

## Approximation of a Reaction-Diffusion Equation with a Nonlocal Term

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### 1 Introduction.

We consider a scalar bistable reaction-diffusion equation

$$(RD) \quad \epsilon u_t = \epsilon^2 \Delta u + f(u) - v, \quad t > 0, x \in \Omega,$$

under the Neumann boundary condition

$$(BC) \quad \frac{\partial u}{\partial \mathbf{n}} = 0, \quad t > 0, x \in \partial\Omega.$$

Here  $u$  is an order parameter while  $v$  an additional parameter (acting as inhibitors).  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $\mathbf{n}$  stands for the outward unit normal vector on the boundary  $\partial\Omega$ . The nonlinear term  $f$  is assumed to be the negative derivative of a smooth double-well potential  $W$ :  $f(u) = -W'(u)$ . A typical example is  $f(u) = u - u^3$ . The parameter  $\epsilon > 0$  is supposed to be very small, and we intend to study the problem above as the singular perturbation problem.

We will treat in this paper a situation in which the spacial average of the order parameter is preserved:

$$(PP) \quad \frac{1}{|\Omega|} \int_{\Omega} u(t, x) dx \equiv m \quad (\text{constant}), \quad t \geq 0,$$

i.e., a case where  $v$  in (RD) is given by

$$(NL) \quad v(\cdot) = \frac{1}{|\Omega|} \int_{\Omega} f(u(\cdot, x)) dx.$$

When  $\epsilon > 0$  is very small, the solution  $u(t, x)$  of (RD) with an appropriate initial condition creates a sharp transition layer with width of  $O(\epsilon)$  and it is expected to move according to some motion laws, called interface equations. Our purpose of this paper is (1) to derive interface equations from (RD); and (2) to investigate how solutions of interface equations evolve.

**Remark 1.** From a variational point of view, the equation (RD) is characterized as the  $L^2(\Omega)$ -gradient system for the energy functional of van der Waals type

$$E^\epsilon(u) := \int_{\Omega} \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) dx$$

subject to the constraint (PP), and the nonlocal term  $v$  is regarded as the Lagrange multiplier (see [2] for example).

## 2 Derivation of interface equations.

Throughout the remaining part of this paper, an interface is meant to be a smooth, closed,  $N - 1$  dimensional hypersurface embedded in  $\Omega \subset \mathbb{R}^N$ . We will derive some interface equations from (RD) by the method of matched asymptotic expansions (see [9] for more details).

### 2.1 Preliminaries.

We now present precise assumptions on  $f$  and prepare some notations for our problem.

(A1) The function  $f$  is  $C^\infty$  on  $\mathbb{R}$  and the curve  $f(u) - v = 0$  consists of three sub-branches of solutions

$$\mathcal{C}^- = \{(u, v) \mid u = h^-(v), v \in I^- := (\underline{v}, \infty)\},$$

$$\mathcal{C}^+ = \{(u, v) \mid u = h^+(v), v \in I^+ := (-\infty, \bar{v})\},$$

and

$$\mathcal{C}^0 = \{(u, v) \mid u = h^0(v), v \in I^0 := I^- \cap I^+ = (\underline{v}, \bar{v})\},$$

satisfying  $f'(h^\pm(v)) < 0$  (or equivalently  $h_v^\pm(v) < 0$ ) on  $I^\pm$ .

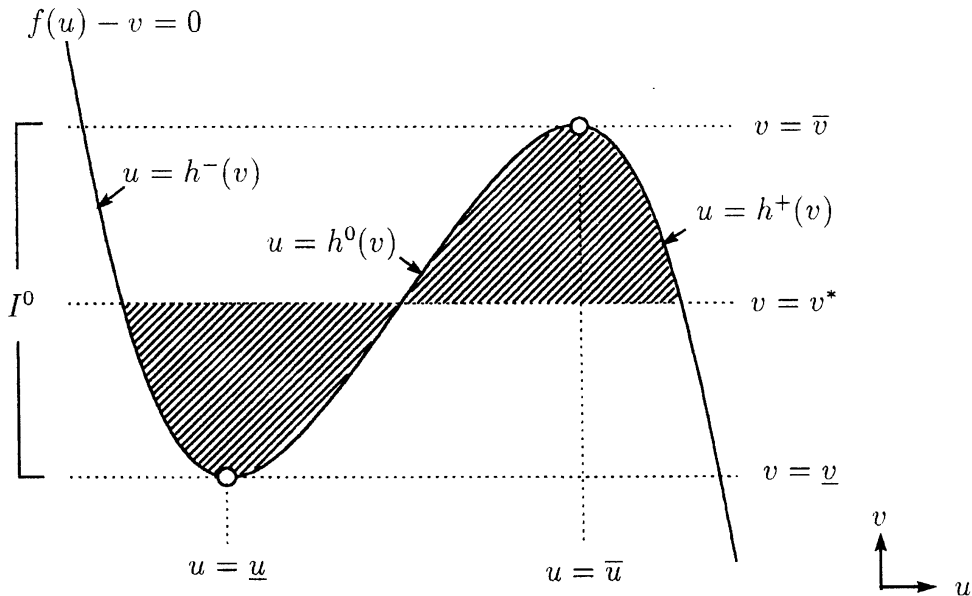
(A2) For each  $v \in I^0$ , it holds that  $h^-(v) < h^0(v) < h^+(v)$ .

