

BOUNDED HARMONIC FUNCTIONS ON UNLIMITED COVERING SURFACES

大同工大 瀬川重男 (Shigeo Segawa)

京都産大・理 正岡弘照 (Hiroaki Masaoka)

1. Introduction

Let W be an open Riemann surface possessing a Green's function. Consider a p -sheeted *unlimited* covering surface $\widetilde{W}$ of W with projection map π . It is easily seen that $\widetilde{W}$ also possesses a Green's function (cf. e.g. [A-S]). We denote by $HP(R)$ ($HB(R)$, resp.) the class of positive (bounded, resp.) harmonic functions on an open Riemann surface R . It is obvious that the inclusion relation

$$HX(W) \circ \pi := \{h \circ \pi : h \in HX(W)\} \subset HX(\widetilde{W})$$

holds for $X = P, B$. The main purpose of this paper is to give a necessary and sufficient condition, in terms of Martin boundary, in order that the relation $HX(W) \circ \pi = HX(\widetilde{W})$ holds for $X = P, B$.

For an open Riemann surface R , we denote by R^* , Δ^R and Δ_1^R the Martin compactification, the Martin boundary and the minimal Martin boundary of R , respectively. It is known that the projection map π of $\widetilde{W}$ to W is extended to $\widetilde{W}^*$ continuously and $\pi(\Delta^{\widetilde{W}}) = \Delta^W$ (cf. [M-S2]). For each $\zeta \in \Delta^W$, put

$$\Delta_1^{\widetilde{W}}(\zeta) = \Delta_1^{\widetilde{W}} \cap \pi^{-1}(\zeta) = \{\tilde{\zeta} \in \Delta_1^{\widetilde{W}} : \pi(\tilde{\zeta}) = \zeta\},$$

which is the set of minimal boundary points of $\widetilde{W}$ lying over $\zeta \in \Delta^W$. Our main results are the followings.

Theorem 1. *In order that the relation $HP(W) \circ \pi = HP(\widetilde{W})$ holds, it is necessary and sufficient that $\Delta_1^{\widetilde{W}}(\zeta)$ consists of a single point for every $\zeta \in \Delta_1^W$.*

Theorem 2. *In order that the relation $HB(W) \circ \pi = HB(\widetilde{W})$ holds, it is necessary and sufficient that $\Delta_1^{\widetilde{W}}(\zeta)$ consists of a single point for ω_z^W almost all $\zeta \in \Delta_1^W$, where ω_z^W is a harmonic measure on Δ^W with respect to W and $z \in W$.*

Proofs of Theorems 1 and 2 will be given in §3 and §4, respectively.

Let D be the unit disc $\{|z| < 1\}$. In §5, we will be concerned with p -sheeted unlimited covering surfaces of D which illustrate Theorems 1 and 2. We will prove the following.

Proposition. Set $A = \{(1 - 2^{-n-1})e^{i2\pi k/2^{n+2}} : n = 1, 2, \dots, k = 1, \dots, 2^{n+2}\}$. If $\widetilde{D}$ is a p -sheeted unlimited covering surface of D with projection map π such that there is a branch point of $\widetilde{D}$ of order $p - 1$ (or multiplicity p) over every $z \in A$ and there are no branch points of $\widetilde{D}$ over $D \setminus A$, then $HP(D) \circ \pi = HP(\widetilde{D})$.

We will show a bit more (cf. Theorem 5.1). Modifying the above $\widetilde{D}$, we will also give a p -sheeted unlimited covering surface $\widetilde{D}_1$ of D with projection map π such that $HP(D) \circ \pi \neq HP(\widetilde{D}_1)$ and $HB(D) \circ \pi = HB(\widetilde{D}_1)$.

2. Martin boundary of p -sheeted unlimited covering surfaces

Let W be an open Riemann surface possessing a Green's function and $\widetilde{W}$ a p -sheeted unlimited covering surface of W with projection map π . Since the pullback of a Green's function on W by π is a nonconstant positive superharmonic function on $\widetilde{W}$, we see that $\widetilde{W}$ possesses a Green's function (cf. e.g. [A-S], [S-N]). For Martin compactification, Martin boundary and minimal Martin boundary, we follow the notation in Introduction. We first note the following (cf. [M-S2]).

Proposition 2.1. The projection map π of $\widetilde{W}$ onto W is extended to the Martin compactification $\widetilde{W}^*$ of $\widetilde{W}$ continuously and $\pi(\Delta^{\widetilde{W}}) = \Delta^W$.

We recall the definition of $\Delta_1^{\widetilde{W}}(\zeta)$ ($\zeta \in \Delta^W$) in Introduction:

$$\Delta_1^{\widetilde{W}}(\zeta) = \Delta_1^{\widetilde{W}} \cap \pi^{-1}(\zeta) = \{\tilde{\zeta} \in \Delta_1^{\widetilde{W}} : \pi(\tilde{\zeta}) = \zeta\}.$$

