dots, \lambda_{m+1}) \xrightarrow{h_2(\mathbb{U}_m)} \bigoplus_{\lambda_{t+1}} \mathcal{E}'(\mathbb{U}_m(\mathbb{S}_{\lambda_{m+1}})) \rightarrow 0, \text{ where } (\mathbb{U}_m = \mathbb{T}_m(\mathbb{S}_{\lambda_1}, \dots, \lambda_t) \text{ and } \mathbb{U}_{m+1} = \mathbb{T}_{m+1}(\mathbb{S}_{\lambda_1}, \dots, \lambda_t). \text{ Moreover, } \mathbb{S}_{\lambda_{m+1}} \in \mathbb{U}_{m+1} - \mathbb{U}_m.$$

Postulated conditions of the existence of isomorphisms in  $(1.3)_1, \{(1.3)_2, (1.3)_2'\}$  and homomorphisms in  $(1.3)_3, (1.3)_3'$  will be called 'Identification condition', 'Disjoint condition'

(\*) Isomorphisms and homomorphisms are those of  $\mathbb{R}$ -cochain complexes.

and 'Incidence condition' ('Mayer-Vietoris condition') respectively. The collection of isomorphisms  $i(S), \dots$  and homomorphisms  $h_k$ 's will be denoted by  $(K(C(S)))$ . When we emphasize the role of  $(K(C(S)))$ , we say that  $C(S)$  is  $(K(C(S))$ -C.C.I. . Let  $C(S)$  and  $(K(C(S)))$  be as above. Then isomorphisms  $i(S), \dots$  and homomorphisms  $h_k$ 's of cochain complexes induce corresponding isomorphisms  $i^*(S), \dots$  and homomorphisms  $h_k^*$ 's of cohomology groups naturally. The collection of  $i^*(S), \dots$  and  $h_k^*$ 's will be denoted by  $(K^*(C(S)))$ .

Equivalences between C.C.I.'s. Let  $(X, S)$  be a prestratified space, and let  $C^1(S)$  be  $(K(C^1(S)))$ -C.C.I. ( $i = 1, 2$ ). Moreover, let  $\alpha^*, \beta^*$  and  $\beta'^*$  be families of  $\mathbb{R}$ -homomorphisms of the following forms:  $\alpha^* = \{\alpha^*(T) : H^*(C'^1(T)) \rightarrow H^*(C'^2(T)), T \in S_C\}$ ,  $\beta^* = \{\beta^*(U) : H^*(E^1(U)) \rightarrow H^*(E^2(U)), U \in S_0\}$  and  $\beta'^* = \{\beta'^*(U) : H^*(E'^1(U)) \rightarrow H^*(E'^2(U)), U \in S_{0C}\}$ . Then we can define the notion of commutativity of  $\{\alpha^*, \beta^*, \beta'^*\}$  with  $(K(C^1(S)))$  in an obvious manner ([6]). We say that  $C^1(S)$  ( $i = 1, 2$ ) are  $\{\alpha^*, \beta^*, \beta'^*\}$ -equivalent if (i)  $\{\alpha^*, \beta^*, \beta'^*\}$  commute with  $(K(C^1(S)))$  ( $i = 1, 2$ ) and if (ii) the homomorphism  $\beta^*(U)$  is an isomorphism for any  $U \in S_0$ .

Now, in our investigations, there are reasons for regarding  $C'(T), T \in S_C$  as 'global' data and  $E(U), U \in S_0$  as 'local' data. The following lemma shows that certain propert-

-les of global data are derived from those of local data.

Lemma 1, Let  $\mathcal{C}^1(\mathcal{S}) = \{C^1(S), E^1(S), E'^1(S)\}$  be  $\mathcal{K}(\mathcal{C}^1(\mathcal{S}))$  -C.C.I. ( $i = 0, 1, 2$ ) of a prestratified space  $(X, \mathcal{S})$ .

(I) If  $H^*(E^0(\bar{U}))$  is a finitely generated  $\mathbb{R}$ -module for each  $(\bar{U}) \in \mathcal{S}_0$ , then  $H^*(C^0(\bar{U}))$  is so for each  $(\bar{T}) \in \mathcal{S}_0$ .

(II) Let  $\mathcal{C}^1(\mathcal{S})$  ( $i = 1, 2$ ) be  $\{\alpha^*, \beta^*, \beta'^*\}$ -equivalent. Then  $\alpha^*(\bar{T})$  is an  $\mathbb{R}$ -isomorphism for each  $(\bar{T}) \in \mathcal{S}_0$ . Here  $\alpha^*, \beta^*, \beta'^*$  are families of  $\mathbb{R}$ -homomorphisms of the forms given in the beginning of n.2.

For the proof of Lemma 1, see [6].

n.3. An exact sequence of Mayer-Vietoris type. Let  $K$  be an algebraically closed field of any characteristic. In n.3. every variety in question is assumed to be a reduced  $K$ -variety. Let  $A^n$ ,  $V$  and  $D$  be an affine space of dimension  $n$ , a variety in  $A$  and a divisor in  $A$ , respectively, such that for each irreducible component  $V_j$  of  $V$ ,  $V_j \not\subset D$  and  $V_j \cap D \neq \emptyset$ . We denote by  $W$  the variety in  $A$  characterized by  $|W| = |V| \cap |D|$ . Now let  $(\mathcal{O})_{I_V, I_W}$  and  $(\mathcal{O})_{I_D} = (\mathcal{O}(h))$  denote respectively the ring  $K[x_1, \dots, x_n]$  and the ideals of  $V, W$ , and  $D$ . The completions  $\varprojlim_n (\mathcal{O})_{I_V}^n$  and  $\varprojlim_n (\mathcal{O})_{I_W}^n$  are denoted by  $\hat{\mathcal{O}}^V$  and  $\hat{\mathcal{O}}^W$  respectively. We denote the localizations  $(\mathcal{O}[h^{-1}])$  and  $\hat{\mathcal{O}}^W[h^{-1}]$  by  $\mathcal{O}[*D]$  and  $\hat{\mathcal{O}}^W[*D]$  respectively. Moreover, let  $\hat{\mathcal{O}}^{V-W}$  and  $\hat{\mathcal{O}}^{W, V-W}$  be respectively the completions defined by  $\varprojlim_n \frac{\mathcal{O}[*D]}{(\mathcal{O}[*D], I_V^n)}$  and  $\varprojlim_n \frac{\hat{\mathcal{O}}^W[*D]}{(\hat{\mathcal{O}}^W[*D], I_V^n)}$ . In the above we regard  $(\mathcal{O})_{I_V}$  as contained in  $\mathcal{O}[*D]$  and  $\hat{\mathcal{O}}^W[*D]$  in a natural fashion. For the graded

ring  $\hat{\Omega}_A$  of K-differential forms over A, let  $\hat{\Omega}_A^V, \hat{\Omega}_A^W, \hat{\Omega}_A^{W,V-W}$  respectively denote  $\hat{\Omega}_A \otimes \hat{\Omega}_A^V, \dots, \hat{\Omega}_A \otimes \hat{\Omega}_A^{W,V-W}$ . Then we have :

27 Lemma 2. For the rings  $\hat{\Omega}_A^V, \dots, \hat{\Omega}_A^{W,V-W}$ , the exact sequence

$$(I) \quad 0 \rightarrow \hat{\Omega}_A^V \xrightarrow{\hat{\rho}_W^V \oplus \hat{\rho}_{V-W}^V} \hat{\Omega}_A^W \oplus \hat{\Omega}_A^{V-W} \xrightarrow{\hat{\rho}_{W,V-W}^W \oplus \hat{\rho}_{W,V-W}^{V-W}} \hat{\Omega}_A^{W,V-W} \rightarrow 0,$$

holds. In (I) the continuous homomorphisms  $\hat{\rho}_W^V, \dots, \hat{\rho}_{W,V-W}^{V-W}$

are determined naturally from topologies of  $\hat{\Omega}_A^V, \dots, \hat{\Omega}_A^{W,V-W}$ .

For the proof of Lemma 2, see [6]. The exact sequence (I) relates cohomology groups  $H^*(\hat{\Omega}_A^V), H^*(\hat{\Omega}_A^{V-W})$  and  $H^*(\hat{\Omega}_A^{W,V-W})$  to the cohomology group  $H^*(\hat{\Omega}_A^V)$ . For a pair (V,W) of smooth varieties (defined over the complex number field  $\mathbb{C}$ ), the idea of relating the cohomology groups of  $W, V-W$  and  $N(W)^*-W$  to that of V may be regarded as one of the basic ideas in the classical (analytic) theories of residues. (Cf. J.Leray[2], P.A.Griffiths[1],...). The sequence (I) might be regarded as a generalization in an algebraic direction of the idea explained above. Moreover, Lemma 2 enables us to attach the algebraic de Rham C.C.I. to the pre-stratified space  $(V, S=(W, V-W))$  in a natural manner. 立休

Remarks about results untouched here. In this note, we have spent several pages explaining ideas used in defining C.C.I.'s. For arguments on C.C.I.'s untouched here, see [6]. In particular, [6] contains examples of C.C.I.'s such

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(\*)  $N(W)$  is a suitable neighbourhood of W in V.

as the singular , the  $C^\infty$  - de Rham , and the P.G.(polynomial growth) de Rham|C.C.I.'s, as well as an application of the sequence (I).

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#### §4. Normalized series of prestratified spaces.

