

Ideas of proving symmetry of Kim-independence

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1 Introduction and Preliminaries

In [1], the notion of Kim-independence was introduced, and it was shown that $NSOP_1$ -theories are characterized as those theories for which Kim-independence has the symmetric property over models. In the proof of this characterization, the authors of [1] used Erdős-Rado theorem, which is a combinatorial set theoretic result on uncountable cardinals. In this article, we try to present a new proof of this fact only using Compactness theorem and Ramsey's theorem. We give an outline of the idea of the proof.

In this article, L is a language and T is a complete L -theory having an infinite model. For simplicity, we assume L is countable. We fix a big saturated model M^* of T and we work in M^* . Small subsets of M^* are denoted by $A, B, C, \dots$. Finite tuples in M^* are denoted by $a, b, c, \dots$. Variables are denoted by $x, y, z, \dots$. Formulas are denoted by $\varphi, \psi, \dots$. Types are denoted by $p, q, r, \dots$ and $S(A)$ is the set of all complete types over A . We say a and b have the same type over A (in symbol $a \equiv_A b$) if there is a type $p \in S(A)$ for which $a, b \models p$. For any $A \subset B$ and $p \in S(B)$, $p|_A = \{\varphi(x) \in p \mid \varphi(x) : L(A)\text{-formula}\}$. Let $\text{Aut}(M^*/A) = \{\sigma : M^* \rightarrow M^* \mid \sigma \text{ is an automorphism over } A\}$. A sequence $\langle a_i \mid i < \alpha \rangle$, where α is an ordinal, is called an indiscernible sequence over A , if for any strictly increasing partial function $f : \alpha \rightarrow \alpha$, there is an $\sigma \in \text{Aut}(M^*/A)$ with $\sigma \supset \{(a_i, a_{f(i)}) \mid i < \alpha\}$.

2 Kim-independence

A complete type p over the domain M^* will be called a global type. The following definitions are from [1].

Definition 1 (A -invariant global type). We say a global type $p(x) \in S(M^*)$ is A -invariant, if

$$\varphi(x, a) \in q \iff \varphi(x, b) \in q.$$

holds, for any $a, b \in M^*$ with $a \equiv_A b$ and any L -formula $\varphi(x, y)$.

Definition 2 (Morley sequence). Let q be an A -invariant global type. $\langle b_i \mid i < \omega \rangle$ will be called an A -Morley sequence (defined by q) if $b_i \models q|_{Ab_{<i}}$, for all $i < \omega$.

Remark 3. For any set A , an A -Morley sequence is an indiscernible sequence over A . This can be shown by an induction on the length of the sequence.

Example 4 ($T = \text{Th}(\mathbb{Q}, <)$). Let $q(x)$ be an M -invariant global type extending $\{x > a \mid a \in M\}$.

1. Suppose that all formulas $x < a$ with $a > M$ belong to q . Then any decreasing sequence $a_0 > a_1 > \dots > a_i > \dots > M$ becomes an M -Morley sequence defined by q .
2. Suppose that all formulas $x > a$ with $a > M$ belong to q . Then any increasing sequence $M < a_0 < a_1 < \dots < a_i < \dots$ becomes an M -Morley sequence defined by q .

Definition 5 (Kim-divide). We say that a formula $\varphi(x, b)$ Kim-divides over A if there are an A -invariant global type q and an A -Morley sequence $I = \langle b_i \mid i < \omega \rangle$ defined by q such that

1. $b_0 = b$,
2. $\{\varphi(x, b_i) \mid i < \omega\}$ is inconsistent.

A type $p \in S(B)$ Kim-divides over A if there is a formula $\varphi(x, b) \in p$ that Kim-divides over A .

Example 6 ($T = \text{Th}(\mathbb{Q}, <)$). Let M be a model of T . Let us consider the formula $a_0 < x < b_0$.

1. Suppose that there is an element $m \in M$ with $a_0 < m < b_0$. Then the formula $a_0 < x < b_0$ does not Kim-divide over M .
2. Suppose that $M < a_0 < b_0$. Let $q(y, z)$ be the global type $\{a < y < z : a \in M^*\}$. Then, the formula $a_0 < x < b_0$ Kim-divides over M by this q .