(A3) For each  $v \in I^0$ , we define

$$\mathcal{S}(v) := \int_{h^-(v)}^{h^+(v)} f(u) - v \, du.$$

Then there exists a unique point  $v^* \in I^0$  such that  $\mathcal{S}(v^*) = 0$  and  $\mathcal{S}'(v^*) < 0$ .

**Remark 2.** We may regard the point  $(h^0(v^*), v^*)$  as the origin  $(0, 0)$  by appropriate translations.



An unknown interface  $\Gamma(t)$ , which is to be determined, is expressed as a smooth embedding from a fixed  $N - 1$  dimensional reference manifold  $\mathcal{M}$  to  $\mathbb{R}^N$ :

$$(2.1) \quad \gamma(t, \cdot) : \mathcal{M} \rightarrow \Gamma(t) \subset \Omega, \quad \mathcal{M} \ni y \mapsto x = \gamma(t, y) \in \Gamma(t).$$

Let  $\Omega^\pm(t)$  be subregions (called bulk regions) in  $\Omega$  decomposed by  $\Gamma(t)$  such as

$$\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t),$$

and  $\nu(t, y) \in \mathbb{R}^N$  the unit normal vector on  $\Gamma(t)$  at  $x = \gamma(t, y)$  pointing into the interior of the bulk region  $\Omega^+(t)$ . In advance we standardize the parametrization as in (2.1) in such a way that  $\gamma_t(t, y)$  is always parallel to  $\nu(t, y)$  [3]. For sufficiently small  $\delta > 0$ , a point  $x$  in a neighborhood  $\{x \in \Omega \mid \text{dist}(x; \Gamma(t)) < \delta\}$  is uniquely represented as

$$(2.2) \quad x = \gamma(t, y) + r\nu(t, y),$$

which gives us a new coordinate system  $(t, r, y)$ . We denote by  $J(t, r, y)$  Jacobian associated with (2.2). Namely,

$$J(t, r, y) = \prod_{i=1}^{N-1} (1 + r\kappa_i(t, y)) =: 1 + \sum_{i=1}^{N-1} H_i(t, y)r^i,$$

where  $\kappa_i(t, y)$  ( $i = 1, \dots, N - 1$ ) stand for the principal curvatures of  $\Gamma(t)$  at  $x = \gamma(t, y)$ .

Let  $u^\epsilon$  be a solution of (RD) for an appropriate initial condition:

$$(2.3) \quad \epsilon u_t^\epsilon(t, x) = \epsilon^2 \Delta u^\epsilon(t, x) + f(u^\epsilon(t, x)) - v^\epsilon(t), \quad t > 0, \quad x \in \Omega,$$

$$(2.4) \quad v^\epsilon(t) = \frac{1}{|\Omega|} \int_{\Omega} f(u^\epsilon(t, x)) dx, \quad t > 0.$$

We define an interface  $\Gamma^\epsilon(t)$  as a level set of the solution  $u^\epsilon$  to (RD). Since transition layers are expected to develop in regions  $\{x \in \Omega \mid u^\epsilon(t, x) \approx h^0(v^*)\}$ , we set (cf. Remark 2)

$$(2.5) \quad \Gamma^\epsilon(t) := \{x \in \Omega \mid u^\epsilon(t, x) = 0\}.$$

On the other hand,  $\Gamma^\epsilon(t)$  is also assumed to be expressed as a graph of a smooth function over the interface  $\Gamma(t)$ :

$$(2.6) \quad \Gamma^\epsilon(t) = \{x \in \Omega \mid x = \gamma(t, y) + \epsilon R^\epsilon(t, y)\nu(t, y), \quad y \in \mathcal{M}\}.$$

$R^\epsilon$ , of course, is a priori unknown and is to be determined.