We denote by $\nu_{\widetilde{W}}(\zeta)$ the (cardinal) number of $\Delta_1^{\widetilde{W}}(\zeta)$. We next fix a point $a \in W$ and a point $\tilde{a} \in \widetilde{W}$ with

$$(2.1) \quad \pi(\tilde{a}) = a.$$

We consider the Martin kernel $k_\zeta^W(\cdot)$ ($k_\zeta^{\widetilde{W}}(\cdot)$, resp.) on W ($\widetilde{W}$, resp.) with pole at ζ ($\tilde{\zeta}$, resp.) and with reference point a ($\tilde{a}$, resp.), that is,

$$k_\zeta^W(z) = \frac{g^W(z, \zeta)}{g^W(a, \zeta)} \quad (k_\zeta^{\widetilde{W}}(\tilde{z}) = \frac{g^{\widetilde{W}}(\tilde{z}, \tilde{\zeta})}{g^{\widetilde{W}}(\tilde{a}, \tilde{\zeta})}, \text{ resp.})$$

for $\zeta \in W$ ($\tilde{\zeta} \in \widetilde{W}$, resp.), where $g^W(\cdot, \zeta)$ ($g^{\widetilde{W}}(\cdot, \tilde{\zeta})$, resp.) is a Green's function on W ($\widetilde{W}$, resp.) with pole at ζ ($\tilde{\zeta}$, resp.). Note that

$$(2.2) \quad k_\zeta^W(a) = k_\zeta^{\widetilde{W}}(\tilde{a}) = 1.$$

In our previous paper [M-S2], we proved the following.

Proposition 2.2. Suppose $\zeta \in \Delta^W$. Then

- (i) If $\zeta \in \Delta^W \setminus \Delta_1^W$, then $\nu_{\tilde{W}}(\zeta) = 0$;
- (ii) If $\zeta \in \Delta_1^W$, then $1 \leq \nu_{\tilde{W}}(\zeta) \leq p$;
- (iii) If $\zeta \in \Delta_1^W$ and $\Delta_1^{\tilde{W}}(\zeta) = \{\tilde{\zeta}_1, \dots, \tilde{\zeta}_n\}$, then there exist positive numbers $c_1, \dots, c_n$ such that

$$(2.3) \quad k_{\zeta}^W \circ \pi = c_1 k_{\tilde{\zeta}_1}^{\tilde{W}} + \dots + c_n k_{\tilde{\zeta}_n}^{\tilde{W}}.$$

In the relation (2.3) above, by (2.1) and (2.2), we have

$$(2.4) \quad \sum_{i=1}^n c_i = 1.$$

Let s be a positive superharmonic function on W and E is a subset of W . We denote by ${}^w\hat{R}_s^E$ the *balayage* of s with respect to E on W . We here give the definitions of *minimal thinness* and *minimal fine neighborhood* (cf. [B]).

DEFINITION 2.1. Let ζ be a point of Δ_1^W and E a subset of W . We say that E is *minimally thin* at ζ if ${}^w\hat{R}_{k_{\zeta}^W}^E \neq k_{\zeta}^W$.

DEFINITION 2.2. Let ζ be a point of Δ_1^W and U a subset of W . We say that $U \cup \{\zeta\}$ is a *minimal fine neighborhood* of ζ if $W \setminus U$ is minimally thin at ζ .

The following is easily verified from Proposition 3.1 of our previous paper [M-S2] (see also [M]).

Proposition 2.3. Let $\tilde{\zeta} \in \Delta_1^{\tilde{W}}$ and $\tilde{U}$ a subset of $\tilde{W}$. Then $\tilde{U} \cup \{\tilde{\zeta}\}$ is a *minimal fine neighborhood* of $\tilde{\zeta}$ if and only if $\pi(\tilde{U}) \cup \{\pi(\tilde{\zeta})\}$ is a *minimal fine neighborhood* of $\pi(\tilde{\zeta})$.

For $\zeta \in \Delta_1^W$, we denote by $\mathcal{M}_W(\zeta)$ the class of connected open sets M such that $W \setminus M$ is minimally thin at ζ . Moreover, for $M \in \mathcal{M}_W(\zeta)$ and a p -sheeted unlimited covering surface $\tilde{W}$ of W with projection map π , we denote by $n_{\tilde{W}}(M)$ the number of connected components of $\pi^{-1}(M)$. Then $\nu_{\tilde{W}}(\zeta)$ is characterized by $n_{\tilde{W}}(M)$ as follows, which is a main result of our previous paper [M-S2].

Proposition 2.4. Suppose $\zeta \in \Delta_1^W$. Then $\nu_{\tilde{W}}(\zeta) = \max_{M \in \mathcal{M}_W(\zeta)} n_{\tilde{W}}(M)$.

3. Proof of Theorem 1

In this section, we give the proof of Theorem 1. For the sake of simplicity, we introduce the following notation:

$$\Delta = \Delta^W, \Delta_1 = \Delta_1^W, \tilde{\Delta} = \Delta^{\tilde{W}}, \tilde{\Delta}_1 = \Delta_1^{\tilde{W}}, \tilde{\Delta}_1(\zeta) = \Delta_1^{\tilde{W}}(\zeta)$$

and

$$k_\zeta = k_\zeta^W, \tilde{k}_\zeta = k_\zeta^{\tilde{W}}.$$

Proof of Theorem 1. Assume that $HP(W) \circ \pi = HP(\tilde{W})$. Let ζ be an arbitrary point in Δ_1 . We need to show that $\tilde{\Delta}_1(\zeta)$ consists of a single point. Take a point $\tilde{\zeta} \in \tilde{\Delta}_1(\zeta)$. By Proposition 2.2 (iii), there exists a positive constant c such that

$$(3.1) \quad c\tilde{k}_{\tilde{\zeta}} \leq k_\zeta \circ \pi$$

on $\tilde{W}$. By assumption, there exists an $h \in HP(W)$ such that

$$(3.2) \quad \tilde{k}_{\tilde{\zeta}} = h \circ \pi$$

on $\tilde{W}$. Hence, by (3.1), we see that $ch \leq k_\zeta$ on W . This with minimality of k_ζ implies that there exists a positive constant c_1 such that

$$(3.3) \quad h = c_1 k_\zeta$$

on W . Hence, by (3.2), we see that $\tilde{k}_{\tilde{\zeta}} = c_1 k_\zeta \circ \pi$ on $\tilde{W}$. From this with (2.1) and (2.2), it follows that $c_1 = 1$. Therefore we obtain

$$(3.4) \quad \tilde{k}_{\tilde{\zeta}} = k_\zeta \circ \pi$$

on $\tilde{W}$. This yields that $\tilde{\Delta}_1(\zeta) = \{\tilde{\zeta}\}$.