##### 4.1. Normalized series of prestratified spaces. Let

$R^n(x)$  be an  $n$ -dimensional euclidean space with coordinates  $(x) = (x_1, \dots, x_n)$ . By a series  $(R)$  of prestratified spaces in  $R^n(x)$ , we mean a collection  $(R) = \{(R), (U, V, S), (U', V', S')\}$  of a series  $R$  of euclidean spaces  $(U(U'))$  of bounded connected domains,  $(V(V'))$  of closed analytic varieties and  $(S(S'))$  of prestratifications, where  $(R), \dots, (S')$  are of the following forms.

(4.1)<sub>1</sub>  $(R) = \{R^1(y_1), R^2(y_2), \dots, R^n(y_1, \dots, y_n)\}$ , where  $(y_1, \dots, y_n)$  is a system of coordinates of  $R^n(x)$ .

(4.1)<sub>2</sub>  $(U) = \{U^j\}_{j=1, \dots, n}$   $(U') = \{U'^j\}_{j=1, \dots, n}$

where  $U^j$  ( $U'^j$ ) is a connected bounded domain in  $R^j(y_1, \dots, y_j)$  ( $j = 1, \dots, n$ ).

(4.1)<sub>3</sub>  $(V) = \{V^j\}_{j=1, \dots, n}$   $(V') = \{V'^j\}_{j=1, \dots, n}$ , where  $(V^j(V'^j))$  is a closed analytic variety in  $U^j(U'^j)$  such that  $\dim V^j (\dim V'^j) \leq j - 1$  ( $j = 1, \dots, n$ ).

(4.1)<sub>4</sub>  $(S_0) = \{S_0^j\}_{j=1, \dots, n}$   $(S'_0) = \{S'^j_0\}_{j=1, \dots, n}$ , where  $(S_0^j(S'^j_0))$  is a prestratification of the pair  $(U^j, V^j)$  ( $U'^j, V'^j$ ) ( $j = 1, \dots, n$ ).

~~For a series  $R = R, (U, V, S_0), (U', V', S'^0_0)$  of prestrati~~

Let  $(R) = \{R, (U, V, S), (U', V', S')\}$  be a series of prestratified spaces in  $R^n(x)$ . We then call the series  $(S_0^j)_{j=1, \dots, n}$  a series of prestratifications of the pair  $(U) = \{U^j\}_{j=1, \dots, n}, (V) = \{V^j\}_{j=1, \dots, n}$  ( $(U') = \{U'^j\}_{j=1, \dots, n}, (V') = \{V'^j\}_{j=1, \dots, n}$ ). Moreover, let  $(S^j) (S'^j)$  denote the induced prestratification of  $V^j (V'^j)$  from  $(S_0^j) (S_0'^j)$ ,  $j = 1, \dots, n$ . We mean by the induced series  $(S) (S')$  of prestratifications of  $(V) (V')$  the series  $(S^j)_{j=1, \dots, n} ((S')^j)_{j=1, \dots, n}$  of prestratifications of  $V^j$ 's ( $V'^j$ 's).

Let  $(R) = \{R, (U, V, S), (U', V', S')\}$  be a series of prestratified spaces in  $R^n(x)$ . For the series  $(R) = \{R^j(y_1, \dots, y_j)\}_{j=1, \dots, n}$ ,  $(S_0^j)_{j=1, \dots, n}$ , we call  $R^j(y_1, \dots, y_j), \dots, (S_0^j)$  the  $j$ -th component of  $(R), \dots, (S_0^j)$  respectively. Also for the induced series  $(S) = \{S^j\}_{j=1, \dots, n} ((S') = \{(S')^j\}_{j=1, \dots, n})$  of  $(V) (V')$  from  $(S_0^j) (S_0'^j)$ , we call  $(S^j) ((S')^j)$  the  $j$ -th component of  $(S) (S')$ .

Let  $(R) = \{R, (U, V, S), (U', V', S')\}$  be a series of prestratified spaces in  $R^n(x)$ . We say that  $(R)$  is admissible if the following conditions  $(4.2)_{1 \sim 3}$  and  $(4.3)_{1 \sim 3}$  are valid.

$(4.2)_1$  Each stratum  $S_\lambda^j \in (S_0^j - S^j) ((S')^j \in (S_0'^j - (S')^j))$ ,  $j = 1, \dots, n$ , is a connected component of  $U^j - V^j (U'^j - V'^j)$ . Here  $(S_0^j) (S_0'^j)$  is the  $j$ -th component of  $(S) (S')$ , and  $(S^j) ((S')^j)$

is the induced prestratification of  $V^j(V'^j)$  from  $(S_0^j, (S_0^j)^j)$ .

(4.2)<sub>2</sub> The  $j$ -th component  $(U^j, (U^j)^j)$  of  $(U, (U^j))$ ,  $j = 1, \dots, n$ , satisfies the (d) - regularization condition. Moreover, the  $j$ -th components  $(U'^j, V'^j, (S_0^j)^j)$  of the triple  $(U, V, (S_0^j))$  is a (d)-envelop of the  $j$ -th components  $(U^j, V^j, S_0^j)$  of  $(U, V, S_0)$ .

(4.2)<sub>3</sub> The  $j$ -th component  $(S_0^j)$  of  $(S_0)$  satisfies the (g)-separation condition,  $j = 1, \dots, n$ .

(4.3)<sub>1</sub> Between components  $U'^j, (U^j)$  and  $U'^{j-1}(U^{j-1})$ ,  $j = 2, \dots, n$ , commutative diagrams of the following forms are valid.

$$\begin{array}{ccc} U'^j & \xrightarrow{\psi'^j} & U'^{j-1} \times I^0 \\ \downarrow \pi_{j-1,j} & & \downarrow p_1^j \\ U'^{j-1} & \xrightarrow{id^j} & U'^{j-1} \end{array}, \quad \begin{array}{ccc} U^j & \xrightarrow{\psi^j} & U^{j-1} \times I^0 \\ \downarrow \pi_{j-1,j} & & \downarrow p_1^j \\ U^{j-1} & \xrightarrow{id^j} & U^{j-1} \end{array}.$$

Here  $\psi'^j$  is a  $C^\infty$ -diffeomorphism and  $\psi^j$  is the restriction \* of  $\psi^j$  to  $U^j$ . Moreover,  $p_1^j(p_2^j)$  denotes the projection from the product  $U'^{j-1} \times I^0$  ( $U^{j-1} \times I^0$ ) to the first factor  $U'^{j-1}(U^{j-1})$ , and  $id^j$  ( $id^j$ ) denotes the identity map on  $U'^j$  ( $U^j$ ).

(4.3)<sub>2</sub> The restriction  $\pi_{j-1,j}(y)|_{V'^j}$  of  $\pi_{j-1,j}(y)$  to  $V'^j$  is compatible with prestratifications  $S^j$  of  $V^j$  and  $S_0^j$  of  $U^j$ ,  $j = 2, \dots, n$ .

(4.3)<sub>3</sub> For each atratum  $S^j$  of the induced prestratification  $S_\lambda^j$  of  $V^j$ , the map  $\pi_{j-1,j}(y) : S_\lambda^j \rightarrow \pi_{j-1,j}(y)(S_\lambda^j)$  is real analytically biholomorphic,  $j = 2, \dots, n$ .

Remark 4.1. Let  $(R) = \{(R), (U, V, S), (U', V', S')\}$  be an admissible series of prestratifications in  $R^n(x)$ . Then the conditions  $(4.2)_{1 \sim 3}$  and  $(4.3)_{1 \sim 3}$  imply the following.

$(4.4)_1$  The restriction  $\pi_{j-1,j}|_{V^j}$  of  $\pi_{j-1,j}(y)$  to  $V^j$  is compatible with prestratifications  $(S^j)$  of  $V^j$  and  $(S_0^j)$  of  $U^{j-1}$ ,  $j = 2, \dots, n$ . Here  $V^j, S^j, S_0^j, U^{j-1}$  are  $j$ -th components of  $(V)$ , the induced series  $(S)$  of prestratifications of  $(V)$  from  $(S_0)$ , and the  $(j-1)$ -th component of  $(U)$  respectively.

$(4.4)_2$  For each stratum  $S_\lambda^j \in (S^j)$  the projection  $\pi_{j-1,j}(y) : S_\lambda^j \rightarrow \pi_{j-1,j}(y)(S_\lambda^j)$  is (real analytically) biholomorphic,  $j = 2, \dots, n$ .

$(4.4)_3$  If  $(S^1)$  consists of strata which are connected, then any stratum of  $(S_0^j)$ ,  $j = 2, \dots, n$ , is a connected analytic manifold.