## 2.2 Outer expansion.

We separate the whole domain  $\Omega$  into two components  $\Omega^{\epsilon, \pm}(t)$  by the interface  $\Gamma^\epsilon(t)$  such as  $\Omega = \Omega^{\epsilon, -}(t) \cup \Gamma^\epsilon(t) \cup \Omega^{\epsilon, +}(t)$ , and substitute the formal expansions

$$(2.7) \quad U^\epsilon(t, x) = U^{\epsilon, \pm}(t, x) = \sum_{j \geq 0} \epsilon^j U^{j, \pm}(t, x), \quad v^\epsilon(t) = \sum_{j \geq 0} \epsilon^j v^j(t)$$

into (2.3) in order to see the profile of solutions away from layer regions. Equating to zero the coefficient of each power of  $\epsilon$  in the resulting equation, we obtain the following series of equations:

$$(2.8) \quad f(U^{0,\pm}) - v^0 = 0,$$

$$(2.9) \quad f'(U^{0,\pm})U^{j,\pm} = v^j + F_j^\pm, \quad j \geq 1.$$

Here  $F_j^\pm$  stand for functions depending on  $U^{k,\pm}$  ( $0 \leq k < j$ ) only.

As the solution of (2.8), noting that **(A1)**, we choose

$$(2.10) \quad U^{0,\pm}(t, x) := h^\pm(v^0(t)).$$

Once we make this choice,  $U^{j,\pm}$  ( $j \geq 1$ ) can be successively expressed by (2.9) as

$$(2.11) \quad U^{j,\pm}(t, x) = h_v^\pm(v^0(t))v^j(t) + V_j^\pm(t)$$

with  $V_j^\pm$  being some functions depending on  $v^k$  ( $0 \leq k < j$ ). Therefore once  $v^j$  is known,  $U^{j,\pm}$  are determined completely.  $v^j$  ( $j \geq 0$ ) will be determined later so that the  $C^1$ -matching conditions are satisfied (cf. subsection 2.5). We note, in particular, that the outer solution  $U^\epsilon(t, x)$  is independent of  $x$ , and therefore is denoted simply as  $U^\epsilon(t)$  in the sequel.

### 2.3 Inner expansion.

To deal with layer phenomena near  $r = \epsilon R^\epsilon(t, y)$  (cf. (2.2), (2.6)), we use a stretched variable  $z := \epsilon^{-1}[r - \epsilon R^\epsilon(t, y)]$  and recast our problem (2.3) in terms of  $(t, z, y)$ :

$$(2.12) \quad \tilde{u}_{zz}^\epsilon + (\gamma_t \cdot \nu) \tilde{u}_z^\epsilon + f(\tilde{u}^\epsilon) + \epsilon R_t^\epsilon \tilde{u}_z^\epsilon - v^\epsilon + \mathcal{D}^\epsilon \tilde{u}^\epsilon = 0, \quad z \in (-\delta/\epsilon - R^\epsilon, \delta/\epsilon - R^\epsilon),$$

where  $\mathcal{D}^\epsilon$  stands for a differential operator including  $R^\epsilon$ .

We will seek an asymptotic solution to (2.12) of the form

$$(2.13) \quad \tilde{u}^\epsilon(t, z, y) = U^\epsilon(t, x)|_{x=\gamma(t,y)+(\epsilon z + \epsilon R^\epsilon(t,y))\nu(t,y)} + \phi^\epsilon(t, z, y) = U^\epsilon(t) + \phi^\epsilon(t, z, y),$$

i.e., we will determine  $\phi^\epsilon$  in such a way that  $\tilde{u}^\epsilon$  in (2.13) asymptotically satisfies (2.12) for  $z \in (-\infty, \infty)$ . We substitute the formal expansions

$$(2.14) \quad R^\epsilon(t, y) = R^1(t, y) + \epsilon R^2(t, y) + \epsilon^2 R^3(t, y) + \cdots,$$

$$(2.15) \quad \tilde{u}^\epsilon(t, z, y) = \tilde{u}^{\epsilon,\pm}(t, z, y) = U^{\epsilon,\pm}(t) + \phi^{\epsilon,\pm}(t, z, y)$$

$$= \sum_{j \geq 0} \epsilon^j U^{j,\pm}(t) + \sum_{j \geq 0} \epsilon^j \phi^{j,\pm}(t, z, y) =: \sum_{j \geq 0} \epsilon^j \tilde{u}^{j,\pm}(t, z, y)$$

together with the expansion for  $v^\epsilon$  into (2.12) to obtain some series of equations for  $\tilde{u}^{j,\pm}$  and  $\phi^{j,\pm}$  in  $\pm z \in (0, \infty)$ . We now exhibit equations for  $\tilde{u}^{j,\pm}$  only:

$$(2.16) \quad \tilde{u}_{zz}^{0,\pm} + (\gamma_t \cdot \nu) \tilde{u}_z^{0,\pm} + f(\tilde{u}^{0,\pm}) - v^0 = 0,$$

$$(2.17) \quad \tilde{u}_{zz}^{j,\pm} + (\gamma_t \cdot \nu) \tilde{u}_z^{j,\pm} + f'(\tilde{u}^{0,\pm}) \tilde{u}^{j,\pm} = v^j - R_t^j \tilde{u}_z^{0,\pm} + \mathcal{F}_j^\pm, \quad j \geq 1.$$

Here  $\mathcal{F}_j^\pm$  stand for functions depending on  $R^k, v^k, \tilde{u}^{k,\pm}$  ( $0 \leq k < j$ ) with  $R^0 := \gamma$ .

