

On the convergency to the limit cycle  
in the dynamical system of Multivibrator

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Abstract

The dynamics of a multivibrator circuit with two slow and two fast degrees of freedom in the singular limit depends on only one parameter. For some values of it, the trajectories of the system take place jumping. This paper, by using a quasi-potential in this system, shows that the trajectories converge to the limit cycle where repeated jumping behaviour is observed.

1. Introduction.

The fast vector field gives 2-dimensional manifold in  $R^4$  at the singular limit ( $\varepsilon \rightarrow 0$ ) and the slow vector field on the manifold describes the dynamics. Therefore, we consider the natural projection  $\Pi_2: R^4 \rightarrow R^2$ , determined by the translated coordinate. Then, it is easy to analyze the system by using approximated FET (field effect transistor) characteristic as a piecewise linear function. This paper proves the convergence of the trajectories to the limit cycle through the function. Judging the convergency, we use a quasi-potential for the system instead of the trajectory itself.

2. The dynamical system of multivibrator.

In this system, we get the following equations([1]):

$$\begin{pmatrix} \dot{v}_3 \\ \dot{v}_6 \end{pmatrix} = -1/(CR_c) \begin{pmatrix} v_3 + v_{10} \\ v_6 + v_{11} \end{pmatrix} \quad (\dot{\phantom{x}} = d/dt) \quad \equiv F(v_3, v_6, v_{10}, v_{11}), \quad (1)$$

$$\varepsilon \begin{pmatrix} \dot{v}_{10} \\ \dot{v}_{11} \end{pmatrix} = \begin{pmatrix} -(v_3 + v_{10})/R_c + f(-v_6 - v_{11}) - v_{10}/R_s \\ -(v_6 + v_{11})/R_c + f(-v_3 - v_{10}) - v_{11}/R_s \end{pmatrix} \quad \equiv G(v_3, v_6, v_{10}, v_{11}), \quad (2)$$

where  $\varepsilon$  is a parasitic inductance (any  $\varepsilon \in R^1_+$ ) and  $f$  is defined as follows ( $f \in C^0$ ):

$$f(x) = \begin{cases} 0 & (x \geq \alpha) \\ (\beta/\alpha)x - \beta & (|x| \leq \alpha) \\ -2\beta & (x \leq -\alpha) \end{cases} \quad (3)$$

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Then, we assume that

$$\beta/\alpha > 1/R_s + 1/R_c \equiv 1/R. \quad (4)$$

We denote the following notations:

$$\Sigma \equiv \{(v_3, v_6, v_{10}, v_{11}); G=0\}, \quad (5)$$

$$\Sigma r \equiv \{(v_3, v_6, v_{10}, v_{11}) \in \Sigma; \partial G / \partial (v_{10}, v_{11}) \text{ is nondegenerate}\},$$

$$(\partial \Sigma = \{(v_3, v_6, v_{10}, v_{11}) \in \Sigma; |\partial G / \partial (v_{10}, v_{11})| = 0\}, \quad (6)$$

$$\Sigma s \equiv \{(v_3, v_6, v_{10}, v_{11}) \in \Sigma; \text{ real eigenvalues of } \partial G / \partial (v_{10}, v_{11}) \text{ are negative}\}, \quad (7)$$

$$\Sigma h \equiv \Sigma r \setminus \Sigma s, \quad (8)$$

$$\Sigma f \equiv \text{Cl } \Sigma s \cap \text{Cl } \Sigma h. \quad (9)$$

The eigenvalues of  $\partial G / \partial (v_{10}, v_{11})$  at  $p \in \Sigma$  are

$$\lambda = -(1/R_s + 1/R_c) \pm (f'(-v_3 - v_{10})f'(-v_6 - v_{11}))^{1/2}. \quad (10)$$

Therefore,  $p \in \Sigma s$  if and only if

$$\begin{aligned} f'(-v_3 - v_{10})f'(-v_6 - v_{11}) &\geq 0 \text{ and} \\ f'(-v_3 - v_{10})f'(-v_6 - v_{11}) &< (1/R_s + 1/R_c)^2, \end{aligned} \quad (11)$$

or

$$f'(-v_3 - v_{10})f'(-v_6 - v_{11}) < 0 \quad (f' \text{ is the derivative of } f). \quad (12)$$