Let  $(R) = \{(R), (U, V, S), (U', V', S')\}$  be an admissible series of prestratifications in  $R^n(x)$ . Moreover, let  $(S) (S')$  denote the induced series of prestratifications of  $(V) (V')$  from  $(S_0) (S'_0)$ . Let  $S_\lambda^{j,j} \in (S^j)$ ,  $j = 2, \dots, n$ . By a representation datum of  $S_\lambda^{j,j}$ , we mean a collection  $\mathcal{P}(S_\lambda^{j,j}) = \{f(S_\lambda^{j,j}), g(S_\lambda^{j,j})\}$  of sets  $f(S_\lambda^{j,j}), g(S_\lambda^{j,j})$  of analytic functions, where  $f(S_\lambda^{j,j})$  and  $g(S_\lambda^{j,j})$  are of the following forms.

$$(4.5)_1 \quad f(S_\lambda^{j,j}) = f(S_\lambda^{j,j} : x_1, \dots, x_n, y_1, \dots, y_{j-1})$$

$$(4.5)_1 \quad \mathcal{F}(S_\lambda'^j) = \{f_t(S_\lambda'^j: y_1, \dots, y_{n_\lambda'^j}, y_{n_\lambda'^j+t})\}_{t=1, \dots, n_\lambda'^j}$$

where  $f_t(S_\lambda'^j)$ ,  $t=1, \dots, n_\lambda'^j$  is a monic polynomial in  $y_{n_\lambda'^j+t}$  such that (i) coefficients of  $f_t(S_\lambda'^j)$  are analytic functions in  $U_{n_\lambda'^j} \subset R(y_1, \dots, y_{n_\lambda'^j})$  and (ii)  $f_t(S_\lambda'^j)$  vanishes on  $S_\lambda'^j$ . In the above  $R(y_1, \dots, y_{n_\lambda'^j})$  is the  $n_\lambda'^j$ -th component of  $(R)$  and  $U_{n_\lambda'^j}$  is the  $n_\lambda'^j$ -th component of  $(U)$ .

$$(4.5)_2 \quad \mathcal{F}(S_\lambda'^j) = \{f_t(S_\lambda'^j: y_1, \dots, y_j)\}_{t=1, \dots, k_\lambda'^j},$$

where  $k_\lambda'^j \geq j - n_\lambda'^j$  and  $f_t(S_\lambda'^j)$  vanishes on  $S_\lambda'^j$ .

For a representation datum  $\mathcal{F}(S_\lambda'^j) = \{\mathcal{F}(S_\lambda'^j), \mathcal{G}(S_\lambda'^j)\}$ ,  $S_\lambda'^j \in \mathcal{S}^j$ , let  $\Delta(\mathcal{F}(S_\lambda'^j))$  denote the closed analytic variety in  $U^j$  defined to be the zero locus of the following functions.

$$(4.5)_1 \quad \mathcal{F}(S_\lambda'^j), \quad J\left(\frac{f^I(S_\lambda'^j)}{y_{n_\lambda'^j+1}, \dots, y_j}\right), \quad \text{where } I = (i_1, \dots, i_{n_\lambda'^j})$$

and  $f^I(S_\lambda'^j)$  is the abbreviation of  $\{f_{i_1}(S_\lambda'^j), \dots, f_{i_{n_\lambda'^j}}(S_\lambda'^j)\}$

In the above we write  $j - n_\lambda'^j$  as  $n_\lambda'^j$ .

Moreover, for  $m \in \mathbb{Z}^{+n^j}$ , let  $D_m(\mathcal{F}(S_\lambda'^j))$  denote the locally closed analytic variety in  $U^j$  defined as follows.

$$(4.5)_2 \quad D_m(\mathcal{F}(S_\lambda'^j)) = \left\{ Q^j \in U^j : \left( \frac{\partial^{k_s} f_s(S^j)}{\partial y_s^{k_s}} \right) (Q^j) = 0, \quad 0 \leq k_s \leq m_s - 1, \quad \left( \frac{\partial^{m_s} f_s(S^j)}{\partial y_s^{m_s}} \right) (Q^j) \neq 0, \quad s = n_\lambda'^j + 1, \dots, j \right\}.$$

The

A collection  $\mathcal{F}^j = \{\mathcal{F}^j, \mathcal{G}^j\}$  of  $\mathcal{F}^j = \{\mathcal{F}(S_\lambda'^j), S_\lambda'^j \in \mathcal{S}^j\}$

~~is a collection~~

and  $(F')^j = \{f'(S'_\lambda{}^j), S'_\lambda{}^j \in (S')^j\}$ , where  $(f'(S'_\lambda{}^j), f'(S'_\lambda{}^j))$  is a representation datum of  $(S')^j$ , will be called a representation datum of  $(S')^j$ . Moreover, by a representation datum of  $(S')^j$  we mean a collection of series  $(F) = \{F, F'\}$ , where  $(F) = \{F^j\}_{j=2, \dots, n}$ .

and  $(F') = \{F'^j\}_{j=2, \dots, n}$  consist of representation data  $(F^j, F'^j)$  ( $j = 2, \dots, n$ ). For a representation datum  $(F) = (F, F')$  as above  $(F)^j = (F^j, F'^j)$  will be called the j-th component of  $(F)$ .

Let  $R^n(x)$  be an n-dimensional euclidean space with coordinates  $(x)$ . By a normalized series of prestratified spaces in  $R^n(x)$ , we mean a pair  $(R, F)$  of an admissible series  $R$  of prestratified spaces in  $R^n(x)$  and a representation datum  $(F)$  of  $(R)$  satisfying the following conditions (4.6)

(4.6)<sub>1</sub> For each stratum  $S'_\lambda{}^j \in (S')^j$ ,  $j = 2, \dots, n$ , the zero locus  $V(f'(S'_\lambda{}^j))$  of  $f'(S'_\lambda{}^j)$  is expressed as the union of strata in  $(S')^j$  of dimension at most  $n'_\lambda{}^j$ . Moreover, for any stratum  $S'_\lambda{}^j \subset V(f'(S'_\lambda{}^j))$  of dimension  $n'_\lambda{}^j$ ,  $f'(S'_\lambda{}^j) = f'(S'_\lambda{}^j)$ .

(4.6)<sub>2</sub> Ramification conditions. For each stratum  $S'_\lambda{}^j \in (S')^j$ ,  $j = 2, \dots, n$ ,

$$S'_\lambda{}^j \cap (f'(S'_\lambda{}^j)) = \phi.$$

(4.6)<sub>3</sub> Higher discriminant conditions For any pair  $(S'_\mu{}^j, S'_\lambda{}^j)$  of strata in  $(S')^j$  satisfying  $S'_\mu{}^j < S'_\lambda{}^j$ ,

Now our position is to define a notion of n.s. of pr.

4 - 6

Such that the following conditions (4.6)<sub>1,2,3</sub> are satisfied.

See (1.1)

there exists a (uniquely determined) element  $m \in \mathbb{Z}^{+n}{}^j$ ,  $n = j - d$  in such a manner that

$$s_{\lambda}^{'j} \in D_m(\mathbb{F}(s_{\lambda}^{'j}))$$

holds.

Of course,  $(s_{\lambda}^{'j})$ ,  $(\mathbb{F}(s_{\lambda}^{'j}))$ ,  $(\mathbb{F}'(s_{\lambda}^{'j}))$  in (4.6) are the  $j$ -th component of  $(S)$  and the representation datum of the stratum  $s_{\lambda}^{'j} \in (S)^{'j}$  respectively.

Let  $R^n(x)$  be a euclidean space of dimension  $n$ , and let  $(R, \mathbb{F})$  be a normalized series of prestratifications in  $R^n(x)$ . We write the admissible series  $(R)$  as  $(R) = \{(R, (U, V, S), (U', V', S'))\}$ . Moreover, let  $(S) (S')$  denote the induced series of prestratifications of  $(V) (V')$  from  $(S_0) (S'_0)$ . Let  $s_{\lambda}^{'n} (s_{\lambda}^{'n})$  be a stratum of the  $n$ -th component  $(S)^{'n} (S')^{'n})$  of  $(S) (S')$ . We say that the normalized series  $(R, \mathbb{F})$  satisfies the differentiability condition for  $(S)^{'n} (S')^{'n})$  if the following condition is valid.

(4.7) For any  $j > \dim(s_{\lambda}^{'n})$  ( $j > \dim(s_{\lambda}^{'n})$ ), the length  $l(s_{\lambda}^{'j})$  ( $l(s_{\lambda}^{'j})$ ) of  $s_{\lambda}^{'j} = \pi_{j,n}(s_{\lambda}^{'n})$  ( $s_{\lambda}^{'j} = \pi_{j,n}(s_{\lambda}^{'n})$ ) in  $(S)^{'j}$  ( $(S')^{'j})$  is equal to one. Here  $(S)^{'j}$  and  $(S')^{'j}$  are  $j$ -th components of  $(S)$  and the  $j$ -th component of  $(S')$  respectively.

Note that if  $(R, \mathbb{F})$  satisfies the differentiability condition for  $s_{\lambda}^{'n} \in (S)^{'n}$  ( $s_{\lambda}^{'n} \in (S')^{'n}$ ), then  $l(s_{\lambda}^{'n})$  in  $(S)^{'n}$  ( $l(s_{\lambda}^{'n})$  in  $(S')^{'n}$ ) is equal to one. We are using the terminology of

'differentiability condition' for the condition (4.7), because the condition (4.7) assures the differentiability of certain maps generally' on  $(S_\lambda^n, S_\lambda'^n)$ . (Cf [C]... Also see the later arguments of this paper.)  $\uparrow$  C.A.D.R.A.

1. Let  $(U, V)$  be a pair of a bounded domain  $U$  in a euclidean space  $R^n(x)$  and a closed analytic variety in  $U$ . Moreover let  $(W) = \{W_t\}$  be a finite set of closed analytic subvarieties  $W_t$ 's of  $V$ . We mean by a normalized series  $(R, F)$  of prestratified spaces attached to  $(U, V, W)$  a normalized series  $(R, F) = (R, (U, V, S))$ , of prestratified spaces in  $R^n(x)$  satisfying the following conditions.

(4.8)<sub>1</sub>  $(U, V) = (U^n, V^n)$ , where  $U^n$  and  $V^n$  are  $n$ -th components of  $U$  and  $V$  respectively.

