

The singularity on interpolation by rational spline functions

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Let P_n be the collection of all polynomials of degree n , and R_l^r the collection of all rational functions with the form $p(x)/q(x)$, where $p \in P_r$, and $q \in P_l$.

Denote by T the following partition of the interval $[a, b]$:

$$T: a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

If a real function $R(x)$ defined on $[a, b]$ satisfies

- 1° $R(x) \in R_l^r$, in each interval $[x_j, x_{j+1}]$;
- 2° $R(x) \in C^s[a, b]$,

then $R(x)$ is said to be a rational spline of type $-(r, l)^s$ with respect to the partition T .

In this paper, we shall discuss the rational splines of type $-(2, 1)^1$, and type $-(2, 1)^2$ with the forms

$$R(x) = p_{1j}(x) + \frac{(x - x_j)(x - x_{j+1})}{q_{1j}(x)}, \quad x_j \leq x \leq x_{j+1}, \quad j = 0, \cdots, n-1, \quad (1)$$

where $p_{1j}(x)$ and $q_{1j}(x) \in P_1$.

Suppose that the interpolation conditions are

$$\begin{cases} R(x_j) = y_j, & R'(x_j) = y'_j, \\ R(x_{j+1}) = y_{j+1}, & R'(x_{j+1}) = y'_{j+1}; \end{cases} \quad (2)$$

$$\begin{cases} R(x_j) = y_j, & R(x_{j+1}) = y_{j+1}, \\ R'(x_j) = [f(x_{j-1}, x_j) + f(x_j, x_{j+1})]/2, \\ R'(x_{j+1}) = [f(x_j, x_{j+1}) + f(x_{j+1}, x_{j+2})]/2; \end{cases} \quad (3)$$

$$\begin{cases} R(x_j) = y_j, & R(x_{j+1}) = y_{j+1}, \\ R'(x_j) = y'_j, & R''(x_j) = y''_j, \end{cases} \quad (4)$$

respectively, where $f(x_j, x_{j+1})$ denotes the divided difference of the first degree, etc.

Denote by $R_{1j}(x)$, $R_{2j}(x)$, and $R_{3j}(x)$ the rational spline functions of satisfying the interpolation conditions (1)-(2), (1)-(3), and (1)-(4) respectively.

R.H. Wang and S.T. Wu([1][2]) have obtained the following rational piecewise functions which are satisfying the interpolation conditions (1)-(2), (1)-(3), and (1)-(4) respectively,

$$R_{1j}(x) = y_j + f(x_j, x_{j+1})(x - x_j) + \frac{(x - x_j)(x - x_{j+1})[y'_j - f(x_j, x_{j+1})][y'_{j+1} - f(x_j, x_{j+1})]}{(x - x_j)[y'_j - f(x_j, x_{j+1})] + (x - x_{j+1})[y'_{j+1} - f(x_j, x_{j+1})]}, \quad (5)$$

$$R_{2j}(x) = y_j + f(x_j, x_{j+1})(x - x_j) + \frac{\{(x - x_j)(x - x_{j+1})[f(x_{j-1}, x_j) - f(x_j, x_{j+1})][f(x_{j+1}, x_{j+2}) - f(x_j, x_{j+1})]\}}{2[f(x_{j-1}, x_j) + f(x_{j+1}, x_{j+2}) - 2f(x_j, x_{j+1})]x - 2[f(x_{j-1}, x_j)x_j + f(x_{j+1}, x_{j+2})x_{j+1} - f(x_j, x_{j+1})(x_j + x_{j+1})]}, \quad (6)$$

$$R_{3j}(x) = y_j + f(x_j, x_{j+1})(x - x_j) + \frac{2[y'_j - f(x_j, x_{j+1})]^2(x - x_j)(x_{j+1} - x)}{2[y'_j - f(x_j, x_{j+1})](x_{j+1} - x) + y''_j(x_j - x_{j+1})(x - x_j)}. \quad (7)$$

Denote by $R_i(x)$ ($i = 1, 2, 3$) the rational spline functions:

$$R_i(x) = \{R(x) \in (2, 1)^1 \mid R(x)|_{[x_j, x_{j+1}]} = R_{ij}(x), \quad j = 0, \dots, n-1\}, \quad i = 1, 2;$$

$$R_3(x) = \{R(x) \in (2, 1)^2 \mid R(x)|_{[x_j, x_{j+1}]} = R_{3j}(x), \quad j = 0, \dots, n-1\}.$$

It is not hard to prove the following lemmas:

[Lemma 1] $R_1(x)$ has the singular point in the interval $[x_j, x_{j+1}]$, if and only if

$$\text{sign}\{[y'_j - f(x_j, x_{j+1})] \cdot [y'_{j+1} - f(x_j, x_{j+1})]\} > 0. \quad (8)$$

[Lemma 2] $R_2(x)$ has the singular point in the interval $[x_j, x_{j+1}]$, if and only if

$$\text{sign}\{[f(x_{j-1}, x_j) - f(x_j, x_{j+1})] \cdot [f(x_{j+1}, x_{j+2}) - f(x_j, x_{j+1})]\} > 0. \quad (9)$$