We impose the following conditions:

- Boundary conditions at  $z = 0$  (cf. (2.5)):

$$(2.18) \quad \tilde{u}^{j,\pm}(t, 0, y) = U^{j,\pm}(t) + \phi^{j,\pm}(t, 0, y) = 0.$$

- Boundary conditions at  $z = \pm\infty$  (outer-inner matching conditions):

$$(2.19) \quad \phi^{j,\pm}(t, z, y) \rightarrow 0 \text{ exponentially as } z \rightarrow \pm\infty.$$

- $C^1$ -matching conditions at  $z = 0$ :

$$(2.20) \quad \tilde{u}_z^{j,-}(t, 0, y) = \tilde{u}_z^{j,+}(t, 0, y).$$

## 2.4 Expansion of nonlocal term.

(2.4) is recast as follows:

$$(2.21) \quad \begin{aligned} & \dot{U}^{\epsilon,-}|\Omega^-| + \dot{U}^{\epsilon,+}|\Omega^+| \\ &= (\dot{U}^{\epsilon,+} - \dot{U}^{\epsilon,-}) \sum_{i \geq 0} \int_{\mathcal{M}} \frac{H_i(t, y)}{i+1} (\epsilon R^\epsilon(t, y))^{i+1} dS_y^{\gamma(t, \cdot)} \\ &+ \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{\epsilon,-} + (\gamma_t \cdot \nu) \phi_z^{\epsilon,-} + \epsilon R_t^\epsilon \phi_z^{\epsilon,-} + \mathcal{D}^\epsilon \phi^{\epsilon,-}] J^\epsilon dz dS_y^{\gamma(t, \cdot)} \\ &+ \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{\epsilon,+} + (\gamma_t \cdot \nu) \phi_z^{\epsilon,+} + \epsilon R_t^\epsilon \phi_z^{\epsilon,+} + \mathcal{D}^\epsilon \phi^{\epsilon,+}] J^\epsilon dz dS_y^{\gamma(t, \cdot)} \\ &+ O(\epsilon^{-1} e^{-\delta/\epsilon}). \end{aligned}$$

Here  $J^\epsilon(t, z, y) := J(t, r, y)|_{r=\epsilon z + \epsilon R^\epsilon(t, y)}$  and  $dS_y^{\gamma(t, \cdot)}$  stands for the volume element on  $\mathcal{M}$  induced from  $dS_x^{\Gamma(t)}$ , the surface element on  $\Gamma(t)$  at  $x$ , by the embedding  $\gamma(t, \cdot)$ . These are denoted simply as  $dS_x$  and  $dS_y$  in the sequel.

We substitute the outer and inner expansions into (2.21) to obtain some series of equations:

$$(2.22) \quad \begin{aligned} \dot{U}^{0,-}|\Omega^-| + \dot{U}^{0,+}|\Omega^+| &= \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{0,-} + (\gamma_t \cdot \nu) \phi_z^{0,-}] dz dS_y \\ &+ \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{0,+} + (\gamma_t \cdot \nu) \phi_z^{0,+}] dz dS_y, \end{aligned}$$

$$(2.23) \quad \begin{aligned} & \dot{U}^{j,-}|\Omega^-| + \dot{U}^{j,+}|\Omega^+| \\ &= (\dot{U}^{0,+} - \dot{U}^{0,-}) \int_{\mathcal{M}} R^j dS_y \\ &+ \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{0,-} + (\gamma_t \cdot \nu) \phi_z^{0,-}] \kappa R^j dz dS_y \\ &+ \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{0,+} + (\gamma_t \cdot \nu) \phi_z^{0,+}] \kappa R^j dz dS_y \\ &+ \int_{\mathcal{M}} \int_{-\infty}^0 [\phi_{zz}^{j,-} + (\gamma_t \cdot \nu) \phi_z^{j,-} + R_t^j \phi_z^{0,-}] dz dS_y \\ &+ \int_{\mathcal{M}} \int_0^\infty [\phi_{zz}^{j,+} + (\gamma_t \cdot \nu) \phi_z^{j,+} + R_t^j \phi_z^{0,+}] dz dS_y + \mathcal{I}^j, \quad j \geq 1. \end{aligned}$$

Here  $\kappa := \kappa_1 + \cdots + \kappa_{N-1}$  is the mean curvature of  $\Gamma(t)$  at  $x = \gamma(t, y)$ , and  $\mathcal{I}^j(t)$  stands for a function calculated by using functions  $R^k$ ,  $U^{k,\pm}$  and  $\phi^{k,\pm}$  ( $0 \leq k < j$ ).