((4.8)<sub>2</sub> Each subvariety  $W_t$  is the union of strata of  $S^n$  where  $S^n$  is the  $n$ -th component of the induced series  $(S)$  from  $S_0$ .

Let  $R^n(x)$  and  $(U, V, W)$  be as above. Moreover, let  $X$  be an equidimensional subvariety of  $R^n(x)$ . A normalized series  $(R, F) = (R, (U, V, S), (U', V', S'), (F, F'))$  will be said to satisfy the differentiability condition for  $X$  if the following is valid.

(4.7)'  $X$  is the union of strata of  $S^n$ . Moreover, for each stratum  $S_\lambda^n \subset X$  such that  $\dim S_\lambda^n = \dim X$ ,  $(R, F)$  satisfies the differentiability condition.

Remark 4.2. When there are no dangers of confusions, we shall do the following abbreviations of terminologies:

(i) The terminology a normalized series of prestratifications attached to ....' will be abbreviated as a normalized series attached to ...'. (ii) The terminology the induced series of prestratifications of  $(V)$  from  $(S_0, \dots)$  will be abbreviated as the induced series from  $(S_0, \dots)$ .

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Remark 4.3. Let  $(P, F) = ((P, U, V, S), (U', V', S')), (F, F')$  be a normalized series of prestratifications in a euclidean space  $R^n(x)$ . We denote by  $(S)$  and  $(S')$  the induced series from  $(S_0)$  and  $(S'_0)$  respectively. Let  $S_\lambda^j$  be a stratum of the  $j$ -th component  $(S)^j$  of  $(S)$ . Then we use the symbol  $\text{fron}(S_\lambda^j, F'^j)$  for the closed set  $V(f(S_\lambda^j)) - (S_\lambda^j)$ . Here  $f(S_\lambda^j)$  is in the  $j$ -th component  $(F)^j$  of  $(F)$ . Moreover, assume that  $(S_\lambda^j)$  and the  $j$ -th component  $(U^j)$  of  $(U)$  has a common point, and let  $S_\lambda^j$  denote the intersection  $S_\lambda^j \cap U^j$ . Then we use the following symbols: (i) The symbols  $f(S_\lambda^j)$  and  $f'(S_\lambda^j)$  mean the sets  $f(S_\lambda^j)$  and  $f'(S_\lambda^j)$  of analytic functions respectively. (ii) The symbol  $V(f'(S_\lambda^j))$  denotes the intersection  $V(f'(S_\lambda^j)) \cap U^j$ . (iii) The symbol  $\text{fron}(S_\lambda^j, (F')^j)$  denotes the closed set  $V(f'(S_\lambda^j)) - S_\lambda^j$ .

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In the sequel of this section we will mainly be concerned with investigations of properties of normalized series. We begin by defining manifolds of certain types for strata in question.

4.2. Level manifolds  $T'_{\alpha}(S_{\lambda}^j)$ , ..., . Let  $(R, F) = ((R, (U, V, S), (U', V', S')), (F, F'))$  be a normalized series of prestratifications in a euclidean space  $R^n(x)$ . For the series  $R = R^1(y_1), \dots, R^n(y_1, \dots, y_n)$ , projections  $\pi_{j,j'}(y) \in R^j$  will be abbreviated as  $\pi_{j,j'}$ ,  $1 \leq j \leq j' \leq n$ . Moreover, let  $S = \{S^j\}$  and  $S' = \{S'^j\}$  ( $j = 1, \dots, n$ ) we mean induced series from  $S_0$  and  $S'_0$  respectively. Now let  $S_{\lambda}^j$  be a stratum of  $S^j$ ,  $j = 2, \dots, n$ . We first remark that the condition  $(4.6)_2$  implies that the coordinates  $(y_1, \dots, y_{n-j+1})$  are local parameters at any point on  $S_{\lambda}^j$ . For a point  $P_{\lambda}^j \in S_{\lambda}^j$  and a couple  $\alpha$  define an  $(n-j-1)$ -dimensional (real analytic) manifold  $T'_{\alpha}(P_{\lambda}^j)$  by

$$T'_{\alpha}(P_{\lambda}^j) = \left\{ Q^j \in R^j : y_t(Q^j) = y_t(P_{\lambda}^j), t = 1, \dots, n-j \right.$$

and

$$\left. \text{and } t = j, |y_t(Q^j) - y_t(P_{\lambda}^j)| < \sigma \cdot d(P_{\lambda}^j, \text{fron}(S_{\lambda}^j)), t = n-j+1, \dots, j-1 \right\}$$

Moreover, for a couple  $\alpha$ , define a  $(j-1)$ -dimensional manifold  $T_{\alpha}(S_{\lambda}^j)$  by

$$(4.9) \quad T_{\alpha}(S_{\lambda}^j) = \bigcup_{P_{\lambda}^j} T'_{\alpha}(P_{\lambda}^j), \text{ where } P_{\lambda}^j \in S_{\lambda}^j.$$

We call the above manifolds  $T(R')$

We call the manifolds  $T'_\omega(P_\lambda^j)$  and  $T'_\omega(S_\lambda^j)$  as above the  $(y_1, \dots, y_n, y_j)$  - level manifold of  $P_\lambda^j \in S_\lambda^j$  and the  $(y_j)$ -level manifold of  $S_\lambda^j$  respectively. Let  $T'_\omega(P_\lambda^j)$  be the  $(y_1, \dots, y_n, y_j)$  - level manifold of  $P_\lambda^j \in S_\lambda^j$ . For a point  $Q_\lambda^j \in T'_\omega(P_\lambda^j)$  and a couple  $\sigma'$ , define a one dimensional manifold  $T''_\sigma(Q_\lambda^j)$  by

$$T''_\sigma(Q_\lambda^j) = \{ Q_j'' \in R^j(y_1, \dots, y_j) : y_t(Q_j'') = y_t(Q_j^j) \\ t = 1, \dots, j-1, |y_j(Q_j'') - y_j(Q_j^j)| \leq \sigma'_j(Q_j^j), \text{ from } (S_\lambda^j) \}$$

Moreover, for couples  $\sigma', \sigma''$  define a neighbourhood  $T_{\sigma', \sigma''}(S_\lambda^j)$  by

$$(4, 10) \quad T_{\sigma', \sigma''}(S_\lambda^j) = \bigcup_{Q_\lambda^j \in T'_{\sigma'}(S_\lambda^j)} T''_{\sigma''}(Q_\lambda^j), \text{ where } Q_\lambda^j \in T'_{\sigma'}(S_\lambda^j).$$

We call the neighbourhood  $T_{\sigma', \sigma''}(S_\lambda^j)$  as above the  $T$ -neighbourhood of the size  $(\sigma', \sigma'')$ .

The manifolds  $T'_{\omega}(S_\lambda^j), T_{\sigma', \sigma''}(S_\lambda^j), \dots$  as above are convenient inductive arguments of the normalized series  $(\mathbb{R}, \mathbb{F})$  on  $j = 1, \dots, n$ .

We shall define one another type of manifolds, denoted by  $U_{\omega, c}(S_\lambda^j)$ , for  $S_\lambda^j$ : Let  $T'_\omega(S_\lambda^j)$  be the  $(y_j)$ -level manifold of  $S_\lambda^j$  and  $(c) = (c_1, c_2)$  be a couple of positive numbers  $(c_1, c_2)$ . We note that we are doing an exceptional use of the term 'couple' for  $(c)$  from the rule 3.1.

Let  $Q_\lambda^j \in T'_\omega(S_\lambda^j)$ , and let  $P_\lambda^j$  be the point in  $S_\lambda^j$  satisfying

$$P_\lambda^j \in R^j(Q_\lambda^j)$$

the condition :  $Q_\lambda^j \in T_\lambda^j(P_\lambda^j)$  . Define a one dimensional manifold  $U_c(Q_\lambda^j)$  by

$$U_c(Q_\lambda^j) = \{ Q_\lambda'^j \in R^j(y_1, \dots, y_j) : y_t(Q_\lambda'^j) = y_t(Q_\lambda^j), t = 1, \dots, j-1, |y_j(Q_\lambda'^j) - y_j(Q_\lambda^j)| \leq c \cdot d(Q_\lambda^j, P_\lambda^j) \}$$

Moreover, for a couple  $\sigma$  and a couple  $c = (c_1, c_2)$  of ~~the~~ positive numbers, define a manifold  $U_{\sigma, c}(S_\lambda^j)$  of dimension  $j$  by

$$(4.9)_3 \quad U_{\sigma, c}(S_\lambda^j) = \bigcup_{Q_\lambda^j \in T_\sigma^j(S_\lambda^j)} U_c(Q_\lambda^j), \text{ where } Q_\lambda^j \in T_\sigma^j(S_\lambda^j) .$$

The manifolds  $U_{\sigma, c}(S_\lambda^j)$ 's as above will be also convenient in inductive arguments of the series  $(R, F)$  .

Remark to 4.2. Similar arguments as above for strata  $S_\lambda^j$ 's of  $(S)_\lambda^j$  are, of course, possible. However, arguments in 4.2. suffice for the later arguments.

Let  $(R, F)$  be a normalized series (in  $R^n(x)$ ) of the form 4.2.. Moreover, let the symbols  $(S, S')$ ,  $(S^j, S'^j)$ , ... have the same meanings as in §4.2. Our next task is to examine relations between prestratifications  $(S^j)$  and  $(S_0^{j-1})$ ,  $j = 2, \dots, n$ . Here  $(S_0^{j-1})$  means, of course, the  $(j-1)$ -th component of  $(S_0)$ . Our arguments will be divided into several pieces. Before we enter into detailed arguments, we shall do certain remarks on materials concerned below.