Definition 7 (Kim-fork). $\varphi(x, b)$ Kim-forks over A if there are $n < \omega$ and $\psi_0(x, c), \dots, \psi_n(x, c)$ such that

1. $\psi_i(x, c) : \text{Kim-divides over } A$,
2. $M^* \models \forall x [\varphi(x, b) \rightarrow \bigvee_{i \leq n} \psi_i(x, c)]$.

$p \in S(B)$ Kim-forks over A if there is $\varphi(x, b) \in p$ Kim-forks over A .

By definition, If $\varphi(x, a)$ Kim-divides over A , then $\varphi(x, a)$ Kim-forks over A (but not the converse).

3 NSOP₁ theories

Definition 8 (NSOP₁). T has SOP₁ if there exist $\varphi(x, y) \in L$ and a binary tree of tuples $(c_\eta)_{\eta \in 2^{<\omega}}$ such that

1. For all $\beta \in 2^\omega$, $\{\varphi(x, c_{\beta|_m}) \mid m < \omega\}$ is consistent,
2. For all $\gamma \in 2^{<\omega}$ and $\gamma \supseteq \eta^\frown \langle 0 \rangle$, $\{\varphi(x, c_{\alpha^\frown \langle 1 \rangle}), \varphi(x, c_\gamma)\}$ is inconsistent.

T is NSOP₁ if T does not have SOP₁.

“ T is NSOP₁” characterizes an L -formula and two infinite sequence of tuples.

Fact 9 ([1]). Let T be a complete theory. T.F.A.E.

1. T has SOP₁.
2. There are $\langle a_i b_i \mid i < \omega \rangle$ and $\varphi(x, y) \in L$ such that
 - $a_i \equiv_{(ab) < i} b_i$ ($\forall i < \omega$),
 - $\{\varphi(x, a_i) \mid i < \omega\}$ is consistent,
 - $\{\varphi(x, b_i) \mid i < \omega\}$ is 2-inconsistent.

By Fact 9,

Fact 10 ([1]). Let T be a complete theory. T.F.A.E.

1. T is NSOP₁,
2. For all $M \models T$, $\varphi(x, b)$ and M -invariant global type $q \supset \text{tp}(b/M)$, if $\varphi(x, b)$ Kim-divides over M by q , then for all M -invariant global type r satisfies $r|_M = q|_M$, $\varphi(x, b)$ Kim-divides over M by r .

By Fact 10,

Fact 11 ([1], $T : \text{NSOP}_1$). If $\varphi(x, b)$ Kim-forks over M , $\varphi(x, b)$ Kim-divides over M .

By Fact 11,

Fact 12 ($T : \text{NSOP}_1$). For any B and $p \in S(B)$,

$$p \text{ Kim-divides over } M \iff p \text{ Kim-forks over } M.$$

Notation. $a \downarrow_A^K b \iff \text{tp}(a/Ab)$ does not Kim-fork over A .

Fact 13 ([1], $T : \text{NSOP}_1$). $\downarrow^K$ satisfies the following conditions :

1. (Extension over models) If $a \downarrow_M^K b$, then for all c , there exists $a' \equiv_{Mb} a$ satisfies $a' \downarrow_M^K bc$.

2. (Chain condition) If $a \downarrow_M^K b$ and M -Morley sequence $I = \langle b_i \mid i < \omega \rangle$ starts with b , there exists $a' \equiv_{Mb} a$ such that

- $a' \downarrow_M^K I$
- I : an indiscernible sequence over Ma'

In [1], Kaplan and Ramsey proved

Fact 14 ([1]). T.F.A.E.

1. T is NSOP₁.
2. Symmetry : $a \downarrow_M^K b \iff b \downarrow_M^K a$.

4 Ideas proving Symmetry of Kim-independence

I want to prove

Theorem 15 ([1]). If T is NSOP₁, $\downarrow^K$ satisfies symmetry over models, i.e. $a \downarrow_M^K b \implies b \downarrow_M^K a$.

by only using Compactness theorem and Ramsey's theorem. We introduce the notion of finitely satisfiability of types.