As a result,

$$\Sigma s = \{p \in \Sigma; |v_3 + v_{10}| > \alpha \text{ or } |v_6 + v_{11}| > \alpha\}, \quad (13)$$

$$\begin{aligned} \Sigma f = \{p \in \Sigma; |v_3 + v_{10}| = \alpha \text{ and } |v_6 + v_{11}| < \alpha, \text{ or} \\ |v_3 + v_{10}| < \alpha \text{ and } |v_6 + v_{11}| = \alpha\}. \end{aligned} \quad (14)$$

Let  $\Pi_2: R^4 \rightarrow R^2$ ,

$$(v_3, v_6, v_{10}, v_{11}) \rightarrow (v_3 + v_{10}, v_6 + v_{11}) \equiv (x, y), \quad (15)$$

then the following two notations are defined:

$$\Pi_2 \Sigma f' \equiv \{(x, y) \in \Pi_2 \Sigma f; x = \alpha\}, \quad (16)$$

$$\Pi_2 \Sigma f'' \equiv \{(x, y) \in \Pi_2 \Sigma f; y = \alpha\}. \quad (17)$$

|       |                  |                    |                  |
|-------|------------------|--------------------|------------------|
|       | $x = -\alpha$    | $x = \alpha$       |                  |
|       | $\Pi_2 \Sigma s$ | $\Pi_2 \Sigma s$   | $\Pi_2 \Sigma s$ |
| $R^2$ |                  | $\Pi_2 \Sigma f''$ | $y = \alpha$     |
|       | $\Pi_2 \Sigma s$ | $\Pi_2 \Sigma h$   | $\Pi_2 \Sigma s$ |
|       |                  | $\Pi_2 \Sigma f'$  | $y = -\alpha$    |
|       |                  |                    | $\Pi_2 \Sigma s$ |

Moreover, we define  $D^{ij}$  ( $i, j = \pm 0$ ),  $D^{00} = \Pi_2 \Sigma h$ , as the figure,

|       |               |              |               |
|-------|---------------|--------------|---------------|
|       | $x = -\alpha$ | $x = \alpha$ |               |
|       | $D^{-+}$      | $D^{0+}$     | $D^{++}$      |
| $R^2$ | $D^{-0}$      | $D^{00}$     | $D^{+0}$      |
|       | $D^{--}$      | $D^{0-}$     | $D^{+-}$      |
|       |               |              | $y = -\alpha$ |

and

$$\Sigma s^{ij} \equiv (\Pi_2^{-1} D^{ij}) \cap \Sigma s. \quad (18)$$

Next, we show analysis of  $\Sigma f'$ .

Let  $(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$ , then

$$\begin{aligned} a_3 + a_{10} &= \alpha, \quad |a_6 + a_{11}| < \alpha, \\ -\alpha/R_c - \beta(a_6 + a_{11})/\alpha + \beta - a_{10}/R_s &= 0, \\ -(a_6 + a_{10})/R_c &\quad -a_{11}/R_s = 0. \end{aligned} \quad (19)$$

Thus,  $(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$  is parameterized by  $a_3$  as follows:

$$\begin{aligned} a_6 &= (1/R_s + 1/R_c)(a_3 - \alpha)\alpha/\beta - \alpha(\alpha/\beta R_c - 1)(R_s/R_c + 1), \\ a_{10} &= \alpha - a_3, \\ a_{11} &= (\alpha - a_3)\alpha/\beta R_c + \alpha^2 R_s/\beta R_c^2 - \alpha R_s/R_c. \end{aligned} \quad (20)$$

with

$$\alpha(1 + R_s/R_c) - 2\beta R_s < a_3 < \alpha(1 + R_s/R_c), \quad (21)$$

(when  $p = (\alpha, -\alpha)$ ,  $a_3 = \alpha(1 + R_s/R_c) - 2\beta R_s$  and when  $p = (\alpha, \alpha)$ ,  $a_3 = \alpha(1 + R_s/R_c)$ ,

$q \in \Sigma f'$ ,  $p = \Pi_2(q) \in \Pi_2 \Sigma f' \subset \mathbb{R}^2$ ). For the above point in  $\Sigma f'$ , let

$$\Pi_2(a_3, a_6, a_{10}, a_{11}) = (a_3 + a_{10}, a_6 + a_{11}) = (x, y) \in \Pi_2 \Sigma f', \quad (22)$$

then

$$x = \alpha, \quad y = \alpha a_3/\beta R_s - (1/R_s + 1/R_c)\alpha^2/\beta + \alpha, \quad (23)$$

with  $a_3$  satisfying (21). Therefore, by (20), (23),  $\Pi_2|_{\Sigma f'}$  is an embedding.