(i) Let  $S_\lambda^j$  be a stratum of  $\mathcal{S}^j$ ,  $j = 2, \dots, n$ . Then, by a simple computation, we have the following relation.

$\{N_\delta(S_\lambda^j), \tau^j\} \sim \{T_{\sigma, \tau}(S_\lambda^j)\}$ , where  $\mathcal{E}$ ,  $\sigma$  and  $\sigma \circ \tau$  exhaust all the couples obeying the rule  $\text{An } 3.3$ .

(ii) Let  $S_\lambda^{j'} \in \mathcal{S}^{j'}$ , and let  $S_\lambda^j = S_\lambda^{j'} \cup U_\lambda^j$ ,  $j = 2, \dots, n$ . Moreover, let  $D^{j'}$  be an analytic variety in  $U^{j'}$  defined to be the zero locus of an analytic function  $g$  in  $U^{j'}$ . Here  $U^j$  and  $U^{j'}$  are the  $j$ -th components of  $\mathcal{U}$  and  $\mathcal{U}'$  respectively. By the inequality of Lojasiewicz,  $d(P_\lambda^j, V(f')(S_\lambda^{j'})) \sim d(P_\lambda^j, D^{j'})$  for any  $P_\lambda^j \in S_\lambda^j$ . Here  $(f')(S_\lambda^{j'})$  is in the  $j$ -th component  $(F')^j$  of  $(F')$ . Assume that  $D^{j'} \cap S_\lambda^{j'} = \emptyset$ . Then from the obvious relation  $D^{j'} \cap V(f')(S_\lambda^{j'}) = D^{j'} \cap (V(f')(S_\lambda^{j'}) - S_\lambda^{j'})$  and the condition  $(4.2)_3$ , we have

$$(4.10) \quad d(P_\lambda^j, \text{fron}(S_\lambda^j)) \leq d(P_\lambda^j, D^{j'}) \text{ for any } P_\lambda^j \in S_\lambda^j.$$

(ii)' Let  $S_\lambda^{j'}, S_\mu^{j'}$  be strata in  $\mathcal{S}^{j'}$  such that  $S_\lambda^{j'} \subset S_\mu^{j'}$ . Moreover, let  $m = (m_1, \dots, m_{n-j'})$  be the element in  $\mathbb{Z}^{n-j'}$  such that  $S_\mu^{j'} \subset D_m^{j'}(f)(S_\lambda^{j'})$ , where  $(f)(S_\lambda^{j'}) \in F^{j'}$ . Let  $D_m^{j'}$  be the variety in  $U^{j'}$  defined to be the zero locus of the function  $\partial^{m_s} f_s(S_\lambda^{j'})$ ,  $s = 1, \dots, n-j'$ . Then from the basic conditions  $(4.6)_4$  and  $(4.10)$  the following fact is valid.

$$(4.10)' \quad d(P_\lambda^j, \text{fron}(S_\lambda^j)) \leq d(P_\lambda^j, D_m^{j'}) \text{ for any } P_\lambda^j \in S_\lambda^j.$$

Now we begin by arguments by investigating the inverse

~~image  $\pi_{j-1,j}^{-1}(S_\lambda^j)$  of  $S_\lambda^j$  in  $\mathcal{S}^{j-1}$ .~~

images  $\pi_{j-1,j}^{-1}(S_\lambda^{j-1})$ 's, where  $S_\lambda^{j-1} \in \mathcal{S}_0^{j-1}$ .

4.3. The inverse image  $\pi_{j-1,j}^{-1}(S_\lambda^{j-1})$ ,  $S_\lambda^{j-1} \in \mathcal{S}_0^{j-1}$ ,  $j=2, \dots, \mathbb{N}$ .  
 Let  $S_\lambda^{j-1} \in \mathcal{S}_0^{j-1}$ . Define the subset  $\mathcal{S}^j(S_\lambda^{j-1})$  of  $\mathcal{S}^j$  by  $\mathcal{S}^j(S_\lambda^{j-1}) = \{S_{\lambda'}^j \in \mathcal{S}^j : \pi_{j-1,j}(S_{\lambda'}^j) = S_\lambda^{j-1}\}$ . For a stratum  $S_{\lambda'}^j \in \mathcal{S}^j(S_\lambda^{j-1})$  let  $y_{\lambda'}^j(S_\lambda^{j-1})$  denote the single valued analytic function on  $S_\lambda^{j-1}$  defined by

$y_{\lambda'}^j(S_\lambda^{j-1}; P_\lambda^{j-1}) = y_{\lambda'}^j(P_\lambda^j)$  for  $P_\lambda^j \in S_{\lambda'}^j$ , where  $P_\lambda^j$  is the point in  $S_{\lambda'}^j$  satisfying  $\pi_{j-1,j}(P_\lambda^j) = P_\lambda^{j-1}$ . Take a point  $P_0^{j-1} \in S_\lambda^{j-1}$ . By a suitable change of suffices, we can assume the following.

$$y_{\lambda_1}^j(P_0^{j-1}) < y_{\lambda_2}^j(P_0^{j-1}) < \dots < y_{\lambda_k}^j(P_0^{j-1}).$$

Then from the connectedness of  $S_\lambda^{j-1}$  and the very basic condition of the disjointness of strata (cf. § 1), we know that the following series of inequalities

$$(4.11)' \quad y_{\lambda_1}^j(P_\lambda^{j-1}) < y_{\lambda_2}^j(P_\lambda^{j-1}) < \dots < y_{\lambda_k}^j(P_\lambda^{j-1})$$

holds for any  $P_\lambda^{j-1} \in S_\lambda^{j-1}$ .

From the above observations, we can define an order :

$$(4.11) \quad S_{\lambda_1}^j < S_{\lambda_2}^j < \dots < S_{\lambda_k}^j$$

in an accordance with the series (4.11)',

In the above argument we made an essential use of the existence of a 'natural' order in the fibers  $\pi_{j-1,j}^{-1}(P_\lambda^{j-1})$ ,  $P_\lambda^{j-1} \in S_\lambda^{j-1}$ . The existence of the 'natural' order (4.11) in  $\mathcal{S}^j(S_\lambda^{j-1})$  is led by a very simple observation, but makes later arguments on quantitative properties of real analytic

varieties smooth.

Let  $\tilde{\mathcal{S}}^j(S_\lambda^{j-1})$  denote the set composed of connected components of  $\pi_{j-1,j}^{-1}(S_\lambda^{j-1})|_{U^j} - |\mathcal{S}^j(S_\lambda^{j-1})|$ , where  $\pi_{j-1,j}^{-1}(S_\lambda^{j-1})|_{U^j}$  denotes the intersection  $\pi_{j-1,j}^{-1}(S_\lambda^{j-1}) \cap U^j$ . Then from the existence of the order (4.11) and the condition (4.3)<sub>1</sub> as a commutative diagram of the following form (4.12) is valid for each  $\tilde{S}_\lambda^j \in \tilde{\mathcal{S}}^j(S_\lambda^{j-1})$ .

$$(4.12) \quad \begin{array}{ccc} \tilde{S}_\lambda^j & \xrightarrow{\tilde{\varphi}_\lambda^j} & S_\lambda^{j-1} \times I^0 \\ \downarrow \pi_{j-1,j}^{-1} & & \downarrow \text{pr}_1 \\ S_\lambda^{j-1} & \xrightarrow{\text{id}} & S_\lambda^{j-1} \end{array}$$

In the above  $\tilde{\varphi}_\lambda^j$  is a  $C^\infty$  diffeomorphism, and  $\text{pr}_1, \text{id}..$  are the projection to the first factor  $S_\lambda^{j-1}$  and the identity map on  $S_\lambda^{j-1}$  respectively. Also, from the order (4.11), we can define an order

$$(4.11)' \quad \tilde{S}_{\lambda_1}^j < \tilde{S}_{\lambda_2}^j < \tilde{S}_{\lambda_3}^j$$

in the set  $\tilde{\mathcal{S}}^j(S_\lambda^{j-1})$  in a natural and obvious manner. Moreover, for each  $\tilde{S}_\lambda^j \in \tilde{\mathcal{S}}^j(S_\lambda^{j-1})$  at most two strata  $S_{\lambda'}^j \in \mathcal{S}^j(S_\lambda^{j-1})$  are contained in  $\tilde{S}_\lambda^j - \tilde{S}_{\lambda'}^j$ . A stratum  $S_{\lambda'}^j \in \mathcal{S}^j(S_\lambda^{j-1})$  contained in  $\tilde{S}_\lambda^j - \tilde{S}_{\lambda'}^j$  will be called a horizontal component of the boundary of  $\tilde{S}_\lambda^j$ .