Conversely, assume that $\nu_{\tilde{W}}(\zeta) = 1$ for every $\zeta \in \Delta_1$. We only need to show $HP(\tilde{W}) \subset HP(W) \circ \pi$, since the reversed inclusion is trivial. By assumption, we set $\tilde{\Delta}_1(\zeta) = \{\tilde{\zeta}\}$ for each $\zeta \in \Delta_1$. By Proposition 2.2 (iii) and (2.4), we have

$$(3.5) \quad \tilde{k}_{\tilde{\zeta}} = k_\zeta \circ \pi$$

for every $\zeta \in \Delta_1$. Take an arbitrary $\tilde{h}$ in $HP(\tilde{W})$. By the Martin representation theorem (cf. e.g. [1], [2] and [3]), there exists a Radon measure $\tilde{\mu}$ on $\tilde{\Delta}$ with $\tilde{\mu}(\tilde{\Delta} \setminus \tilde{\Delta}_1) = 0$ such that

$$(3.6) \quad \tilde{h} = \int \tilde{k}_{\tilde{\zeta}} d\tilde{\mu}(\tilde{\zeta}).$$

Choose arbitrary two points $\tilde{z}_1$ and $\tilde{z}_2$ in $\tilde{W}$ with $\pi(\tilde{z}_1) = \pi(\tilde{z}_2)$. In view of (3.5) and (3.6), we obtain

$$\tilde{h}(\tilde{z}_1) = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z}_1) d\tilde{\mu}(\tilde{\zeta}) = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z}_2) d\tilde{\mu}(\tilde{\zeta}) = \tilde{h}(\tilde{z}_2).$$

Therefore we deduce that $\tilde{h} \in HP(W) \circ \pi$ for every $\tilde{h} \in HP(\tilde{W})$, and hence $HP(\tilde{W}) \subset HP(W) \circ \pi$.

The proof is herewith complete. $\square$

4. Proof of Theorem 2

In this section, we give the proof of Theorem 2. Let $\omega_z(\cdot)$ ($\tilde{\omega}_{\tilde{z}}(\cdot)$, resp.) be the harmonic measure on Δ ($\tilde{\Delta}$, resp.) with respect to W ($\tilde{W}$, resp.) and $z \in W$ ($\tilde{z} \in \tilde{W}$, resp.). It is well-known that harmonic measure is a Radon measure (cf. e.g. [C-C]). It is also well-known that $\omega_z(\cdot)$ ($\tilde{\omega}_{\tilde{z}}(\cdot)$, resp.) can be extended to the outer measure on Δ ($\tilde{\Delta}$, resp.) by

$$\omega_z(E) = \inf\{\omega_z(B) : B \text{ is a Borel set with } E \subset B\}$$

$$(\tilde{\omega}_{\tilde{z}}(\tilde{E}) = \inf\{\tilde{\omega}_{\tilde{z}}(\tilde{B}) : \tilde{B} \text{ is a Borel set with } \tilde{E} \subset \tilde{B}\}, \text{ resp.})$$

for a subset E ($\tilde{E}$, resp.) of Δ ($\tilde{\Delta}$, resp.). It is known that $h(z) = \omega_z(E)$ is a nonnegative harmonic function on W for every $E \subset \Delta$. By minimum principle, it is obvious that, for an arbitrary $E(\subset \Delta)$ ($\tilde{E} \subset \tilde{\Delta}$, resp.), $\omega_z(E) = 0$ ($\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$, resp.) for a $z \in W$ ($\tilde{z} \in \tilde{W}$, resp.) if and only if $\omega_z(E) = 0$ ($\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$, resp.) for all $z \in W$ ($\tilde{z} \in \tilde{W}$, resp.). Let f be a real-valued function on the Martin boundary Δ^R of an open Riemann surface R . We denote by H_f^R ($\tilde{H}_f^R$, resp.) the solution (upper solution, resp.) of Dirichlet problem on $R(= W \text{ or } \tilde{W})$ with boundary values f in the sense of Perron-Wiener-Brelot. We first prove the following.