### 3. The jumping orbits starting from $\Sigma f'$ .

For  $q = (a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$ , we denote

$$Rq^2 \equiv \{(a_3, a_6, v_{10}, v_{11}); v_{10}, v_{11} \in \mathbb{R}^1\}, \quad (24)$$

then

$$Cl \Sigma h \cap Rq^2 = \{q\}. \quad (25)$$

In fact,  $(a_3, a_6, v_{10}, v_{11}) \in \Sigma h \cap Rq^2$ , then

$$\begin{aligned} (a_3 + v_{10})/R_c + (a_6 + v_{11})\beta/\alpha + v_{10}/R_s - \beta &= 0, \\ (a_6 + a_{11})/R_c + (a_3 + v_{11})\beta/\alpha + v_{11}/R_s - \beta &= 0. \end{aligned} \quad (26)$$

This equation (26) has a unique solution  $(v_{10}, v_{11})$  by the assumption (4). Since,

$$q \in Cl \Sigma h \cap Rq^2, \quad (27)$$

we have (25).

On the set  $\Sigma s^{0+} \cap Rq^2$  ( $q = (a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$ ),

let  $(a_3, a_6, v_{10}, v_{11}) \in \Sigma s^{0+} \cap Rq^2$ , then

$$\begin{aligned} (a_3 + v_{10})/R_c + v_{10}/R_s &= 0, \\ (a_6 + v_{11})/R_c + (a_3 + v_{10})\beta/\alpha + v_{11}/R_s - \beta &= 0, \end{aligned} \quad (28)$$

therefore,

$$\begin{aligned} v_{10} &= -a_3 R_s / (R_s + R_c), \\ v_{11} &= (-R_s a_6 + \beta R_s R_c) / (R_s + R_c) - R_s R_c^2 a_3 \beta / \alpha (R_s + R_c)^2. \end{aligned} \quad (29)$$

Let  $\Pi_2(\Sigma s^{0+} \cap Rq^2) \equiv (x, y) \in B \subset \mathbb{R}^2$  ( $x = a_3 + v_{10}$ ,  $y = a_6 + v_{11}$ ), then

$$\begin{aligned} B: \quad x &= R_c a_3 / (R_s + R_c), \\ y &= -R_s R_c^2 \beta a_3 / \alpha (R_s + R_c)^2 + R_c a_6 / (R_s + R_c) + R_s R_c \beta / (R_s + R_c), \end{aligned} \quad (30)$$

and then  $(\alpha, \alpha) \in \text{Cl}B$ .

On the set  $\Sigma s^{-+} \cap Rq^2$ ,  $(a_3, a_6, v_{10}, v_{11}) \in \Sigma s^{-+} \cap Rq^2$ ,

$$(a_3 + v_{10})/R_c + v_{10}/R_s = 0,$$

$$(a_6 + v_{11})/R_c + v_{11}/R_s - 2\beta = 0, \quad (31)$$

therefore,

$$v_{10} = -R_s a_3 / (R_s + R_c), \quad v_{11} = (2\beta R_s R_c - R_s a_6) / (R_s + R_c). \quad (32)$$

Let  $\Pi_2(\Sigma s^{-+} \cap Rq^2) \equiv (x, y) \in A \subset R^2$  ( $x = a_3 + v_{10}, y = a_6 + v_{11}$ ), then

$$A: \quad x = R_c a_3 / (R_s + R_c), \quad (33)$$

$$y = (2\beta R_s R_c + R_c a_6) / (R_s + R_c).$$

Since,  $a_6$  is a linear function of  $a_3$  by (20), the set A and set B are linear segments. Furthermore, for  $x = R_c a_3 / (R_s + R_c) = -\alpha$ , the y's in (30) and (33) are equal, thus  $\text{Cl}A \cup \text{Cl}B$  is connected. By the assumption (4) and (21), we have

$$R_s(\alpha/R - 2\beta) < a_3 < 0 < \alpha(1 + R_s/R_c). \quad (34)$$

Let  $\Gamma_1$  be the set of the points in the traces (fast orbits) starting from  $\Sigma f'$ . Then,  $\Gamma_1$  is a 2-dimensional manifold with boundary (like a belt). We will show that  $A \cup B \subset \partial\Gamma_1$  and that a trace in  $\Gamma_1$  is a curve with end points in  $\Sigma f'$  and  $A \cup B$ .