4.4. The going down property. Comparisons of frontiers and distances. ~~In 4.4.~~

and distances . As before , let  $(R, F)$  be a normalized series of the form given in 4.2, and let  $(S^j, S_0^j)$  have the same meanings as in §4.2. First we show the following

Proposition 4.1. (GOING DOWN THEOREM FOR NORMALIZED SERIES). Being  $(R, F), (S_0, S, S_0^j, S^j, \dots)$  as above, any triplet  $(S_\mu^{j-1}, S_\lambda^{j-1}, S_\lambda^j) \in S_0^{j-1} \times S_0^{j-1} \times S^j, j = 2, \dots, n,$  satisfying the following diagram

$$(a) \quad \begin{array}{ccc} & S_\lambda^j & \\ & \downarrow & \\ S_\mu^{j-1} & < & S_\lambda^{j-1} \end{array}$$

can be completed a quatlet  $(S_\mu^{j-1}, S_\lambda^{j-1}, S_\mu^j, S_\lambda^j) \in S_0^{j-1} \times S_0^{j-1} \times S^j \times S^j$  satisfying a diagram of the following form.

$$(b) \quad \begin{array}{ccc} S_\mu^j & < & S_\lambda^j \\ \downarrow & & \downarrow \\ S_\mu^{j-1} & < & S_\lambda^{j-1} \end{array}$$

In the above diagrams (a), (b) should be understood respectively as follows: (a) '  $S_\mu^{j-1} < S_\lambda^{j-1}, \pi_{j-1, j}(S_\lambda^j) = S_\lambda^{j-1}$ , (b) '  $S_\mu^{j-1} < S_\lambda^{j-1}, S_\mu^j < S_\lambda^j, \pi_{j-1, j}(S_\lambda^j) \neq S_\lambda^{j-1}, \text{ where } \pi_{j-1, j}(S_\lambda^j) = S_\mu^{j-1}$ .

Remark to Proposition 4.1. The stratum  $S_\mu^j$ , satisfying the diagram (b) may not be unique.

In Proposition 4.1. We borrow the terminology of Going down theorem from the ring theory . (See M. Nagata I I) local rings  
Note, however, that the context of Proposition 4.1. corresponds to that of the Going up theorem rather than to that of Going down theorem in the ring theory .

Proof of Proposition 4.1. Let  $(S_{\mu}^{j-1}, S_{\lambda}^{j-1}, S_{\lambda'}^j) \in \mathbb{S}_0^{j-1} \times \mathbb{S}_0^{j-1} \times \mathbb{S}^j$  be a triplet,  $j = 2, \dots, n$ , such that the diagram

$$\begin{array}{ccc} & & S_{\lambda'}^j \\ & \swarrow & \downarrow \\ S_{\mu}^{j-1} & \prec & S_{\lambda}^{j-1} \end{array}$$

holds. For the stratum  $S_{\lambda'}^j \in \mathbb{S}^j$  satisfying  $S_{\lambda}^j \cap U^j = S_{\lambda'}^j$ , let  $V(f_{\lambda'}^j(S_{\lambda'}^j))$  denote the zero locus in  $U^j$  of the monic polynomial  $f_{\lambda'}^j(S_{\lambda'}^j) \in \mathbb{C}(S_{\lambda'}^j)$ . Take a point  $P_{\lambda}^{j-1} \in S_{\lambda}^{j-1}$ . Then  $\pi_{j-1,j}^{-1}(P_{\lambda}^{j-1}) \cap V(f_{\lambda'}^j(S_{\lambda'}^j))$  consists of finite points  $P_{\lambda_s}^j$ ,  $s = 1, \dots, s_0$ . Take neighbourhoods  $N(P_{\lambda_s}^j)$ 's of  $P_{\lambda_s}^j$ 's in  $U^j$  and a neighbourhood  $N(P_{\lambda}^{j-1})$  suitably. Then we can assume that (i)  $N(P_{\lambda_s}^j) \cap N(P_{\lambda_{s'}}^j) = \emptyset$ , if  $s \neq s'$  and that (ii)  $\pi_{j-1,j}^{-1}(N(P_{\lambda}^{j-1})) \cap V(f_{\lambda'}^j(S_{\lambda'}^j)) \subset \bigcup_s N(P_{\lambda_s}^j)$ . Moreover, we can assume that if  $P_{\lambda_s}^j \in V(f_{\lambda'}^j(S_{\lambda'}^j))$ , then  $N(P_{\lambda_s}^j) \cap V(f_{\lambda'}^j(S_{\lambda'}^j)) \neq \emptyset$ . Take a sequence  $\{Q_t^{j-1}\}_{t=1, \dots, \infty}$  of points in  $S_{\lambda}^{j-1}$  such that  $\lim_t Q_t^{j-1} = P_{\lambda}^{j-1}$ . Let  $Q_t^j$  denote the point characterized by  $\pi_{j-1,j}^{-1}(Q_t^{j-1}) \cap S_{\lambda'}^j$ . Then we can choose a subsequence  $\{Q_{1_t}^j\}_{t=1, \dots, \infty}$  of  $\{Q_t^j\}$  which converges to one of the points  $P_{\lambda_s}^j$ . Because  $Q_{1_t}^j$ 's are in  $S_{\lambda'}^j$ , the limit:  $\lim_t Q_{1_t}^j = P_{\lambda_s}^j$  is in  $S_{\lambda'}^j$ . Then clearly the stratum  $S_{\mu}^j \in \mathbb{S}^j$  containing  $P_{\lambda_s}^j$  satisfies the diagram

$$\begin{array}{ccc} S_{\mu}^j & \preceq & S_{\lambda}^j \\ \downarrow & & \downarrow \\ S_{\mu}^{j-1} & \preceq & S_{\lambda}^{j-1} \end{array}$$

Q.E.D.

Now let  $S_{\lambda}^{j-1}, S_{\mu}^{j-1}, S_{\lambda'}^j \in \mathbb{S}_0^{j-1} \times \mathbb{S}_0^{j-1} \times \mathbb{S}^j$ ,  $j = 2, \dots, n$ , be a

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a triplet satisfying the diagram

$$\begin{array}{c} S_{\lambda'}^j \\ \downarrow \\ S_{\lambda}^{j-1} < S_{\lambda}^{j-1} \end{array}$$

Let  $(S_{\lambda}^j(S_{\lambda}^{j-1}))_{\overline{S}_{\lambda}^j}$  denote the subset of  $(S_{\lambda}^j(S_{\lambda}^{j-1}))$  composed of strata contained in  $\overline{S}_{\lambda}^j$ . Proposition 4.1. assures that  $(S_{\lambda}^j(S_{\lambda}^{j-1})) \neq \emptyset$ . By a simple observation we know that

$$(4.13) \quad (S_{\lambda}^j(S_{\lambda}^{j-1}))_{\overline{S}_{\lambda}^j} = \pi_{j-1,j}(\text{fron}(S_{\lambda}^{j-1})) \cap \overline{S}_{\lambda'}^j.$$

Let  $(S_{\lambda}^{j-1}, S_{\lambda'}^{j-1}) \in (S_{\lambda}^{j-1}) \times (S_{\lambda'}^{j-1})$  such that  $\pi_{j-1,j}(S_{\lambda'}^{j-1}) = S_{\lambda}^{j-1}$ . Then from (4.13) we know easily that

$$(4.13) \quad \text{fron}(S_{\lambda'}^j) = \pi_{j-1,j}(\text{fron}(S_{\lambda}^{j-1})) \cap \overline{S}_{\lambda'}^j.$$

Let  $S_{\lambda}^{j-1}, S_{\lambda'}^{j-1}$  be as above. Moreover, let  $S_{\lambda}^{'j-1}$  be the stratum in  $(S_{\lambda}^{j-1})_0$  such that  $S_{\lambda}^{'j-1} \cap U^j = S_{\lambda}^{j-1}$ . Define an analytic function  $g(S_{\lambda}^{'j-1})$  in  $U^{'j-1}$  by

$$g(S_{\lambda}^{'j-1}) = \prod_{\mu} f'_{\mu}(S_{\lambda}^{'j-1}), \text{ where } S_{\mu}^{'j-1} \in \text{fron}(S_{\mu}^{'j-1}).$$

Then, by the inequality of Lojasiewicz, we have

$$g(S_{\lambda}^{'j-1}; P_{\lambda}^{j-1}) \sim d(P_{\lambda}^{j-1}, V(g(S_{\lambda}^{'j-1})))$$

$$\tilde{d}(P_{\lambda}^{j-1}, V(g(S_{\lambda}^{'j-1}))) \sim V(f')(S_{\lambda}^{'j-1}, \overline{S}_{\lambda'}^{'j-1}) \text{ for any } P_{\lambda}^{j-1} \in S_{\lambda}^{j-1}.$$

In the above  $V(g(S_{\lambda}^{'j-1}))$  denotes the zero locus of  $g(S_{\lambda}^{'j-1})$  in  $U^{'j-1}$ . On the otherhand from (4.6)<sub>1</sub> and (4.2)<sub>3</sub> we know that

$$\begin{aligned} \tilde{d}(P_{\lambda}^{j-1}, \text{fron}(S_{\lambda}^{j-1})) &\sim \tilde{d}(P_{\lambda}^{j-1}, V(g(S_{\lambda}^{'j-1})) \cap V(f')(S_{\lambda}^{'j-1})) \\ &\text{for any } P_{\lambda}^{j-1} \in S_{\lambda}^{j-1}. \end{aligned}$$

Therefore we have

therefore

$$(4.14)' \quad g(P_{\lambda}^{j-1}) \sim \tilde{d}(P_{\lambda}^{j-1}, \text{fron}(S_{\lambda}^{j-1})) \text{ for any } P_{\lambda}^{j-1} \in S_{\lambda}^{j-1}.$$

Now define an analytic function  $\tilde{g}$  in  $U^j$  be  $\tilde{g}(P^j) = g(\pi_{j-1,j}(P^j))$ , where  $P^j \in U^j$ . Then, for the stratum  $S_{\lambda}^{j-1}$  satisfying  $S_{\lambda}^{j-1} \cap U^j = S_{\lambda}^j$ , we have

$$\tilde{g}(P_{\lambda}^j) \sim d(P_{\lambda}^j, V((f')^{-1}(S_{\lambda}^{j-1})) \pi_{j-1,j}^{-1}(V(g(S_{\lambda}^{j-1})))$$

for any  $P_{\lambda}^j \in S_{\lambda}^j$ .