Lemma 4.1. *Let $\tilde{E}$ be a subset of $\tilde{\Delta}$. Then $\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$ if and only if $\omega_z(\pi(\tilde{E})) = 0$.*

Proof. Suppose that $\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$. By definition, there exists a Borel set $\tilde{B} \subset \tilde{\Delta}$ with $\tilde{E} \subset \tilde{B}$ such that

$$(4.1) \quad \tilde{\omega}_{\tilde{z}}(\tilde{B}) = H_{1_{\tilde{B}}}^{\tilde{W}}(\tilde{z}) = 0,$$

where $1_{\tilde{B}}$ is the characteristic function of $\tilde{B}$ on $\tilde{\Delta}$. Let $\tilde{s}$ be an arbitrary positive superharmonic function on $\tilde{W}$ such that $\liminf_{\tilde{z} \rightarrow \tilde{\zeta}} \tilde{s}(\tilde{z}) \geq 1$ for every $\tilde{\zeta} \in \tilde{B}$. Set

$$s(z) := \sum_{\tilde{z} \in \pi^{-1}(z)} m(\tilde{z}) \tilde{s}(\tilde{z}),$$

where $m(\tilde{z})$ is multiplicity of π at $\tilde{z}$. Then $s(z)$ is a positive superharmonic function on W and $\liminf_{z \rightarrow \zeta} s(z) \geq 1$ for every $\zeta \in \pi(\tilde{B})$. Hence $s(z) \geq \overline{H}_{1_{\pi(\tilde{B})}}^W(z)$. From this and the fact $\overline{H}_{1_{\pi(\tilde{B})}}^W(z) \geq \omega_z(\pi(\tilde{B}))$ (cf. e.g. [C-C]), it follows that

$$s(z) \geq \omega_z(\pi(\tilde{B})) \geq \omega_z(\pi(\tilde{E})).$$

Therefore, by letting $s(z)$ arbitrarily small in view of (4.1), we obtain $\omega_z(\pi(\tilde{E})) = 0$.

Suppose $\omega_z(\pi(\tilde{E})) = 0$. By definition, there exists a Borel set $B \subset \Delta$ with $B \supset \pi(\tilde{E})$ such that

$$(4.2) \quad \omega_z(B) = H_{1_B}^W(z) = 0.$$

Let s be an arbitrary positive superharmonic function on W such that $\liminf_{z \rightarrow \zeta} s(z) \geq 1$ for every $\zeta \in B$. Then $s \circ \pi(\tilde{z})$ is a positive superharmonic function on $\tilde{W}$ and $\liminf_{\tilde{z} \rightarrow \tilde{\zeta}} s \circ \pi(\tilde{z}) \geq 1$ for every $\tilde{\zeta} \in \pi^{-1}(B)$. Hence $s \circ \pi(\tilde{z}) \geq \bar{H}_{1_{\pi^{-1}(B)}}^{\tilde{W}}(\tilde{z})$. From this and the fact $\bar{H}_{1_{\pi^{-1}(B)}}^{\tilde{W}}(\tilde{z}) \geq \tilde{\omega}_{\tilde{z}}(\pi^{-1}(B))$, it follows that

$$s \circ \pi(\tilde{z}) \geq \tilde{\omega}_{\tilde{z}}(\pi^{-1}(B)) \geq \tilde{\omega}_{\tilde{z}}(\pi^{-1}(\pi(\tilde{E}))) \geq \tilde{\omega}_{\tilde{z}}(\tilde{E}).$$

Therefore, letting $s \circ \pi(\tilde{z})$ arbitrarily small in view of (4.2), we obtain $\tilde{\omega}_{\tilde{z}}(\tilde{E}) = 0$.

The proof is herewith complete. $\square$

We next consider the sets

$$N_1 := \{\zeta \in \Delta_1 : \nu_{\tilde{W}}(\zeta) = 1\}$$

and

$$N_2 := \Delta_1 \setminus N_1 = \{\zeta \in \Delta_1 : \nu_{\tilde{W}}(\zeta) \geq 2\}.$$

Put $\tilde{N}_1 = \pi^{-1}(N_1) \cap \tilde{\Delta}_1$ and $\tilde{N}_2 = \pi^{-1}(N_2) \cap \tilde{\Delta}_1$. By means of Proposition 2.2, it is easily seen that $\tilde{N}_1 \cup \tilde{N}_2 = \tilde{\Delta}_1$ and $\pi(\tilde{N}_i) = N_i$ ($i = 1, 2$). We denote by $\tilde{d}(\cdot, \cdot)$ the metric on $\tilde{W}^*$ defined by

$$d(z, \zeta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \frac{k_z(z_n)}{1 + k_z(z_n)} - \frac{k_{\zeta}(z_n)}{1 + k_{\zeta}(z_n)} \right|,$$

where $\{z_n : n = 1, 2, \dots\}$ is a dense subset of $\tilde{W}$. Set $\tilde{U}_r(\tilde{z}_0) = \{\tilde{z} \in \tilde{W}^* : \tilde{d}(\tilde{z}, \tilde{z}_0) < r\}$ for $\tilde{z}_0 \in \tilde{W}^*$ and $r > 0$.

Lemma 4.2. *Suppose $\omega_z(N_2) > 0$. Then there exists a $\tilde{\zeta}_0 \in \tilde{N}_2$ such that $\tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_r(\tilde{\zeta}_0)) > 0$ for every $r > 0$.*