## 2.5 $C^1$ -matching.

We note that the following problem

$$(2.24) \quad \begin{cases} Q_{zz} + cQ_z + f(Q) - v = 0, & z \in (-\infty, \infty), \\ Q(\pm\infty) = h^\pm(v), \quad Q(0) = 0, \end{cases}$$

has a unique solution pair  $(Q(z; v), c(v))$  for each  $v \in I^0$ . Then (2.16) with (2.18)-(2.20) have unique solutions if and only if

$$(2.25) \quad \gamma_t(t, y) \cdot \nu(t, y) = c(v^0(t)) \quad v^0(t) \in I^0,$$

and solutions are given by

$$(2.26) \quad \tilde{u}^{0,\pm}(t, z, y) = Q(z; v^0(t)), \quad \pm z \in (0, \infty).$$

Once (2.25) is satisfied and we have (2.26), we can successively show the existence and uniqueness of  $\phi^{j,\pm}$  satisfying (2.19) for all  $j \geq 0$ .

As for  $\tilde{u}^{j,\pm}$  ( $j \geq 1$ ), equations (2.17) with (2.18)-(2.20) have unique solutions if and only if a solvability condition of (2.17)

$$\int_{-\infty}^{\infty} e^{cz} Q_z (v^j - R_t^j Q_z + \mathcal{F}_j) dz = 0$$

is satisfied, which is equivalent to

$$(2.27) \quad R_t^j(t, y) = c'(v^0(t)) v^j(t) + \rho_j(t, y)$$

with  $\rho_j$  being a function calculated by using  $R^k$ ,  $v^k$  and  $\tilde{u}^k$  ( $0 \leq k < j$ ). For instance,  $\rho_1$  is given by

$$(2.28) \quad \rho_1 = -\kappa + \frac{\int_{-\infty}^{\infty} e^{c(v^0)z} Q_z(z; v^0) Q_v(z; v^0) dz}{\int_{-\infty}^{\infty} e^{c(v^0)z} [Q_z(z; v^0)]^2 dz} \dot{v}^0.$$

On the other hand, (2.22) and (2.23) with (2.18)-(2.20) respectively yield

$$(2.29) \quad \dot{v}^0(t) = \frac{h^+(v^0(t)) - h^-(v^0(t))}{h_v^-(v^0(t))|\Omega^-(t)| + h_v^+(v^0(t))|\Omega^+(t)|} c(v^0(t)) |\Gamma(t)|,$$

$$(2.30) \quad \dot{v}^j(t) = \int_{\mathcal{M}} a(t, y) R^j(t, y) dS_y + b(t) v^j(t) + \sigma_j(t).$$

Here  $a$  and  $b$  are some functions depending only on  $(\Gamma, v^0)$  given by

$$(2.31) \quad a := \frac{[h^+(v^0) - h^-(v^0)] c(v^0) \kappa + [h_v^+(v^0) - h_v^-(v^0)] \dot{v}^0}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|},$$

$$\begin{aligned}
(2.32) \quad b &:= \frac{h^+(v^0) - h^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} c'(v^0) |\Gamma| \\
&+ \frac{(h_v^+(v^0) - h_v^-(v^0)) c(v^0) |\Gamma| - (h_{vv}^-(v^0)|\Omega^-| + h_{vv}^+(v^0)|\Omega^+|) \dot{v}^0}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|},
\end{aligned}$$

while  $\sigma_j$  stands for a function computed by employing  $R^k$ ,  $v^k$  and  $\phi^{k,\pm}$  ( $0 \leq k < j$ ). For instance,  $\sigma_1$  is given by

$$\begin{aligned}
\sigma_1 = & - \frac{h^+(v^0) - h^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} \int_{\mathcal{M}} \kappa dS_y \\
& + \left[ h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+| \right]^{-1} \times \left[ c(v^0) \left( \int_{-\infty}^{\infty} z Q_z(z; v^0) dz \right) \int_{\mathcal{M}} \kappa dS_y \right. \\
& - \frac{d}{dt} \left( h_v^-(v^0) \dot{v}^0 \right) |\Omega^-| - \frac{d}{dt} \left( h_v^+(v^0) \dot{v}^0 \right) |\Omega^+| \\
& - \left( \int_{-\infty}^0 (Q_v(z; v^0) - h_v^-(v^0)) dz + \int_0^{\infty} (Q_v(z; v^0) - h_v^+(v^0)) dz \right) \dot{v}^0 |\Gamma| \\
& + \left( h_v^+(v^0)^2 - h_v^-(v^0)^2 \right) c(v^0) (\dot{v}^0)^2 |\Gamma| \\
& \left. + \left( h^+(v^0) - h^-(v^0) \right) \frac{\int_{-\infty}^{\infty} e^{c(v^0)z} Q_z(z; v^0) Q_v(z; v^0) dz}{\int_{-\infty}^{\infty} e^{c(v^0)z} [Q_z(z; v^0)]^2 dz} \dot{v}^0 |\Gamma| \right].
\end{aligned}$$