Put  $D^{11} \equiv \Pi_2^{-1} D^{11} \subset R^4$  and  $\Gamma_1^{11} \equiv \Gamma_1 \cap D^{11}$ , then the fast vector field  $Y^{00}$  on  $\Gamma_1^{00}$  is defined by

$$\begin{aligned} Y^{00}: \quad \dot{v}_3 &= 0, \quad \dot{v}_6 = 0, \\ \dot{v}_{10} &= -(a_3 + v_{10})/R_c - (a_6 + v_{11})\beta/\alpha - v_{10}/R_s + \beta, \\ \dot{v}_{11} &= -(a_6 + v_{11})/R_c - (a_3 + v_{10})\beta/\alpha - v_{11}/R_s + \beta, \end{aligned} \quad (35)$$

for any fixed  $(a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$ . By the map  $\Pi_2 | \Gamma_1^{00}: \Gamma_1^{00} \rightarrow R^2$ ,  $Y^{00}$  is induced to a vector field  $(\Pi_2)_* Y^{00}$  on  $D^{00} \subset R^2$  as follows:

$$\begin{aligned} (\Pi_2)_* Y^{00}: \quad \dot{x} &= -x/R - \beta y/\alpha + (a_3/R_s + \beta), \\ \dot{y} &= -\beta x/\alpha - y/R + (a_6/R_s + \beta). \end{aligned} \quad (36)$$

The orbit of (36) with an initial point  $p = \Pi_2(q) = (a_3 + a_{10}, a_6 + a_{11})$  is the curve  $Cq(k)$ ,

$$Cq(k) = (a_3 + a_{10} - k, a_6 + a_{11} + k) = (\alpha - k, a_6 + a_{11} + k). \quad (37)$$

Because, by (35), (36),

$$\begin{aligned} \dot{x} + \dot{y} &= -(\beta/\alpha + 1/R)(x + y) + (a_3 + a_6)/R_s + 2\beta \\ &= -(\beta/\alpha + 1/R)(a_3 + a_{10} + a_6 + a_{11}) + 2\beta = \dot{v}_{10} + \dot{v}_{11} = 0. \end{aligned} \quad (38)$$

As an unstable direction at  $q \in D^{00} \cap \Sigma$  is  $(v_{10}, v_{11}) = (k, -k)$ , it is a jumping direction at  $p \in \Sigma f'$ .

The above map  $\Pi_2 | \Gamma_1^{00}: \Gamma_1^{00} \rightarrow D^{00}$  is a diffeomorphism. Because,  $q = (a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$  is coordinated by  $a_3$  ((20)), and locally,  $\Gamma_1^{00}$  is coordinated by  $(a_3, a_6)$  and any orbit of  $Y^{00}$  is mapped onto an orbit of (36). This map of the orbit is regular at each point, by (35), (36). Moreover, it is diffeomorphic, since the curve  $Cq(k)$  has no self-intersection and the vector

$(\dot{x}, \dot{y})$  is nonsingular at each point on  $Cq(k)$ . As  $\Pi_2: \Sigma f' \rightarrow \Pi_2(\Sigma f')$ ,  $q \in \Sigma f' \rightarrow (a_3 + a_{10}, a_6 + a_{11})$  is a diffeomorphism, if  $q \neq q'$ , then  $Cq(k) \neq Cq'(k)$  and  $Cq(k) \cap Cq'(k) = \emptyset$ .

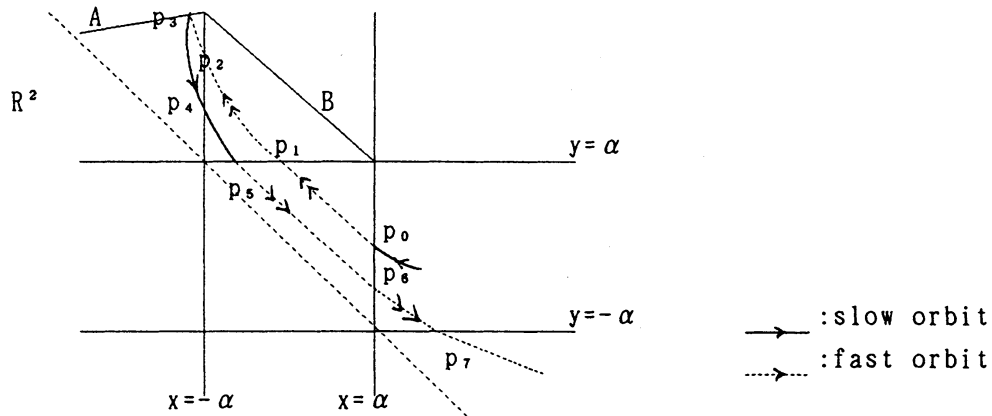
The fast vector field  $Y^{0+}$  on  $\Gamma_1^{0+}$  and  $(\Pi_2)_* Y^{0+}$  on  $\Delta_B \subset D^{0+}$  are as follows;