Notation. We denote $\text{tp}(a/A) = \{\varphi(x) : L(A)\text{-formula, } M^* \models \varphi(a)\}$.

Definition 16. $p(x) \in S(A)$ is finitely satisfiable in B if for any $n < \omega$ and $\varphi_0(x), \dots, \varphi_n(x) \in p$, there is $b \in B$ satisfies

$$M^* \models \bigwedge_{i \leq n} \varphi(b).$$

Let α be an ordinal. $I = \langle a_i \mid i < \alpha \rangle$ is coheir sequence over A if for any $i < \alpha$, $\text{tp}(a_i/Aa_{<i})$ is finitely satisfiable in A and I is an indiscernible sequence over A .

My main idea proving Theorem 15 is using Fact 9. First, I proved

Lemma 17 ($T : \text{NSOP}_1$). We put $p(x, a) = \text{tp}(b/Ma)$. If $a \downarrow_M^K b$, then for all $n < \omega$, there is a sequence $(a_i a'_i)_{i < n}$ satisfies the following conditions :

1. $a_i \equiv_M a'_i \equiv_M a \ (\forall i < n)$,
2. $a_i \equiv_{M(aa')_{>i}} a'_i \ (\forall i < n)$,
3. $\bigcup_{i < n} p(x, a_i) : \text{consistent}$,
4. $(a'_i)_{i < n} : \text{For all } i < n, \text{tp}(a'_i/Ma'_{<i}) \text{ is finitely satisfiable in } M$.

Proof. We confirm only $n = 2$. But same method is applicable for all $n < \omega$. Let κ be a sufficiently large cardinal. Since $a \downarrow_M^K b$, there is $I_0 = (b_i)_{i < \kappa}$ starts with b satisfies the following conditions,

- $a \downarrow_M^K I_0$,
- I_0 : coheir sequence over M and Ma -indsicernible sequence.

Since $a \downarrow_M^K I_0$, there is a'' and $I_1 = (c_i I'_i)_{i < \kappa}$ starts with $a I_0$ satisfies the following conditions,

- $a' \equiv_{M I_0} a$,
- $a' \downarrow_M^K I_1$,
- I'_1 : coheir sequence over M and Ma' -indsicernible sequence.

Let $I_2 = (c'_i I''_i)_{i < \kappa}$ be an coheir sequence over M starts with $a' I_1$. Let $a_1 = c'_0$ and $a'_1 = c'_1$. Since $(c_i)_{i < \kappa}$ is sufficiently long, there is $i < j < \kappa$ such that $c_i \equiv_{M a_1 a'_1} c_j$. Let $a_0 = c_i$ and $a'_0 = c_j$. $\square$

Question 18. For all $n < \omega$, Can we take $(a_i a'_i)_{i < n}$ satisfies the following condition ? :

1. For all $m \leq n$ and $(b_i b'_i)_{i < m}$ are taken by Lemma 17, $(a_i a'_i)_{i < m} \equiv_M (b_i b'_i)_{i < m}$.
 $\implies$ If $m < 2$, $(a_i a'_i)_{i < m} \equiv_M (b_i b'_i)_{i < m}$ but the other case can't satisfy this condition.
2. $\text{tp}(a'_i / M a'_{< i}) \subset \text{tp}(a'_{i+1} / M a'_{< i+1})$ for all $i < n - 1$?
 $\implies$ For all $n < \omega$ and $(a_i a'_i)_{i < n}$, $\text{tp}(a'_0 / M) \subset \text{tp}(a'_i / M a'_{< i})$, but the other case can't satisfy this condition.

We explain another idea.

Lemma 19. Let T be a complete theory. Let $M \subset A$, where $M \models T$ and $p(x) \in S(A)$ be a type finitely satisfiable in M . Let $q(X) \in S(M)$, where X is a set of variables with $x \in X$. Suppose that $p(x) \cup q(X)$ is consistent, in other words, $p|_M \subset q(X)$. Then, there is a type $q^*(X) \in S(A)$ such that

1. $q^*(X)$ is finitely satisfiable in M , and
2. $q^*(X) \supset p(x) \cup q(X)$.

