

On a probability distribution of a binomial type generated by a mean

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1. Means and paths. In this note, we use operator means, in particular, the Kubo-Ando mean [6] plays a central role: A binary operation $\mathfrak{m}$ on positive operators on a Hilbert space is called the *Kubo-Ando (operator) mean* if $\mathfrak{m}$ satisfies the following axioms:

monotonicity: $A \leq C, B \leq D \Rightarrow A \mathfrak{m} B \leq C \mathfrak{m} D.$

semicontinuity: $A_n \downarrow A, B_n \downarrow B \Rightarrow A_n \mathfrak{m} B_n \downarrow A \mathfrak{m} B.$

transformer inequality: $T^*(A \mathfrak{m} B)T \leq T^*AT \mathfrak{m} T^*BT.$

normalization: $A \mathfrak{m} A = A.$

By semicontinuity, we may assume positive operators are invertible. The *representing function* $f_{\mathfrak{m}}(x) = 1 \mathfrak{m} x$ for a Kubo-Ando mean $\mathfrak{m}$ is operator monotone (concave) on $(0, \infty)$ and $\mathfrak{m}$ is represented by

$$A \mathfrak{m} B = A^{\frac{1}{2}} f_{\mathfrak{m}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

A *path* $A \mathfrak{m}_t B$ means parametrized operator means which is usually differentiable for t with $A \mathfrak{m}_0 B = A$ and $A \mathfrak{m}_1 B = B$. A path is called *symmetric* if

$$A \mathfrak{m}_t B = B \mathfrak{m}_{1-t} A$$

holds for all $t \in [0, 1]$. Typical example is (*quasi-arithmetic*) *power means* for $r \in [-1, 1]$:

$$A \#_{r,t} B = A^{\frac{1}{2}} \left((1-t)I + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r \right)^{\frac{1}{r}} A^{\frac{1}{2}},$$

which include important means:

arithmetic mean: $A \nabla_t B = A \#_{1,t} B = (1-t)A + tB$

geometric mean: $A \#_t B = A \#_{0,t} B \equiv \lim_{\varepsilon \rightarrow 0} A \#_{\varepsilon,t} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$

harmonic mean: $A !_t B = A \#_{-1,t} B = ((1-t)A^{-1} + tB^{-1})^{-1}.$

Moreover the above paths are *interpolational* in the sense that

$$(A \#_{r,p} B) \#_{r,t} (A \#_{r,q} B) = A \#_{r,(1-t)p+ tq} B$$

for all $p, q, t \in [0, 1]$.

2. Thompson metric. Let $\mathcal{A}^+$ be the positive invertible elements in a unital C^* -algebra $\mathcal{A}$, which is discussed as differentiable manifold by Corach-Porta-Recht [3, ?]. Corach himself reformulated it in [4]. They showed the above manifold $\mathcal{A}^+$ is the Finsler space with a Finsler metric

$$L(X; A) = \|X\|_A = \|A^{-1/2} X A^{-1/2}\| :$$

Then the geodesic is the shortest path with respect to this metric: The length $\ell(\gamma)$ of path $\gamma(t)$ is defined by

$$\ell(\gamma) \equiv \int_0^1 L(\gamma'(t); \gamma(t)) dt = \int_0^1 \|\gamma(t)^{-1/2} \gamma'(t) \gamma(t)^{-1/2}\| dt.$$

If $\gamma(t)$ is a path from A to B , then

$$\begin{aligned} d(A, B) &\equiv \inf_{\gamma} \ell(\gamma) = \ell(A \#_t B) = \|\log(A^{-1/2} B A^{-1/2})\| \\ &= \log(\max\{\|A^{-1/2} B A^{-1/2}\|, \|B^{-1/2} A B^{-1/2}\|\}) \\ &= \log(\max\{r(A^{-1} B), r(B^{-1} A)\}). \end{aligned}$$

Also the homogeneity of $\mathcal{A}^+$ implies

$$d(A, B) = d(X^* A X, X^* B X) = d(I, A^{-1/2} B A^{-1/2})$$

for invertible X . The metric d makes $\mathcal{A}^+$ a complete metric space and it is called the *Thompson (part) one* [12, 10].

3. Lawson-Lim's operator mean. Recently, Lawson-Lim [8, 9, 7] defines multivariable operator means parametrized by $t \in [0, 1]$ which is an extension of Ando-Li-Mathius' geometric operator mean [1]: For a symmetric path $\mathbf{m}_t$ in Kubo-Ando means, it is defined inductively:

$$(n = 2): \quad \mathbf{m}[2, t](A_1, A_2) = A_1 \mathbf{m}_t A_2$$

$$(n + 1): \quad \mathbf{m}[n + 1, t](A_1, \dots, A_{n+1}) = \lim_{r \rightarrow \infty} A_{\mathbf{m}}(r)_k \text{ if the limit exists}$$

$$\text{where} \quad \begin{cases} A_{\mathbf{m}}(r)_k = \mathbf{m}[n, t]((A_{\mathbf{m}}(r-1)_j)_{j \neq k}) \\ (A_{\mathbf{m}}(1)_k = A_k). \end{cases}$$

Then they showed that $\#[n, t](A_1, \dots, A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathius' one for $t = 1/2$. In [5], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n, t](A_1, \dots, A_n) = \sum_{k=1}^n t[n]_k A_k.$$

Also the harmonic mean is

$$! [n, t](A_1, \dots, A_n) = \left(\sum_{k=1}^n t[n]_k A_k^{-1} \right)^{-1}.$$

If A_k are commuting, then the geometric mean is

$$\#[n, t](A_1, \dots, A_n) = \prod_{k=1}^n A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1 \#_t B_1, A_2 \#_t B_2) \leq d(A_1, B_1) \nabla_t d(A_2, B_2)$$

of the Thompson metric:

$$\begin{aligned} d(\#[n, t](A_1, \dots, A_n), \#[n, t](B_1, \dots, B_n)) &\leq \nabla[n, t](d(A_1, B_1), \dots, d(A_n, B_n)) \\ &= \sum_{k=1}^n t[n]_k d(A_k, B_k), \end{aligned}$$

which shows the existence of the Lawson-Lim geometric mean.