$$Y^{0+}: \begin{aligned} \dot{v}_3 &= 0, \quad \dot{v}_6 = 0, \\ \dot{v}_{10} &= -(a_3 + v_{10})/R_c - v_{10}/R_*, \\ \dot{v}_{11} &= -(a_6 + v_{11}) - (a_3 + v_{10})\beta/\alpha - v_{11}/R_* + \beta, \end{aligned} \quad (39)$$

$$(\Pi_2)_* Y^{0+}: \begin{aligned} \dot{x} &= -x/R + a_3/R_*, \\ \dot{y} &= -\beta x/\alpha - y/R + a_6/R_* + \beta, \end{aligned} \quad (40)$$

on  $\Delta_B$ , where

$$\Delta_B = \{(x, y); \alpha < y < (\alpha/\beta R - R\beta/\alpha)x + (R\beta - \alpha^2/\beta R + \alpha) \text{ and } x+y > 0 \text{ and } -\alpha < x < \alpha\}. \quad (41)$$



The solution of (40) with the initial condition

$$(x(0), y(0)) = p_1 = (x_1, y_1), \quad x_1 = a_3 + a_{11}, \quad y_1 = \alpha, \quad (42)$$

is given as follows:

$$\begin{aligned} x(t) &= (x_1 - Ra_3/R_*)e^{-t/R} + Ra_3/R_*, \\ y(t) &= ((\beta^2 R^2/\alpha^2 - R/R_*)a_6 + \alpha - \beta^2 R^2/\alpha)e^{-t/R} + (\beta/\alpha)(-x_1 + Ra_3/R_*)te^{-t/R} \\ &\quad - \beta R^2 a_3/\alpha R_* + R(a_6/R_* + \beta). \end{aligned} \quad (43)$$

By (40), (43) and (21), on  $\Delta_B$ ,

$$\dot{x} = (a_3/R_* - x_1/R)e^{-t/R} < (\alpha - x_1)e^{-t/R}/R < 0, \quad (-\alpha < x_1 < \alpha). \quad (44)$$

By (43), (30),

$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (Ra_3/R_*, -\beta R^2 a_3/\alpha R_* + R(a_6/R_* + \beta)) \in B. \quad (45)$$

By the property of  $\Pi_2$  and (39), (40), any orbit of  $Y^{0+}$  starting from  $p_1 = (x_1, \alpha)$  arrives at a point  $p_2$ :

$$(i) \quad p_2 = (-\alpha, y_2) \in \Delta_B \quad \text{if } R_*(\alpha/R - 2\beta) < a_3 < -\alpha R_*/R, \quad (46)$$

$$\text{or } (ii) \quad p_2 \in B \quad \text{if } -\alpha R_*/R < a_3 < \alpha R_*/R.$$

Similarly as  $\Pi_2|_{\Gamma_1}$ ,  $\Pi_2|_{\Gamma_1^{0+}}: \Gamma_1^{0+} \rightarrow \Delta_B$  is an orbit preserving

diffeomorphism. The fast vector field  $Y^{-+}$  on  $\Gamma_1^{-+}$  and  $(\Pi_2)_* Y^{-+}$  on  $\Delta_A \subset D^{-+}$

are as follows:

$$Y^{+-}: \begin{aligned} \dot{v}_3 &= 0, \quad \dot{v}_6 = 0, \\ \dot{v}_{10} &= -(a_3 + v_{10})/R_c - v_{10}/R_s, \\ \dot{v}_{11} &= -(a_6 + v_{11})/R_c - v_{11}/R_s + 2\beta, \end{aligned} \quad (47)$$

$$(\Pi_2)_* Y^{+-}: \begin{aligned} \dot{x} &= -x/R + a_3/R_s, \\ \dot{y} &= -y/R + a_6/R_s + 2\beta, \end{aligned} \quad (48)$$

on  $\Delta_A$ , where

$$\Delta_A = \{(x, y); y < \alpha x / \beta R + (2R\beta - \alpha^2 / \beta R + \alpha) \text{ and } x + y > 0 \text{ and } x < -\alpha\}. \quad (49)$$