In fact, by the formula of $R_1(x)$ on $[x_j, x_{j+1}]$ given in (5), the singularity of $R_1(x)$ on $[x_j, x_{j+1}]$ can appear only at point

$$x^* = \frac{[y'_j - f(x_j, x_{j+1})]x_j + [y'_{j+1} - f(x_j, x_{j+1})]x_{j+1}}{[y'_j - f(x_j, x_{j+1})] + [y'_{j+1} - f(x_j, x_{j+1})]} := \lambda x_j + (1 - \lambda)x_{j+1},$$

where $\lambda = [y'_j - f(x_j, x_{j+1})]/[(y'_j - f(x_j, x_{j+1})) + (y'_{j+1} - f(x_j, x_{j+1}))]$. So, it is easy to see that $x^* \in (x_j, x_{j+1})$ if and only if

$$\text{sign}[(y'_j - f(x_j, x_{j+1})) \cdot (y'_{j+1} - f(x_j, x_{j+1}))] > 0.$$

By the similar argument, we can prove Lemma 2.

It notes that if $y'_j - f(x_j, x_{j+1})$ or $y'_{j+1} - f(x_j, x_{j+1}) = 0$, then $R_1(x)$ will be a linear function in the interval $[x_j, x_{j+1}]$, so it should have no singular point.

By using the above Lemmas, we have

[Theorem 1] Let the interpolation function $y = f(x) \in C^2[a, b]$. If $R_i(x)$ ($i = 1, 2$) exists the singular point in the interval $[x_j, x_{j+1}]$, then $y = f(x)$ has the inflection point in the open interval (x_j, x_{j+1}) .

Proof Let $R_1(x)$ exist the singular point in $[x_j, x_{j+1}]$. By Lemma 1, without loss the generality, suppose that the following inequalities hold

$$y'_j - f(x_j, x_{j+1}) > 0, \quad y'_{j+1} - f(x_j, x_{j+1}) > 0. \quad (10)$$

It follows Lagrange's mean value theorem, that there exists $\xi \in (x_j, x_{j+1})$, such that

$$f'(\xi) = f(x_j, x_{j+1}). \quad (11)$$

By (10) and (11), there exist η and ζ of satisfying

$$f'(x_j) - f(x_j, x_{j+1}) = f''(\eta)(x_j - \xi), \quad f'(x_{j+1}) - f(x_j, x_{j+1}) = f''(\zeta)(x_{j+1} - \xi)$$

respectively, where $x_j < \eta < \xi < \zeta < x_{j+1}$. Hence

$$f''(\eta) \cdot f''(\zeta) < 0,$$

and there exists at least one inflection point of $f(x)$ in (η, ζ) . This completes the proof of this theorem for $R_1(x)$.

Similarly we can prove the theorem for the case of $R_2(x)$.

[Theorem 2] Let the interpolation function $y = f(x) \in C^2[a, b]$. If $R_3(x)$ exists the singular point in the interval $[x_j, x_{j+1}]$, then the original interpolation function $y = f(x)$ has the inflection point in the open interval (x_j, x_{j+1}) .

In fact, because of the singular point of $R_3(x)$ may be only appearing at

$$\bar{x} = \frac{y_j''(x_{j+1} - x_j)x_j + 2(y_j' - f(x_j, x_{j+1}))x_{j+1}}{y_j''(x_{j+1} - x_j) + 2(y_j' - f(x_j, x_{j+1}))}.$$

By the same argument shown in the proof of Lemma 1, we have

$$\text{sign}\{[y_j''(x_{j+1} - x_j)] \cdot [y_j' - f(x_j, x_{j+1})]\} > 0,$$

provided that $R_3(x)$ exists the singular point in $[x_j, x_{j+1}]$.

It follows Lagrange's mean value theorem, that there exists $\xi \in (x_j, x_{j+1})$, such that

$$f'(\xi) = f(x_j, x_{j+1}).$$

By Lagrange's mean value theorem once again, there is a point $\eta \in (x_j, \xi)$, such that

$$f'(x_j) - f'(\xi) = f''(\eta)(x_j - \xi).$$

So

$$\text{sign}\{f''(x_j) \cdot f''(\eta)\} < 0.$$

Hence, there exists at least one inflection point of $f(x)$ in (x_j, η) .

This complete the proof of this theorem.

In addition, we may prove that although the interpolation function $f(x)$ has the inflection points in $[a, b]$, however, provided that we are taking all inflection points as the knots of the rational spline function, then the singularity can be avoided to appear. For example, let x_j be an inflection point of $f(x)$, x_{j+1} be not, and there is no another inflection point between x_j and x_{j+1} . Then the first derivative of $f(x)$ will be monotone in the interval $[x_j, x_{j+1}]$. So the inequality (8) will be not satisfied. By Lemma 1, hence, there is no singular point in (x_j, x_{j+1}) .

Therefore, we have

[Theorem 3] For any given interpolation function $f(x) \in C^2[a, b]$, we can construct a partition T of the interval $[a, b]$, such that the rational spline function $R_1(x)$, $R_2(x)$, and $R_3(x)$ based on the partition T have no singularity.

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References

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