Then we obtain the formulae for $t[n]_k$ in [5]:

Lemma.

$$\begin{aligned} t[n]_n &= \frac{t}{1 + (n-2)t} \\ t[n]_1 &= \frac{1-t}{1 + (n-2)(1-t)} = \frac{1-t}{(n-1) - (n-2)t} \end{aligned}$$

Theorem.

$$\begin{aligned} \text{(i)} \quad t[n]_{n-m} &= \frac{m(m+1) + 2m(n-2m-2)t + (n^2 - (4m+1)n + 4m(m+1))t^2}{(n-1)(m + (n-2m)t)(m+1 + (n-2(m+1))t)} \\ \text{(ii)} \quad \sum_{j>n-m-1} t[n]_j &= t[n]_n + \dots + t[n]_{n-m} = \frac{(m+1)(m + (n-2m-1)t)}{(n-1)(m+1 + (n-2m-2)t)}. \end{aligned}$$

Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j < k+1} t[n]_j = 1 - \frac{(n-k)(n-k-1+(2k-n+1)t)}{(n-1)(n-k+(2k-n)t)}.$$

Proof. Suppose the formula for $F_N(k)$ is valid for all k . Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \dots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^{n-1}$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^{n-1}$ and

$$a_{n+1} = a_1 - (1-v) \sum_{k=0}^{n-1} (v-w)^k \longrightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. Therefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. $\square$

Now we give the table for the density function $t[n]_k$:

$1-t$			t			
$\frac{1-t}{2-t}$		$\frac{1-t+t^2}{(2-t)(1+t)}$		$\frac{t}{1+t}$		
$\frac{1-t}{3-2t}$		$\frac{3-4t+2t^2}{3(3-2t)}$		$\frac{1+2t^2}{3(1+2t)}$		$\frac{t}{1+2t}$
$\frac{1-t}{4-3t}$	$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$	$\frac{3-2t+2t^2}{2(3-t)(2+t)}$	$\frac{1+t+4t^2}{2(2+t)(1+3t)}$		$\frac{t}{1+3t}$	
$\frac{1-t}{5-4t}$	$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$	$\frac{2-2t+t^2}{5(2-t)}$	$\frac{1+t^2}{5(1+t)}$	$\frac{1+2t+7t^2}{5(1+t)(1+4t)}$		$\frac{t}{1+4t}$
$\frac{1-t}{6-5t}$	$\frac{15-25t+11t^2}{3(5-3t)(6-5t)}$	$\frac{10-12t+5t^2}{3(4-t)(5-3t)}$	$\frac{2-t+t^2}{(4-t)(3+t)}$	$\frac{3+2t+5t^2}{3(3+t)(2+3t)}$	$\frac{1+3t+11t^2}{3(2+3t)(1+5t)}$	$\frac{t}{1+5t}$

The table for $t[n]_k$

Appendix : binomial mean $\mathbf{m}[n]_t$ for $\mathbf{m}_t$. From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

$$\mathbf{m}[2]_t(A_1, A_2) = A_1 \mathbf{m}_t A_2$$

$$\mathbf{m}[3]_t(A_1, A_2, A_3) = (\mathbf{m}[2]_t(A_1, A_2)) \mathbf{m}_t(\mathbf{m}[2]_t(A_2, A_3))$$

$$\mathbf{m}[n+1]_t(A_1, \dots, A_{n+1}) = (\mathbf{m}[n]_t(A_1, \dots, A_n)) \mathbf{m}_t(\mathbf{m}[n]_t(A_2, \dots, A_{n+1})).$$

This path is *symmetric* in the sense of

$$\mathbf{m}[n]_t(A_1, \dots, A_n) = \mathbf{m}[n]_{1-t}(A_n, \dots, A_1)$$

The binomial arithmetic mean is

$$\nabla[n]_t(A_1, \dots, A_n) = \sum_{k=1}^n {}_{n-1}C_{k-1} (1-t)^{n-k} t^{k-1} A_k,$$

and the barycenter is the usual arithmetic mean:

$$\int_0^1 \nabla[n]_t(A_1, \dots, A_n) dt = \sum_{k=1}^n {}_{n-1}C_{k-1} B(n-k+1, k) A_k = \frac{1}{n} \sum_{k=1}^n A_k$$

where $B(p, q)$ is the beta function. As in [11], a multivariable extension of *logarithmic mean*

$$L[2](a, b) = \frac{b-a}{\log b - \log a}$$

is a fascinating one. Considering

$$L[2](A, B) = \int_0^1 A \#_t B dt$$

holds in Kubo-Ando means, we might define

$$L[n](A_1, \dots, A_n) = \int_0^1 \# [n]_t(A_1, \dots, A_n) dt.$$

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