The solution of (48) with the initial condition

$$(x(0), y(0)) = p_2 = (x_2, y_2), \quad x_2 = -\alpha, \quad (50)$$

is given as follows:

$$\begin{aligned} x(t) &= -(\alpha + Ra_3/R_s) e^{-t/R} + Ra_3/R_s, \\ y(t) &= (y_2 - Ra_6/R_s - 2R\beta) e^{-t/R} + R(a_6/R_s + 2\beta). \end{aligned} \quad (51)$$

By (51) and (33), in the case of (i)  $p_2 = (-\alpha, y_2) \in \Delta_B$ ,

$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (Ra_3/R_s, R(a_6/R_s + 2\beta)) \in A. \quad (52)$$

Similarly as  $\Pi_2|_{\Gamma_1^{00}}$  and  $\Pi_2|_{\Gamma_1^{0+}}, \Pi_2|_{\Gamma_1^{-+}}: \Gamma_1^{-+} \rightarrow \Delta_A$  is an orbit preserving diffeomorphism. From the above discussions, we have proved the following Lemma1..

#### Lemma1.

Let  $Y$  be the fast vector field defined by

$$Y: \begin{aligned} \dot{v}_3 &= 0, \quad \dot{v}_6 = 0, \\ (\dot{v}_{10}, \dot{v}_{11}) &= G(v_3, v_6, v_{10}, v_{11}), \end{aligned} \quad (53)$$

then the orbit of  $Y$  with the initial point  $q_0 = (a_3, a_6, a_{10}, a_{11}) \in \Sigma f'$  arrives at an equilibrium point (sink) of  $Y$  contained an arc  $\underline{A} \cup \underline{B}$  in  $\Sigma s$ , where  $\underline{A} \equiv \Pi_2^{-1}(A) \cap \Sigma s^{-+}$ ,  $\underline{B} \equiv \Pi_2^{-1}(B) \cap \Sigma s^{0+}$ .

#### 4. The slow orbits in $\Sigma s$ .

The slow vector field  $X$  at  $(v_3, v_6, v_{10}, v_{11}) \in \Sigma s$  is as follows:

$$X: \begin{aligned} \dot{v}_3 &= -(v_3 + v_{10})/CR_c, \\ \dot{v}_6 &= -(v_6 + v_{11})/CR_c, \\ \dot{v}_{10} &= 0, \quad \dot{v}_{11} = 0, \end{aligned} \quad (54)$$

$$\Sigma s^{-+}: \begin{aligned} \dot{v}_{10} &= (v_3 + v_{10})/R_c + v_{10}/R_s = 0, \\ \dot{v}_{11} &= (v_6 + v_{11})/R_c + v_{11}/R_s - 2\beta = 0 \end{aligned} \quad (55)$$

The tangent space  $T\Sigma s^{-+}$  at any point in  $\Sigma s^{-+}$  is as follows:

$$T\Sigma s^{-+}: \begin{aligned} (dv_3 + dv_{10})/R_c + (dv_{10})/R_s &= 0, \\ (dv_6 + dv_{11})/R_c + (dv_{11})/R_s &= 0. \end{aligned} \quad (56)$$

For each  $q = (b_3, b_6, b_{10}, b_{11}) \in \Sigma s$ , let  $\Pi_z: T_q R^4 \rightarrow T_q \Sigma s$  be the natural projection  $R^4 = T_q \Sigma s + T_q Rq^2 \rightarrow T_q \Sigma s$ , where  $Rq^2 \equiv \{(b_3, b_6, v_{10}, v_{11}); v_{10}, v_{11} \in \mathbb{R}^1\}$ .