Proof. By virtue of Lemma 4.1, we have $\tilde{\omega}_{\tilde{z}}(\tilde{N}_2) > 0$, since $\pi(\tilde{N}_2) = N_2$. Contrary to the assertion, assume that, for every $\tilde{\zeta} \in \tilde{N}_2$, there exists an $r_{\tilde{\zeta}} > 0$ such that $\tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_{r_{\tilde{\zeta}}}(\tilde{\zeta})) = 0$. Then, by the Lindelöf covering theorem, there exists a sequence $\{\tilde{\zeta}_j\}_{j=1}^{\infty}$ in $\tilde{N}_2$ such that $\tilde{N}_2 \subset \bigcup_{j=1}^{\infty} \tilde{U}_{r_{\tilde{\zeta}_j}}(\tilde{\zeta}_j)$. Hence we have

$$\tilde{\omega}_{\tilde{z}}(\tilde{N}_2) \leq \sum_{j=1}^{\infty} \tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_{r_{\tilde{\zeta}_j}}(\tilde{\zeta}_j)) = 0,$$

which is a contradiction. $\square$

Here, we again recall the definition of $\tilde{\Delta}_1(\zeta)$:

$$\tilde{\Delta}_1(\zeta) = \tilde{\Delta}_1 \cap \pi^{-1}(\zeta) = \{\tilde{\zeta} \in \tilde{\Delta}_1 : \pi(\tilde{\zeta}) = \zeta\}.$$

Lemma 4.3. *Let $\tilde{\xi}$ be a point in $\tilde{N}_2$. Then there exists a $\rho > 0$ such that $\tilde{\Delta}_1(\zeta) \setminus \tilde{U}_\rho(\tilde{\xi})$ is not empty for every $\zeta \in N_2 \cap \pi(\tilde{U}_\rho(\tilde{\xi}))$.*

Proof. Set $\pi(\tilde{\xi}) = \xi$. Then, by definition, $\xi \in N_2$. Assume that the assertion is false. Then there exists a sequence $\{\zeta_j\}_{j=1}^\infty$ in $N_2 \setminus \{\pi(\tilde{\xi})\}$ such that

$$(4.3) \quad \tilde{d}(\tilde{\Delta}_1(\zeta_j), \tilde{\xi}) < 1/j.$$

From this it follows that

$$(4.4) \quad \lim_{j \rightarrow \infty} k_{\zeta_j} = k_\xi.$$

By Proposition 2.2 and (2.4), for each j , there exist positive constants $c_{j1}, \dots, c_{jn_j}$ with $\sum_{i=1}^{n_j} c_{ji} = 1$ such that

$$(4.5) \quad k_{\zeta_j} \circ \pi = \sum_{i=1}^{n_j} c_{ji} \tilde{k}_{\tilde{\zeta}_{ji}},$$

where $\tilde{\Delta}_1(\zeta_j) = \{\tilde{\zeta}_{j1}, \dots, \tilde{\zeta}_{jn_j}\}$. Then, in view of (4.3), we see that

$$\lim_{j \rightarrow \infty} \tilde{k}_{\tilde{\zeta}_{ji_j}} = \tilde{k}_{\tilde{\xi}}$$

independently of choice of i_j in $\{1, \dots, n_j\}$. This with (4.4) and (4.5) implies that

$$k_\xi \circ \pi = \tilde{k}_{\tilde{\xi}}.$$

Therefore, by means of Proposition 2.2, we obtain $\tilde{\Delta}_1(\xi) = \{\tilde{\xi}\}$, which contradicts $\xi \in N_2$. This completes the proof. $\square$

We can restate Theorem 2, in terms of the set N_2 , as follows: *The relation $HB(W) \circ \pi = HB(\tilde{W})$ holds if and only if $\omega_z(N_2) = 0$.*

Proof of Theorem 2. We first prove ‘if’ part. Suppose $\omega_z(N_2) = 0$. Then, by Lemma 4.1,

$$(4.6) \quad \tilde{\omega}_{\tilde{z}}(\tilde{N}_2) = 0.$$

Take an arbitrary $\tilde{h} \in HB(\tilde{W})$. We only need to show $\tilde{h} \in HB(W) \circ \pi$. Adding a constant to $\tilde{h}$, we may assume that $\tilde{h} > 0$ on $\tilde{W}$. Let $c(> 0)$ be the supremum of $\tilde{h}$ on

$\widetilde{W}$. By the Martin representation theorem, there exist Radon measures $\tilde{\mu}$ and $\tilde{\chi}$ on $\tilde{\Delta}$ with $\tilde{\mu}(\tilde{\Delta} \setminus \tilde{\Delta}_1) = 0$ and $\tilde{\chi}(\tilde{\Delta} \setminus \tilde{\Delta}_1) = 0$ such that

$$(4.7) \quad \tilde{h}(\tilde{z}) = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\mu}(\tilde{\zeta})$$

and

$$(4.8) \quad 1 = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\chi}(\tilde{\zeta}).$$

Then

$$c \int \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\chi}(\tilde{\zeta}) = c \geq \tilde{h}(\tilde{z}) = \int \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\mu}(\tilde{\zeta}).$$

Hence, by uniqueness of representing measure, we have

$$(4.9) \quad c\tilde{\chi} \geq \tilde{\mu}.$$

Note that $\tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\chi}(\tilde{\zeta}) = d\tilde{\omega}_{\tilde{z}}(\tilde{\zeta})$ (cf. [C-C, p.140]). From this and (4.9) it follows that

$$\int_{\tilde{N}_2} \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\mu}(\tilde{\zeta}) \leq c \int_{\tilde{N}_2} \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\chi}(\tilde{\zeta}) = c \int_{\tilde{N}_2} d\tilde{\omega}_{\tilde{z}}(\tilde{\zeta}) = c\tilde{\omega}_{\tilde{z}}(\tilde{N}_2).$$

This with (4.6) yields that

$$\int_{\tilde{N}_2} \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\mu}(\tilde{\zeta}) = 0.$$

Therefore, by (4.7) and the fact $\tilde{N}_1 \cup \tilde{N}_2 = \tilde{\Delta}_1$, we have

$$\tilde{h}(\tilde{z}) = \int_{\tilde{N}_1} \tilde{k}_{\tilde{\zeta}}(\tilde{z}) d\tilde{\mu}(\tilde{\zeta}).$$

Since $\tilde{k}_{\tilde{\zeta}} \in HP(W) \circ \pi$ for every $\tilde{\zeta} \in \tilde{N}_1$, this implies that $\tilde{h} \in HP(W) \circ \pi \cap HB(\widetilde{W}) \subset HB(W) \circ \pi$.