We finally arrived at the following interface equations:

$$(IE^0) \quad \gamma_t \cdot \nu = c(v^0), \quad \dot{v}^0 = \frac{h^+(v^0) - h^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} c(v^0) |\Gamma|,$$

$$(IE^j) \quad R_t^j = c'(v^0) v^j + \rho_j, \quad \dot{v}^j = \int_{\mathcal{M}} a R^j dS_y + b v^j + \sigma_j, \quad j \geq 1.$$

### 3 Analysis of interface equations.

We are now ready to study the interface equations. Let us begin with the 0-th order equation (IE<sup>0</sup>).

#### 3.1 0-th order equation.

The equation is as follows:

$$(IE^0\text{-a}) \quad \mathbf{v}(x; \Gamma(t)) = c(v(t)), \quad t > 0, \quad x \in \Gamma(t),$$

$$(IE^0\text{-b}) \quad \dot{v}(t) = \frac{h^+(v(t)) - h^-(v(t))}{h_v^-(v(t))|\Omega^-(t)| + h_v^+(v(t))|\Omega^+(t)|} c(v(t)) |\Gamma(t)|, \quad t > 0,$$

$$(IE^0\text{-c}) \quad \Gamma(0) = \Gamma_0, \quad v(0) = v_0.$$

Here  $\mathbf{v}(x; \Gamma(t)) := \gamma_t(t, y) \cdot \nu(t, y)$  is the normal velocity of  $\Gamma(t)$  at  $x = \gamma(t, y)$ . We note that the superscript '0' in  $v^0(t)$  has been suppressed.

It immediately turns out, due to (IE<sup>0</sup>-a), that the normal speed is independent of the position  $x \in \Gamma(t)$  and is regulated by the (0-th order) nonlocal term  $v$ . Thanks to the identity

$$(3.1) \quad \frac{d}{dt} |\Omega^-(t)| = -\frac{d}{dt} |\Omega^+(t)| = \int_{\Gamma(t)} \mathbf{v}(x; \Gamma(t)) dS_x,$$

the interface equation (IE<sup>0</sup>) implies

$$(3.2) \quad h^-(v(t)) \frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t)) \frac{|\Omega^+(t)|}{|\Omega|} \equiv m_0, \quad t \geq 0,$$

where  $m_0 = m_0(\Gamma_0, v_0)$  is given by

$$(3.3) \quad m_0 := h^-(v_0) \frac{|\Omega_0^-|}{|\Omega|} + h^+(v_0) \frac{|\Omega_0^+|}{|\Omega|}$$

with  $\Omega_0^\pm$  being initial bulk regions such as  $\Omega = \Omega_0^- \cup \Gamma_0 \cup \Omega_0^+$ . We note that (3.2) corresponds to (PP) for (RD) as  $\epsilon \rightarrow 0$  (cf. (2.10)).

We recast (IE<sup>0</sup>) as a system of ordinary differential equations after the manner of Sakamoto [11]. For a given initial interface  $\Gamma_0$  we express  $\Gamma(t)$  as the graph of a function  $r(t, y)$  over  $\Gamma_0$ :  $\gamma(t, y) = \gamma(0, y) + r(t, y)\nu(0, y)$ . Then some elementary calculations yield  $\nu(t, y) \equiv \nu(0, y)$  and  $r(t, y) \equiv r(t)$ , and therefore (IE<sup>0</sup>-a) is recast as  $\dot{r}(t) = c(v(t))$ . On the other hand, the surface area of an interface  $\{x \in \Omega \mid x = \gamma(0, y) + r\nu(0, y), y \in \mathcal{M}\}$  is given by

$$g(r) := \int_{\mathcal{M}} J(0, r, y) dS_y^0 = |\Gamma_0| + \sum_{i=1}^{N-1} \left( \int_{\mathcal{M}} H_i(0, y) dS_y^0 \right) r^i, \quad dS_y^0 := dS_y^{\gamma(0, \cdot)},$$

so we have  $|\Gamma(t)| = g(r(t))$ . Moreover, (3.2) together with  $|\Omega^-(t)| + |\Omega^+(t)| \equiv |\Omega|$  implies that the volume of the bulk regions are represented in terms of  $v$  as

$$(3.4) \quad |\Omega^-| = \frac{h^+(v) - m_0}{h^+(v) - h^-(v)} |\Omega|, \quad |\Omega^+| = \frac{m_0 - h^-(v)}{h^+(v) - h^-(v)} |\Omega|,$$

from which the first factor in the right hand side of (IE<sup>0</sup>-b) is rewritten as  $h(v(t))$  with

$$(3.5) \quad h(v) = h(v; v_0) := \frac{1}{|\Omega|} \frac{[h^+(v) - h^-(v)]^2}{h_v^-(v)[h^+(v) - m_0] + h_v^+(v)[m_0 - h^-(v)]}.$$