For  $q = (v_3, v_6, v_{10}, v_{11}) \in \Sigma s^{-+}$ ,

$$Xq = (-1/CR_c)((v_3+v_{10}), (v_6+v_{11}), 0, 0), \quad (57)$$

we have the followings: put  $\Pi_2 Xq^{-+} = (x_3, x_6, x_{10}, x_{11})$ , then there is a point  $(0, 0, y_{10}, y_{11}) \in TqRq^2$  satisfying

$$\begin{bmatrix} x_3 \\ x_6 \\ x_{10} \\ x_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y_{10} \\ y_{11} \end{bmatrix} + \begin{bmatrix} -(v_3+v_{10})/CR_c \\ -(v_6+v_{11})/CR_c \\ 0 \\ 0 \end{bmatrix}, \quad (58)$$

hence, by (56),

$$\Pi_2 Xq^{-+} = (1/CR_c) \begin{bmatrix} -(v_3+v_{10}) \\ -(v_6+v_{11}) \\ (v_3+v_{10})R_*/(R_*+R_c) \\ (v_6+v_{11})R_*/(R_*+R_c) \end{bmatrix}. \quad (59)$$

On the other hand,

$$\Pi_2 \begin{bmatrix} v_3 \\ v_6 \\ v_{10} \\ v_{11} \end{bmatrix} = \begin{bmatrix} v_3+v_{10} \\ v_6+v_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_3 \\ v_6 \\ v_{10} \\ v_{11} \end{bmatrix}, \quad (60)$$

hence

$$(\Pi_2)_* \begin{bmatrix} dv_3 \\ dv_6 \end{bmatrix} = \begin{bmatrix} dv_3+dv_{10} \\ dv_6+dv_{11} \end{bmatrix}. \quad (61)$$

By the map  $\Pi_2 | \Sigma s^{++} : \Sigma s^{++} \rightarrow D^{++}$  (diffeomorphism),

$$\begin{aligned} (\Pi_2)_* \Pi_2 Xq^{-+} &= (1/CR_c) \begin{bmatrix} -(v_3+v_{10})+(v_3+v_{10})R_*/(R_*+R_c) \\ -(v_6+v_{11})+(v_6+v_{11})R_*/(R_*+R_c) \end{bmatrix} \\ &= (-1/C(R_*+R_c)) \begin{bmatrix} v_3+v_{10} \\ v_6+v_{11} \end{bmatrix}, \end{aligned} \quad (62)$$

therefore,

$$\begin{aligned} (\Pi_2)_* \Pi_2 Xq^{-+}: \dot{x} &= -x/C(R_*+R_c), \\ \dot{y} &= -y/C(R_*+R_c), \end{aligned} \quad (63)$$

and then a time scaled equation of (63) is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix} \quad (64)$$

The solution of (64) with the initial condition  $(x(0), y(0)) = p_3 = (x_3, y_3) \in A$  is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_3 e^{-t} \\ y_3 e^{-t} \end{bmatrix} \quad (65)$$

From (20) and (33),

$$(x_3, y_3) = (a_3 R_c / (R_* + R_c), \alpha a^3 / \beta R a - \alpha^2 / \beta R + 2\beta R + \alpha). \quad (66)$$

There is a  $t > 0$  such that  $(x(t), y(t)) = p_4 = (-\alpha, y_4) \in (\partial \Delta_A) \cap (\partial \Delta_B)$ . In fact,

$$x_4 = x(t) = a_3 (R_c / (R_* + R_c)) e^{-t} = -\alpha, \quad (e^{-t} = -\alpha R_* / R a_3), \quad (67)$$

$$y_4 = y(t) = -\alpha^2 / \beta R + (\alpha^3 / \beta R^2 - 2\alpha\beta - \alpha^2 / R) R_* / a_3.$$

In  $\Sigma s^{0+}$ ,

$$\begin{aligned}\Sigma s^{0+}: v_{10} &= (v_3 + v_{10})/R_c + v_{10}/R_s = 0, \\ v_{11} &= (v_6 + v_{11})/R_c + \beta (v_3 + v_{10})/\alpha + v_{11}/R_s - \beta = 0.\end{aligned}\quad (68)$$

Similarly as in  $\Sigma s^{-}$ , we have, for  $q = (v_3, v_6, v_{10}, v_{11}) \in \Sigma s^{0+}$ ,

$$\Pi_2 X q^{0+} = (1/CR_c) \begin{bmatrix} -(v_3 + v_{10}) \\ -(v_6 + v_{11}) \\ (v_3 + v_{10})R_s/(R_s + R_c) \\ (v_3 + v_{10})\beta R^2/\alpha R_s + (v_6 + v_{11})R_s/(R_s + R_c) \end{bmatrix}, \quad (69)$$

$$\begin{aligned}(\Pi_2)_* \Pi_2 X q^{0+}: \dot{x} &= -x/C(R_s + R_c) \\ \dot{y} &= (\beta R x/\alpha - y)/C(R_s + R_c).\end{aligned}\quad (70)$$