We next prove ‘only if’ part. Suppose $\omega_z(N_2) > 0$. Then, by Lemma 4.2, there exists a $\tilde{\xi} \in \tilde{N}_2$ such that

$$(4.10) \quad \tilde{\omega}_{\tilde{z}}(\tilde{N}_2 \cap \tilde{U}_r(\tilde{\xi})) > 0$$

for every $r > 0$. Moreover, by Lemma 4.3, there exists $\rho > 0$ such that

$$(4.11) \quad \tilde{\Delta}_1(\zeta) \setminus \tilde{U}_{\rho}(\tilde{\xi}) \neq \emptyset$$

for every $\zeta \in N_2 \cap \pi(\tilde{U}_{\rho}(\tilde{\xi}))$. Set

$$\tilde{E}_1 = \tilde{N}_2 \cap \tilde{U}_{\rho/2}(\tilde{\xi}).$$

Then, by (4.10) and Lemma 4.1, we have

$$(4.12) \quad \omega_z(\pi(\tilde{E}_1)) > 0.$$

Set

$$\tilde{E}_2 = \tilde{N}_2 \cap \pi^{-1}(\pi(\tilde{U}_{\rho/2}(\tilde{\xi}))) \setminus \tilde{U}_{\rho}(\tilde{\xi}).$$

Inview of (4.11), we find that

$$(4.13) \quad \pi(\tilde{E}_1) = \pi(\tilde{E}_2).$$

Put $\tilde{h}(\tilde{z}) = \tilde{\omega}_{\tilde{z}}(\tilde{E}_1)$. Then $\tilde{h}(\tilde{z})$ is a bounded harmonic function on $\tilde{W}$. We only need to show $\tilde{h} \notin HB(W) \circ \pi$. By the Fatou-Naim-Dood theorem (cf. [C-C, p.152]), $\tilde{h}(\tilde{z})$ has fine limit 1 (0, resp.) at almost all $\tilde{\zeta}$ in $\tilde{E}_1$ ($\tilde{E}_2$, resp.) with respect to $\tilde{\omega}_{\tilde{z}}$, since $\tilde{E}_1 \cap \tilde{E}_2 = \emptyset$. Accordingly there exists a subset $\tilde{F}_1$ ($\tilde{F}_2$, resp.) of $\tilde{E}_1$ ($\tilde{E}_2$, resp.) with $\tilde{\omega}_{\tilde{z}}(\tilde{F}_1) = 0$ ($\tilde{\omega}_{\tilde{z}}(\tilde{F}_2) = 0$, resp.) such that, for every $\tilde{\zeta}$ in $\tilde{E}_1 \setminus \tilde{F}_1$ ($\tilde{E}_2 \setminus \tilde{F}_2$, resp.),

$$(4.14) \quad \mathcal{F} - \lim_{\tilde{z} \rightarrow \tilde{\zeta}} \tilde{h}(\tilde{z}) = 1 \quad (\mathcal{F} - \lim_{\tilde{z} \rightarrow \tilde{\zeta}} \tilde{h}(\tilde{z}) = 0, \text{ resp.})$$

Then, by Lemma 4.1, $\omega_z(\pi(\tilde{F}_1) \cup \pi(\tilde{F}_2)) = 0$. Hence, by (4.12) and (4.13), there exist points $\tilde{\zeta}_1 \in \tilde{E}_1 \setminus \tilde{F}_1$ and $\tilde{\zeta}_2 \in \tilde{E}_2 \setminus \tilde{F}_2$ with $\pi(\tilde{\zeta}_1) = \pi(\tilde{\zeta}_2)$. This with (4.14) implies that there exists an open subset $\tilde{O}_1$ ($\tilde{O}_2$, resp.) of $\tilde{W}$ such that $\tilde{O}_1 \cup \{\tilde{\zeta}_1\}$ ($\tilde{O}_2 \cup \{\tilde{\zeta}_2\}$, resp.) is a minimal fine neighborhood of $\tilde{\zeta}_1$ ($\tilde{\zeta}_2$, resp.) and that

$$(4.15) \quad \inf_{\tilde{z} \in \tilde{O}_1} \tilde{h}(\tilde{z}) \geq \frac{2}{3} \quad (\sup_{\tilde{z} \in \tilde{O}_2} \tilde{h}(\tilde{z}) \leq \frac{1}{3}, \text{ resp.}).$$

Then, by virtue of Proposition 2.3, we see that $(\pi(\tilde{O}_1) \cap \pi(\tilde{O}_2)) \cup \{\pi(\tilde{\zeta}_1)\}$ is a minimal fine neighborhood of $\pi(\tilde{\zeta}_1) = \pi(\tilde{\zeta}_2)$, and hence $\pi(\tilde{O}_1) \cap \pi(\tilde{O}_2) \neq \emptyset$. Therefore, by (4.15), there exists a subset $\tilde{U}_j$ of $\tilde{O}_j$ ($j = 1, 2$) with $\pi(\tilde{U}_1) = \pi(\tilde{U}_2)$ such that

$$\inf_{\tilde{z} \in \tilde{U}_1} \tilde{h}(\tilde{z}) \geq \frac{2}{3} \quad (\sup_{\tilde{z} \in \tilde{U}_2} \tilde{h}(\tilde{z}) \leq \frac{1}{3}, \text{ resp.}).$$

This means that $\tilde{h} \notin HB(W) \circ \pi$.