Proof. Let $\Pi(X) = \{\neg\theta(X) \mid \theta(X) : L(A)\text{-formula, } \theta \text{ isn't satisfiable in } M\}$ and $\Gamma(X)$ be $p(x) \cup q(X) \cup \Pi(X)$. We claim that $\Gamma(X)$ is consistent. Suppose otherwise, We can find $\varphi_p(x) \in p$, $\varphi_q(x) \in q$, $n < \omega$ and $\psi_0(X), \dots, \psi_n(X) \in \Pi(X)$ such that

$$\varphi_p(x) \wedge \varphi_q(X) \models \bigvee_{i \leq n} \neg\psi_i(X).$$

But this is contradiction since $\exists X \setminus x [\varphi_p(x) \wedge \varphi_q(X)] \in p$ and p is finitely satisfiable in M . $\square$

Proposition 20. Let T be a NSOP₁ theory and $M \models T$. Let $r(x, y) = \text{tp}(ab/M)$, where $a \downarrow_M^K b$. Then for any $n < \omega$, there is a tree $(a_\eta)_{\eta \in 2^{\leq n}}$ such that

1. $a_{\eta \restriction m} a_\eta \models \gamma$, for any $\eta \in \omega^n$ and $m < n$;
2. For any $i < \omega$ and $i \frown \eta \in \omega^{<n}$, $I_{i \frown \eta} = \langle a_{j \frown i \frown \eta} \mid j < \omega \rangle$ is indiscernible sequence over $M \cup \{a_{i \frown \eta}\} \cup \{a_{\nu \frown k \frown \eta} \mid k < i, \nu \in \omega^{<n-1}\}$.
3. $a_{0^{n-2} \frown 1}, a_{0^{n-3} \frown 1}, \dots, a_1$ forms a coheir sequence over M .

Proof. Suppose we already defined a desired tree $(a_\eta)_{\eta \in 2^{\leq n}}$ for $n < \omega$. We rename each a_η to $b_{0 \frown \eta}$. By assumption, $b_{0^{n-1} \frown 1}, b_{0^{n-2} \frown 1}, \dots, b_{0 \frown 1}$ forms a coheir sequence over M . Let $B = \{b_{0^2 \frown \eta} \mid \eta \in \omega^{<n-1}\}$. Since the type $p(x) = \text{tp}(b_{0 \frown 1}/MB)$ is finitely satisfiable in M , there is a coheir extension $p'(x) \in S(M(a_\eta)_{\eta \in \omega^{\leq n}})$ of p . Let $q(X) = \text{tp}((a_\eta)_{\eta \in \omega^{\leq n}}/M)$, where $X = (x_\eta)_{\eta \in \omega^{\leq n}}$ and the variable corresponds to a_η . By Lemma 19, there is a type $p^*(X) \in S(M(a_\eta)_{\eta \in 2^{\leq n}})$ which is finitely satisfiable in M and extends $p'(x_\emptyset) \cup q(X)$. Choose a realization $B_1 = (b_{1 \frown \eta})_{\eta \in \omega^{\leq n}}$ of p^* . Notice that $b_{1 \frown \eta}$ corresponds to x_η . Then we choose $B_i (i \geq 2)$ such that $B_0, B_1, B_2, \dots$ forms a coheir sequence over M . Since $\text{tp}(b_0/M(b_{0 \frown \eta})_\eta)$ is does not Kim-fork over M and Fact 13, there is $r(x) \in S(M(B_i)_{i < \omega})$ satisfies $r(x) \supset \text{tp}(b_0/M(b_{0 \frown \eta})_\eta)$ and does not Kim-fork over M . we choose a realisation $b_\emptyset$ of $r(x)$. Then $(b_\eta)_{\eta \in \omega^{\leq n+1}}$ satisfies the condition 1-3. $\square$

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