In particular, if the initial pair  $(\Gamma_0, v_0)$  is chosen so that  $m_0 \in (\underline{u}, \bar{u})$ , it follows that  $|\Omega^\pm| > 0$  in (3.4) and therefore we have  $h(v) < 0$  for all  $v \in I^0$  (cf. **(A1)**, **(A2)**). Thus the interface equation (IE<sup>0</sup>) are equivalent to the following initial value problem:

$$(ODE^0) \quad \begin{cases} \dot{r} = c(v), \\ \dot{v} = h(v) c(v) g(r), \\ r(0) = 0, \quad v(0) = v_0. \end{cases}$$

By virtue of reformulation above and an equivalent expression of  $c(v)$

$$(3.6) \quad c(v) = -\frac{\mathcal{S}(v)}{\int_{-\infty}^{\infty} [Q_z(z; v)]^2 dz},$$

the interface dynamics are summerized as follows:

- $v \in (v^*, \bar{v}) \implies \dot{r} > 0, \dot{v} < 0;$   
the interface  $\Gamma(t)$  evolves in such a way that the bulk region  $\Omega^-(t)$  grows uniformly.
- $v \in (\underline{v}, v^*) \implies \dot{r} < 0, \dot{v} > 0;$   
the interface  $\Gamma(t)$  evolves in such a way that the bulk region  $\Omega^-(t)$  shrinks uniformly.
- $v = v^* \implies \dot{r} = 0, \dot{v} = 0;$   
the interface  $\Gamma(t)$  does not evolve.

We also obtain the following

**Theorem 3** (Unique existence of solutions). *Let  $\Gamma_0$  be a smooth initial interface, and a pair  $(\Gamma_0, v_0)$  is assumed to satisfy  $v_0 \in I^0$  and  $m_0 \in (\underline{u}, \bar{u})$ . Then the following statements hold true:*

- (1) *There exists a constant  $T > 0$  such that  $(\text{IE}^0)$  has a unique smooth solution pair  $(\Gamma, v)$  on a time interval  $[0, T]$ .*
- (2) *If in addition  $v_0$  is sufficiently close to  $v^*$ , then the unique solution  $(\Gamma, v)$  in (1) exists globally in time.*

*Proof.* (2) immediately follows from the existence of a constant  $R > 0$  such that  $r$ -component  $r(\cdot)$  of the solution to  $(\text{ODE}^0)$  remains in a neighborhood  $(-R, R)$  while the corresponding interface  $\Gamma(\cdot) = \{x \in \Omega \mid x = \gamma(0, y) + r(\cdot)\nu(0, y), y \in \mathcal{M}\}$  is smooth for all  $|r| < R$  when we choose  $v_0 \approx v^*$ .  $\square$

**Theorem 4** (Stability of equilibrium solutions). *Suppose that a pair  $(\Gamma_0, v_0)$  is as in Theorem 3. Then the following statements hold true:*

- (1)  *$(\Gamma_0, v_0)$  is an equilibrium solution of  $(\text{IE}^0)$  if and only if  $v_0 = v^*$ .*
- (2) *The equilibrium solution  $(\Gamma_0, v^*)$  is asymptotically stable relative to  $(\text{ODE}^0)$ .*

*Proof.* (2) We linearize  $(\text{ODE}^0)$  around the corresponding equilibrium solution  $(0, v^*)$  to obtain the eigenvalues 0 and  $h(v^*)c'(v^*)|\Gamma_0| < 0$ .  $\square$

For each  $v \in I^0$ , the nonlinear term  $f(u) - v$  defines a new double-well potential  $\mathcal{W}(u; v)$  with two wells located at  $u = h^\pm(v)$ . Moreover, the potential difference is related to  $\mathcal{S}(v)$  and  $c(v)$  as follows:

$$\mathcal{W}(h^+(v); v) - \mathcal{W}(h^-(v); v) = -\mathcal{S}(v) = c(v) \int_{-\infty}^{\infty} [Q_z(z; v)]^2 dz.$$

Hence it turns out that the 0-th order nonlocal effect equalizes the potential of two wells no matter how the initial state is.

### 3.2 Higher order equations.

The  $j$ -th ( $j \geq 1$ ) order equations are as follows:

$$(IE^j\text{-a}) \quad R_t^j(t, y) = c'(v^0(t)) v^j(t) + \rho_j(t, y), \quad t > 0, y \in \mathcal{M},$$

$$(IE^j\text{-b}) \quad \dot{v}^j(t) = \int_{\mathcal{M}} a(t, y) R^j(t, y) dS_y + b(t) v^j(t) + \sigma_j(t), \quad t > 0,$$

$$(IE^j\text{-c}) \quad R^j(0, y) = R^j(y), \quad v^j(0) = v_0^j.$$

Recall that  $a$  and  $b$  are functions depending only on the solution  $(\Gamma, v^0)$  to  $(IE^0)$  (cf. (2.31), (2.32)), while  $\rho_j$  and  $\sigma_j$  are some functions which can be calculated by using functions with index  $k$  ( $0 \leq k < j$ ) in outer and inner expansions.