The proof is herewith complete. □

5. Harmonic functions on covering surfaces of the unit disc

Let D be the unit disc $\{|z| < 1\}$. In this section, we are concerned with application of Theorem 1 and Thorem 2 in case base surface is D . As is wellknown, the Martin compactification D^* of D is identified with the closure $\overline{D}$ of D with respect to Euclidian topology and the Martin boundary Δ^D of D consists of only minimal points. In this view, we regard $\partial D = \{|z| = 1\}$ as the (minimal) Martin boundary of D .

To state our main result of this section, we introduce some notations. For a discrete subset A of D , we denote by $\mathcal{B}_p(A)$ the class of p -sheeted unlimited covering surface $\tilde{D}$ of D such that there exists a branch point in $\tilde{D}$ of order $p - 1$ (or multiplicity p) over every

$z \in A$ and there exist no branch points in $\widetilde{D}$ over $D \setminus A$. In addition to the Euclidean metric, we consider the pseudohyperbolic metric on D given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|.$$

For $\zeta \in \partial D$ and a positive number $C(< 1)$, we also consider the Stolz type domain with vertex ζ given by

$$S_C(\zeta) = \{z \in D : C|z - \zeta| < 1 - |z|\}.$$

Theorem 5.1. *Let $A = \{a_n : n \in \mathbf{N}\}$ be a discrete subset of D and $\widetilde{D}$ belong to $\mathcal{B}_p(A)$. Suppose that there exists a positive constant $C(< 1)$ satisfying the following two conditions*

- (i) *for every pair (a_m, a_n) in A with $a_m \neq a_n$, $\rho(a_m, a_n) \geq C$;*
- (ii) *for every $\zeta \in \partial D$, there exists a subset $B_\zeta = \{b_n : n \geq n_0\}$ ($n_0 = n_0(\zeta)$) of A such that $b_n \in \{z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n}\} \cap S_C(\zeta)$ for every $n \geq n_0$.*

Then $HP(\widetilde{D}) = HP(D) \circ \pi$, where π is the projection map.

For a bounded Borel subset K of $\mathbf{C}$, we denote by $\lambda(K)$ the logarithmic capacity. As a necessary condition for minimal thinness, the following is available (cf. [LF],[J]).

Lemma 5.1. *Let ζ be in $\partial D = \Delta^D$ and E a relatively closed subset of $S_C(\zeta)$. If E is minimally thin at ζ , then*

$$\sum_{n=1}^{\infty} \frac{1}{\log \frac{1}{\lambda(E_n)}} < \infty,$$

where $E_n = E \cap \{z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n}\}$.

Proof of Theorem 5.1. Let ζ be an arbitrary point in ∂D . By virtue of Theorem 1, we only have to show that $\Delta_1^{\widetilde{D}}(\zeta)$ consists of a single point. Take an arbitrary $M \in \mathcal{M}_D(\zeta)$. Our goal is to show that $\pi^{-1}(M)$ is connected. In fact, in view of Proposition 2.4, connectivity of $\pi^{-1}(M)$ for all $M \in \mathcal{M}_D(\zeta)$ implies $\Delta_1^{\widetilde{D}}(\zeta)$ consists of a single point.

We first assume that there exists an $a_n \in M \cap A \neq \emptyset$. Then, it is easily seen that $\pi^{-1}(M)$ is connected, since $\widetilde{D}$ has a branch point of order $p - 1$ over $a_n \in M$ and M is connected.

We next assume $M \cap A = \emptyset$. Put $F = D \setminus M$. Note that F is minimally thin at ζ and relatively closed in D . For each $n(\geq n_0)$, let F_n be the connected component of F which contains $b_n \in B_\zeta$. We also assume that there exists an F_n ($n \geq n_0$) such that

$$(5.1) \quad d(F_n) < C^2 2^{-n-1},$$

where $d(F_n)$ indicates the diameter of F_n . Then there exists a closed Jordan curve γ_n in $M \setminus A$ such that γ_n surrounds F_n and

$$(5.2) \quad d(F_n) < d(\gamma_n) < C^2 2^{-n-1}.$$

By (i) and (ii), we have

$$|a_m - b_n| \geq C|1 - \overline{b_n}a_m| \geq C(1 - |b_n|) \geq C^2 2^{-n-1},$$

for every $a_m \in A \setminus \{b_n\}$. Hence, by means of (5.2), we see that γ_n surrounds only one point b_n in A . Therefore, $\pi^{-1}(\gamma_n)$ is connected, since $\widetilde{D}$ has a branch point of order $p-1$ over b_n . This with $\gamma_n \in M$ and connectivity of M yields that $\pi^{-1}(M)$ is connected. Accordingly, we complete the proof if we show that there exists an F_n ($n \geq n_0$) satisfying (5.1).