But the right side of the above is equivalent to  $d(P_{\lambda}^j, \text{fron}(S_{\lambda}^j))$

for any  $P_{\lambda}^j \in S_{\lambda}^j$ . Thus we have

$$(4.14)'' \quad \tilde{g}(P_{\lambda}^j) \sim \tilde{d}(P_{\lambda}^j, \text{fron}(S_{\lambda}^j)) \text{ for any } P_{\lambda}^j \in S_{\lambda}^j.$$

From (4.14)' and (4.14)'' we have the following comparison of distances.

$$(4.14) \quad d(P_{\lambda}^j, \text{fron}(S_{\lambda}^j)) \sim d(P_{\lambda}^{j-1}, \text{fron}(S_{\lambda}^{j-1})) \text{, where } P_{\lambda}^j \in S_{\lambda}^j \text{, and } P_{\lambda}^{j-1} = \pi_{j-1,j}(P_{\lambda}^j).$$

Let  $(R, F)$ ,  $S_0, S$ , etc., .... be as in 4.2. 4.4. Now we examine strata  $S^j$ 's in  $S_0^j - S^j$ ,  $j = 2, \dots, n$ .

4.5. Strata  $S^j$ 's in  $S_0^j - S^j$ .

### § 6. Normalized series attached to germs of varieties.

6.1. Formulations of problems. Let  $R^n(x)$  be a euclidean space with coordinates  $(x) = (x_1, \dots, x_n)$  and  $O^n$  the origin of  $R^n(x)$ . By an admissible pair of germs of varieties at  $O^n$ , we mean a pair  $(V, W)$  of germs of varieties at  $O^n$  such that the following condition is valid.

$$(6.1) \quad 1 \leq \dim V \leq n - 1, \quad \text{and} \quad W \subset V.$$

An admissible pair  $(V, W)$  of germs of varieties at  $O^n$  will be called an admissible pair  $(V, W)$  of germs at  $O^n$ , when there is no fear of confusions.

Let  $(V, W)$  be an admissible pair of germs at  $O^n$ . By a normalized series of prestratified spaces attached to  $(V, W)$  we mean a normalized series  $(\mathbb{R}, \mathbb{F}) = ((\mathbb{R}, (\mathbb{U}, \mathbb{V}, \mathbb{W}), (\mathbb{U}', \mathbb{V}', \mathbb{W}')), (\mathbb{R}', \mathbb{F}'))$  in  $R^n(x)$  satisfying the following conditions.

$$(6.2)_1 \quad U^n \supset O^n.$$

$$(6.2)_2 \quad \text{For each stratum } S_\lambda^n \in (\mathbb{S}_0^n) \quad (S_\lambda^n \in \mathbb{S}^n), \quad S_\lambda^n \supset O^n \quad (S_\lambda^n \supset O^n).$$

$$(6.2)_3 \quad \text{Each component } V_\tau \text{ of } V \text{ is an irreducible component of the germ of } V^n \text{ at } O^n.$$

$$(6.2)_4 \quad \text{Each component } V_\lambda \text{ of } V \text{ and each component } W_\mu \text{ of } W \text{ are expressed as the union of germs of strata } S_\lambda^n \text{'s of } \mathbb{S}^n \text{ at } O^n.$$

(6.2)<sub>5</sub> For each  $S_\lambda^j \in \mathbb{S}^j$ , the set  $\mathbb{F}(S_\lambda^j) = \{f_t(S_\lambda^j)\}_t \in \mathbb{F}^j$  consists of Weierstrass polynomials  $f_t(S_\lambda^j)$  in  $y_{n_\lambda^j+t}$  with variables  $(y_1, \dots, y_{n_\lambda^j+t})$ ,  $t = 1, \dots, j - n_\lambda^j$  ( $j = 2, \dots, n$ ).  
In the above we write  $\dim S_\lambda^j$  as  $n_\lambda^j$ .

In (6.2)<sub>1, \dots, 5</sub>,  $\mathbb{U}^n, \mathbb{V}^n$  and  $\mathbb{S}_0^n$  denote the  $n$ -th component of  $\mathbb{U}, \mathbb{V}, \mathbb{S}_0$  respectively. Moreover,  $\mathbb{S}^n$  denotes the  $n$ -th component of the series  $\mathbb{S}$  of prestratifications of  $\mathbb{V}$  induced from  $\mathbb{S}_0$ , and  $(y_1, \dots, y_n)$  denotes the system of coordinates of  $R^n(x)$  defining the series  $\mathbb{R}$ .

When there is no fear of confusions, we call a normalized series  $(\mathbb{R}, \mathbb{F})$  of prestratified spaces attached to  $(V, W)$  simply a normalized series  $(\mathbb{R}, \mathbb{F})$  attached to  $(V, W)$ .

Let  $(V, W)$  be an admissible pair of germs at  $O^n$ . We say that a normalized series  $(\mathbb{R}, \mathbb{F}) = ((\mathbb{R}, \mathbb{U}, \mathbb{V}, \mathbb{S}), \dots)$  in  $R^n(x)$  is attached properly to  $(V, W)$  if, in addition to (6.2)<sub>1 ~ 5</sub>, the following condition is valid.

(6.2)'  $V = V^n$ , where  $V^n$  is the germ of the  $n$ -th component  $V^n$  of  $\mathbb{V}$  at  $O^n$ .

Let  $(V, W)$  be as above, and let  $X$  be an irreducible germ of a variety at  $O^n$  such that  $X \subset V$ . We say that a

$\mathcal{M} = (M_1, \dots, M_n)$  of positive monomials  $M_j$ 's in one variable  
 $(R, F)$  is of type  $\mathcal{M}$

Now our position is to formulate our basic problem in this section: As previously, let  $R^n(x)$  be a Euclidean space with coordinates  $(x)$  and  $O^n$  the origin of  $R^n(x)$ . Then our problem is to prove the following

Theorem 6.1. For an admissible pair  $(V, W)$  of germs at  $O^n$ , there exists a normalized series  $(R, F)$  attached properly to  $(V, W)$ . Moreover, we can assume that  $(R, F)$  satisfies the following conditions.

- (I) The series  $(R, F)$  is of monomial type.
- (II) For any component  $V$  of  $V$ , the series  $(R, F)$  satisfies the differentiability condition.

We will prove Theorem 6.1. in a little later. Here we give a corollary to Theorem 6.1.

Corollary 6.1. Let  $(V, W)$  be as in Theorem 6.1. Then there exists a finite set  $\{(R^s, F^s)\}_s$  of normalized series  $(R^s, F^s)$  attached to  $(V, W)$  such that the following are valid.

- (I)' Each series  $(R^s, F^s)$ ,  $s = 1, \dots$ , is of monomial type.
- (II)' Each  $(R^s, F^s)$ ,  $s = 1, \dots$ , satisfies the differentiability condition for any irreducible component  $V_i$  of  $V$ .

~~$V = D(R^S, F^S)$ , where  $D(R^S, F^S)$  is the ger~~  
~~of  $D(R^S, F^S)$ .~~

$$(III)' \quad \bigcap_S D(R^S, F^S) \subset V.$$

Remark to Corollary 6.1. Let  $\tilde{D}_\mu$  be an irreducible component of  $\bigcap_S D(R^S, F^S)$ . Then the conditions (II)', (III)' imply the following fact.

(I)' There exists an irreducible component  $V_\tau$  of  $V$  such that  $D \not\subseteq V_\tau$ .

Proof of Corollary 6.1. By Theorem 6.1, we can choose a normalized series  $(R, F)$  attached properly to  $(V, W)$  such that (I), (II) hold. We write the irreducible decomposition of  $D(R, F)$  as  $\bigcup_\mu D_\mu$ . Then, between any  $D_\mu$  and any irreducible component  $V_\lambda$  of  $V$ , there is no relation of the form  $V_\lambda \subset D_\mu$ . Next define germs  $V^1, W^1$  by  $V^1 = V \cup D(R, F)$  and  $W^1 = W$ . By Theorem 6.1, there exist normalized series  $(R^1, F^1)$  of monomial type attached properly to  $(V^1, W^1)$  such that the relation  $V_\tau^1 \subset D(R^1, F^1)$  holds for any irreducible component  $V_\tau^1$  of  $V^1$ . (Here  $D(R^1, F^1)$  is, of course, the germ of  $D(R^1, F^1)$  at  $O^n$ ). Therefore, for any irreducible component  $D_\mu$  of  $D(R, F)$  satisfying  $D_\mu \not\subset V$ , we have the following relation.

$$D_\mu \cap D(R^1, F^1) \subseteq D_\mu.$$

If the intersection  $D^1 = D(R, F) \cap D(R^1, F^1)$ , then the proof is completed. Otherwise, define germs  $V^2, W^2$  by  $V^2 = V \cap D^1$  and  $W^2 = W^1$ . We apply Theorem 6.1. to the pair  $(V^2, W^2)$  and do similar arguments for  $(V^2, W^2)$  as above. Clearly repeating these procedures we attain the proof of Corollary 6.1.