Each equation  $(IE^j)$  can be recast as a system of linear non-homogeneous ordinary differential equations. Indeed, by employing a function  $r^j$  given by

$$r^j(t) := R^j(t, y) - R^j(y) - \int_0^t \rho_j(s, y) ds,$$

$(IE^j\text{-a})$  and  $(IE^j\text{-b})$  are respectively expressed as

$$\dot{r}^j(t) = c'(v^0(t)) v^j(t),$$

$$\begin{aligned} \dot{v}^j(t) &= \left( \int_{\mathcal{M}} a(t, y) dS_y \right) r^j(t) + b(t) v^j(t) \\ &\quad + \int_{\mathcal{M}} a(t, y) \left( R^j(y) + \int_0^t \rho_j(s, y) ds \right) dS_y + \sigma_j(t), \end{aligned}$$

from which we obtain an initial value problem of the form

$$(ODE^j) \quad \begin{cases} \dot{r}^j(t) = B(t) v^j(t), \\ \dot{v}^j(t) = C(t) r^j(t) + D(t) v^j(t) + E_j(t), \\ r^j(0) = 0, \quad v^j(0) = v_0^j. \end{cases}$$

Due to this reformulation, we have the following

**Theorem 5** (Unique existence of solutions). *Once the initial pair  $(R^j(y), v_0^j)$  is given, the equations  $(IE^j)$  ( $j \geq 1$ ) are successively solvable on a finite time interval  $[0, T]$ .*

In particular, we can construct a smooth approximate solution  $u_A^\epsilon$  of (RD) in the sense that

$$\begin{aligned} \left\| \epsilon \frac{\partial u_A^\epsilon}{\partial t} - \epsilon^2 \Delta u_A^\epsilon - f(u_A^\epsilon) + \frac{1}{|\Omega|} \int_{\Omega} f(u_A^\epsilon(\cdot, x)) dx \right\|_{L^\infty([0, T] \times \Omega)} &= O(\epsilon^{K+1}), \\ \frac{\partial u_A^\epsilon}{\partial \mathbf{n}} &= 0, \quad (t, x) \in [0, T] \times \partial\Omega, \end{aligned}$$

by means of unique solutions  $(\Gamma, v^0)$  and  $(R^j, v^j)$  of  $(IE^j)$  for  $0 \leq j \leq K$ .

As the solution  $v^0(t)$  approaches the equilibrium state  $v^*$ , the 0-th order equation  $(IE^0)$  becomes powerless to approximate the layer dynamics. In this case, we must move our attention to the equation  $(IE^1)$  for  $(R^1, v^1)$  in order to capture the further dynamics of layers. An investigation in such a direction will be our future work.

## References

- [1] F. Bethuel, G. Huisken, S. Müller and K. Steffen, *Calculus of variations and geometric evolution problems*, Lecture Notes in Math., **1713**, Springer (1999).
- [2] L. Bronsard and B. Stoth, *Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation*, SIAM J. Math. Anal., **28** (1997), no.4, 769-807.
- [3] P. de Mottoni and M. Schatzman, *Geometric evolution of developed interfaces*, Trans. Amer. Math. Soc., **347** (1995), 1533-1589.
- [4] 儀我美一, 界面ダイナミクス — 曲率の効果, 北海道大学数学講究録, No.56 (1998).
- [5] G. Huisken, *The volume preserving mean curvature flow*, J. Reine Angew. Math., **382** (1987), 35-48.
- [6] J. E. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var., **10** (2000), 49-84.
- [7] H. Ikeda, *On the asymptotic solutions for a weakly coupled elliptic boundary value problem with a small parameter*, Hiroshima Math. J., **16** (1986), 227-250.
- [8] K. Okada, *Dynamical approximation of a nonlocal reaction-diffusion equation via interfacial approach*, master thesis, Hiroshima University (2002).
- [9] K. Okada, *Asymptotic expansions for nonlocal reaction-diffusion equations*, Preprint (2002).
- [10] K. Sakamoto, *Asymptotic expansion of interface equation for a reaction-diffusion system*, Tohoku Math. Publications, **8** (1998), 149-158.
- [11] K. Sakamoto, *Spatial homogenization and internal layers in a reaction-diffusion system*, Hiroshima Math. J. **30** (2000), 377-402.
- [12] K. Sakamoto, *Interfaces in activator-inhibitor systems -asymptotics and degeneracy-*, Preprint (2003).