We assume that

$$(5.3) \quad d(F_n) \geq C^2 2^{-n-1}$$

for every $n(\geq n_0)$. Set $E = F \cap S_{\frac{\varepsilon}{2}}(\zeta)$. Note that E is minimally thin at ζ . We denote by F_n^* the connected component of E which contains b_n . Then, in view of (ii) and (5.3), we find that there exists a positive constant $C_1(\leq C^2/2)$ such that

$$(5.4) \quad d(F_n^*) \geq C_1 2^{-n}$$

for every $n(\geq n_0)$. Set $E_n = E \cap \{z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n}\}$. Note that $b_n \in E_n$. Then, by (5.4), we see that, for every $n \geq n_0$, at least one of $\{E_{n-1}, E_n, E_{n+1}\}$ contains a continuum whose diameter is equal to or greater than $C_1 2^{-n-1}$. From this it follows that

$$\max\{\lambda(E_{n-1}), \lambda(E_n), \lambda(E_{n+1})\} \geq C_1 2^{-n-3}$$

for every $n(\geq n_0)$ (cf.[T]). Hence we see that

$$\frac{1}{\log \frac{1}{\lambda(E_{n-1})}} + \frac{1}{\log \frac{1}{\lambda(E_n)}} + \frac{1}{\log \frac{1}{\lambda(E_{n+1})}} \geq \frac{1}{n \log 2 + \log(8/C_1)}$$

for every $n(\geq n_0)$. Therefore we deduce

$$\begin{aligned} \sum_{n=n_0-1}^{\infty} \frac{1}{\log \frac{1}{\lambda(E_n)}} &\geq \frac{1}{3} \sum_{n=n_0}^{\infty} \left(\frac{1}{\log \frac{1}{\lambda(E_{n-1})}} + \frac{1}{\log \frac{1}{\lambda(E_n)}} + \frac{1}{\log \frac{1}{\lambda(E_{n+1})}} \right) \\ &\geq \frac{1}{3} \sum_{n=n_0}^{\infty} \frac{1}{n \log 2 + \log(8/C_1)} = \infty. \end{aligned}$$

By Lemma 5.1, this is absurd, since E is minimally thin at ζ .

The proof is herewith complete. □

Using the notation above, we restate Proposition in Introduction as follows:

Corollary 5.1. *Let $A = \{(1 - 2^{-n-1})e^{i2\pi k/2^{n+2}} : n = 1, 2, \dots, k = 1, \dots, 2^{n+2}\}$ and $\widetilde{D}$ belong to $\mathcal{B}_p(A)$. Then $HP(D) \circ \pi = HP(\widetilde{D})$, where π is the projection map.*

Proof. It is easily seen that A and a positive constant C satisfy the condition (i) of Theorem 5.1. Let ζ be an arbitrary point in ∂D . For every positive integer n , we can choose a positive integer k_n with $1 \leq k_n \leq 2^{n+2}$ such that

$$(5.5) \quad \left| \arg \zeta - \frac{2\pi k_n}{2^{n+2}} \right| \leq \frac{\pi}{2^{n+2}}.$$

Set

$$b_n = (1 - 2^{-n-1})e^{i2\pi k_n/2^{n+2}} \quad (n = 1, 2, \dots).$$

Then, by (5.5), we have

$$(2^{-n-1})^2 \leq |b_n - \zeta|^2 \leq (2^{-n-1})^2 + 4 \sin^2 \frac{\pi}{2^{n+3}}.$$

In view of this with (5.5), it is easily seen that $B_\zeta := \{b_n : n \geq 1\}$ and a positive constant C satisfy the condition (ii) of Theorem 5.1. $\square$

At the last, we give a p -sheeted unlimited covering surface $\widetilde{D}_1$ of D with projection map π such that $HB(D) \circ \pi = HB(\widetilde{D}_1)$ and $HP(D) \circ \pi \neq HP(\widetilde{D}_1)$. Let A be the same as in Corollary 5.1. Set $M = \{|z - \frac{1}{2}| < \frac{1}{2}\}$ and $A_1 = A \setminus M$. Consider a covering surface $D_1 \in \mathcal{B}_p(A_1)$ with projection map π . We now show that $HB(D) \circ \pi = HB(\widetilde{D}_1)$ and $HP(D) \circ \pi \neq HP(\widetilde{D}_1)$. As is proved in the proof of Corollary 5.1, A_1 and a positive constant C satisfy the following two conditions:

- (i) for every pair (a_m, a_n) in A_1 with $a_m \neq a_n$, $\rho(a_m, a_n) \geq C$;
- (ii) for every $\zeta \in \partial D \setminus \{1\}$, there exist a subset $B_\zeta = \{b_n : n \geq n_0\}$ ($n_0 = n_0(\zeta)$) of A_1 such that $b_n \in \{z : 2^{-n-1} \leq |z - \zeta| \leq 2^{-n}\} \cap S_C(\zeta)$ for every $n \geq n_0$.

Therefore the proof of Theorem 5.1 yields that $\nu_{\widetilde{D}_1}(\zeta) = 1$ for every $\zeta \in \partial D \setminus \{1\}$. Hence, by virtue of Theorem 2, we have $HB(D) \circ \pi = HB(\widetilde{D}_1)$. On the other hand, it is easily seen that M belongs to $\mathcal{M}_D(1)$ and $\pi^{-1}(M)$ consists of p components. Hence, by Proposition 2.2 and 2.4, $\nu_{\widetilde{D}_1}(1) = p(> 1)$. Therefore, by Theorem 1, we see that $HP(D) \circ \pi \neq HP(\widetilde{D}_1)$.

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