Now we enter into discussions of the proof of Theorem 6.1. We want to do discussions of the proof of Theorem 6.1 inductively on the dimension of  $V$ . To do inductive arguments of the proof of Theorem 6.1, smooth, it is convenient to consider another problem instead of considering directly Theorem 6.1. The another problem considered here is as follows: Let  $R^n(x)$  denote a euclidean space with coordinates  $(x) = (x_1, \dots, x_n)$  and  $O^n$  the origin  $O^n$  of  $R^n(x)$ . By an admissible series of germs at  $O^n$ , we mean a collection  $(R, V, V', W)$  of a series  $R$  of euclidean spaces and series  $V, V', W$  of germs of varieties. Here  $(R, V, V', W)$  should be of the form (6.5) and should satisfy conditions (6.6).

$$(6.5)_1 \left( R \mid R^{k+1}(y'_1, \dots, y'_{k+1}), \dots, R^n(y'_1, \dots, y'_{k+1}, \dots, y'_n) \right)$$

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where  $(y') = (y'_1, \dots, y'_n)$  is a system of coordinates of  $R^n(x)$ .

(6.5)<sub>2</sub>  $(V) = \{V^j\}$ ,  $(V') = \{V'^j\}$  and  $(W) = \{W^j\}$ ,  $j = k+1, \dots, n$ .  
Here  $V^j, V'^j$  and  $W^j$  are germs of varieties at  $O^j \in R^j(y'_1, \dots, y'_j)$   
,  $j = k+1, \dots, n$ . the origin

(6.6)<sub>1</sub> The integer  $k$  should satisfy the relation:  $1 \leq k \leq n-1$ . Moreover,  $1 \leq \dim V^j \leq k$ ,  $\dim V'^j \leq k$ , and  $V^j \cup V'^j \supset W^j$ ,  $j = k+1, \dots, n$ .

(6.6)<sub>2</sub> Germs  $V^j$  and  $V'^j$  have no common irreducible components.

(6.6)<sub>3</sub> Any irreducible component  $V_\lambda^{j'}$  of  $V^j$  ( $V_\tau^{j'}$  of  $V'^j$ ) is integral over  $\pi_{jj'}^{an}(y')(V^j)$  ( $\pi_{jj'}^{an}(y')(V'^j)$ ),  
 $k+1 \leq j \leq j' \leq n$ .

(6.6)<sub>4</sub>  $\pi_{jj'}^{an}(y')(V^{j'}) \subset V^j$ , and for each irreducible component  $V_\lambda^{j'}$  of  $V^j$ ,  $\pi_{jj'}^{an}(y')(V'^j)$  is an irreducible component of  $V^j$ ,  $k+1 \leq j \leq j' \leq n$ .

(6.6)<sub>5</sub>  $\pi_{jj'}^{an}(y')(V'^j) \subset V^j \cup V'^j$ ,  $\pi_{jj'}^{an}(y')(W^j) \subset W^j$ ,  
 $k+1 \leq j \leq j' \leq n$ .

Remarks to (6.5) and (6.6). (i) The condition (6.6)<sub>1</sub> implies that  $V^j \neq \emptyset$  for  $j = k+1, \dots, n$ . However,  $V'^j, W^j$  may be empty,  $j = k+1, \dots, n$ .

(11) Also note that we are not assuming the following:  
 $\bigcap_{j=1}^n (y^j)(V'^j) \subset V^j.$

The integer  $k$  in (6.6), (6.6) will be called the rank of the admissible series  $(R)$ . Also by the dimension of  $(R)$  ( $\dim(R)$ ) we mean  $\max_j \{V^j \cup V'^j\}, j = k+1, \dots, n$ .

Let  $(R) = \{(R), (V), (V'), (W)\}$  be an admissible series of rank  $k$  of germs of varieties at  $O^n$ . We write  $\{(R), (V), (V'), (W)\}$  explicitly as follows:  $(R) = \{R^{k+1}(y'_{k+1}, \dots, y'_n), \dots, R^n(y'_1, \dots, y'_n)\}$ ,  $(V) = \{V^j\}$ ,  $(V') = \{V'^j\}$ ,  $(W) = \{W^j\}$ ,  $j = k+1, \dots, n$ . Then each pair  $(V^j \cup V'^j, W^j)$  is admissible at  $O^j \in R^j(y'_1, \dots, y'_j)$ ,  $j = k+1, \dots, n$ .

By a normalized series (of prestratified spaces) attached to  $R$ , we mean a normalized series  $((R), (U), (V), (S_0), (U'), (V'), (S'_0), (F), (F'))$  in  $R^n(x)$  such that the following conditions are valid.

(6.7)<sub>1</sub> The coordinates  $(y) = (y_1, \dots, y_n)$  defining the series  $R$  are of the following forms:  $y_{k+2} = y'_{k+2}, \dots, y_n = y'_n$  and  $(y_1, \dots, y_{k+1}) = (y'_1, \dots, y'_{k+1})A + (a)$ , where  $A \in GL(k+1, \mathbb{R})$  and  $(a) \in R^{k+1}$ .

(6.7)<sub>2</sub> The  $j$ -th normalized series  $((R^j), (F^j))$  derived from  $(R, F)$  is attached to  $(V^j \cup V'^j, W^j)$ .

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Let  $\{(\mathbb{R}, \mathbb{V}, \mathbb{V}', \mathbb{W})\}$  be an admissible series of rank  $k$  of germs at  $O^n$ , where  $\mathbb{V}, \mathbb{V}', \mathbb{W}$  are explicitly of the forms:  $\mathbb{V} = \{V^j\}$ ,  $\mathbb{V}' = \{V'^j\}$  and  $\mathbb{W} = \{W^j\}$ ,  $j = k+1, \dots, n$ . Let  $(\mathbb{R}, \mathbb{F})$  be a normalized series attached to  $(\mathbb{R}, \mathbb{V}, \mathbb{V}', \mathbb{W})$ . Then the  $j$ -th normalized series  $(\mathbb{R}^j, \mathbb{F}^j)$  derived from  $(\mathbb{R}, \mathbb{F})$  is attached to  $(\mathbb{V}^j, \mathbb{V}'^j, \mathbb{W}^j)$ . We add the following definitions:

(i) We say that  $(\mathbb{R}, \mathbb{F})$  is attached properly to  $(\mathbb{R}, \mathbb{V}, \mathbb{V}', \mathbb{W})$  if  $(\mathbb{R}^j, \mathbb{F}^j)$  is attached properly to  $(\mathbb{V}^j \cup \mathbb{V}'^j, \mathbb{W}^j)$ ,  $j = k+1, \dots, n$ .

(ii) We say that  $(\mathbb{R}, \mathbb{F})$  satisfies the differentiability condition for  $(\mathbb{R}, \mathbb{V}, \mathbb{V}', \mathbb{W})$  if  $(\mathbb{R}^j, \mathbb{F}^j)$  satisfies the differentiability for each irreducible component of  $\mathbb{V}^j \cup \mathbb{V}'^j$ ,  $j = k+1, \dots, n$ .

(iii) We say that  $(\mathbb{R}, \mathbb{F})$  is of monomial type if the series  $(\mathbb{R}, \mathbb{F})$ , regarded as attached to  $(\mathbb{V}^n \cup \mathbb{V}'^n, \mathbb{W})$ , is of monomial type.

Remark 6.1. Let  $(\mathbb{R}, \mathbb{V}, \mathbb{V}', \mathbb{W})$  be an admissible series of germs at  $O^n$ . We assume that the rank of  $(\mathbb{R}, \mathbb{V}, \mathbb{V}', \mathbb{W}) = n - 1$ . Then  $(\mathbb{V}, \mathbb{V}', \mathbb{W})$  is of the form  $(\mathbb{V}, \mathbb{V}', \mathbb{W})$ , where  $\mathbb{V}, \mathbb{V}', \mathbb{W}$  are germs of varieties at  $O^n$ . Moreover, assume that  $\mathbb{V}' = \emptyset$ .

Then the problem of finding a normalized series attached to the admissible series  $(R, (V, V'), W) = (R^n(y'), V, V', W)$  is entirely same as that of finding a normalized series attached to the admissible pair  $(V, W)$  of germs at  $O^n$ . Now our present problem is to prove the following Lemma for each  $k = 1, 2, \dots$

Lemma 6.1.  $k_d$ . Let  $(R) = \{(R, (V, V'), W)\}$  be an admissible series of <sup>dimension  $d$</sup>  ~~rank  $k$~~  of germs of varieties at the origin  $O^n \in R^n(x)$ .

Then there exists a normalized series  $(R, F)$  attached properly to  $R$ . Moreover, for  $(R, F)$  we can assume the following.

(I")  $(R, F)$  satisfies the differentiability condition for  $R$ .

(II")  $(R, F)$  is of monomial type.

It is clear from Remark 6.1. that the validity of Lemma 6.1  $k_d$  for each  $k = 1, 2, \dots$  implies the validity of Theorem 6.1. Now for the proof of Lemma 6.1  $k_d$  for each  $k = 1, 2, \dots$  it is sufficient to prove the following two facts.

(A) Lemma 6.1<sub>1</sub>.

(B) That the validity of Lemma 6.1  $k_d$ ,  $k' = 1, \dots, k-1$ , implies the validity of Lemma 6.1  $k_d$ , where  $k = 2$ .

The proof of (A) does not contain difficulties. We postpone the proof of (A) for a little while. We first prove (B).

Proof. (B) (A) is proved in (I) and (II) with  $k=1$  and  $k=2$ .