The solution of (70) with the initial condition  $(x(0), y(0)) = p_4 = (x_4, y_4) = (-\alpha, y_4)$  is the followings:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\alpha e^{-t} \\ -\beta R t e^{-t} + y_4 e^{-t} \end{bmatrix}. \quad (71)$$

There is a  $t > 0$  such that  $(x(t), y(t)) = p_5 = (x_5, y_5) \in \partial D^{00} \cap \partial D^{0+}$ , since  $y_4 > \alpha$  and  $y_5 = \alpha$ , by (71)  $y(t) \rightarrow 0$  ( $t \rightarrow \infty$ ).

Lemma 2.

For the point  $p_1 = (x_1, \alpha)$ ,  $p_5 = (x_5, \alpha) \in \partial D^{00} \cap \partial D^{0+}$  defined above, we have  $-\alpha < x_5 < x_1 < \alpha$ .

(proof)

$(x_0, y_0) = p_0 = \Pi_2(q_0) = \Pi_2(a_3, a_6, a_{10}, a_{11}) = (a_3 + a_{10}, a_6 + a_{11})$ , since  $q_0 \in \Sigma f^*$ ,  $x_0 = a_3 + a_{10} = \alpha$ . By applying the case of  $\Sigma s^{-}$ , starting from  $\Sigma s^{0+}$ ,

$$y_0 = a_6 + a_{11} = \alpha a_3/\beta R_s - \alpha^2/\beta R + \alpha, \quad (72)$$

and by using  $p_1 = (x_1, y_1) = (y_0, x_0)$ ,

$$x_1 = \alpha a_3/\beta R_s - \alpha^2/\beta R + \alpha. \quad (73)$$

Since  $y_5 = \alpha$ , it follows that (71) implies

$$-\beta R t + y_4 = \alpha e^t. \quad (74)$$

By using  $e^t = 1 + \theta t$  ( $\theta > 0$ ), the solution of (74) is

$$t = (y_4 - \alpha)/(\alpha \theta + \beta R). \quad (75)$$

From the assumption (4) and (21),

$$R_s(\alpha/R - 2\beta) < a_3, \quad (\alpha/\beta(\alpha/R - 2\beta) < \alpha a_3/\beta R_s, \quad R/(\alpha - 2\beta R) > R_s/a_3). \quad (76)$$

From (75), (76) and (73),

$$x_5 = -\alpha/e^{-t} = -\alpha/(1 + \theta(y_4 - \alpha)/(\alpha \theta + \beta R)), \quad (77)$$

$$x_1 - x_5 = \alpha a_3/\beta R_s - \alpha^2/\beta R + \alpha + \alpha/(1 + \theta(y_4 - \alpha)/(\alpha \theta + \beta R))$$

$$> \alpha(\alpha/R - 2\beta)/\beta - \alpha^2/\beta R + \alpha +$$

$$\begin{aligned}& \alpha(\alpha \theta + \beta R)/(\alpha \theta + \beta R + \theta(-\alpha^2/\beta R + (\alpha^3/\beta R^2 - 2\alpha\beta - \alpha^2/R)R/(\alpha - 2\beta R) - \alpha)) \\ &= -\alpha + \alpha = 0.\end{aligned}\quad (78)$$

□

### 5. A quasi-potential on the slow orbits.

According to the translation of the variables:

$$x \equiv (v_3 + v_{10})/\alpha, \quad y \equiv (v_6 + v_{11})/\alpha, \quad (79)$$

$v_3, v_6, v_{10}, v_{11}$  and  $\partial \Sigma$  are represented by the new coordinate as follows:

$$v_3 = \alpha R_* x / R - R_* f(-\alpha y), \quad v_6 = \alpha R_* y / R - R_* f(-\alpha x), \quad (80)$$

$$v_{10} = -\alpha R_* x / R + R_* f(-\alpha y), \quad v_{11} = -\alpha R_* y / R + R_* f(-\alpha x) \text{ and} \\ \partial \Sigma = \{(x, y) \in \mathbb{R}^2; \begin{bmatrix} 1/R & -f'(-\alpha y) \\ -f'(-\alpha x) & 1/R \end{bmatrix} = 0\}. \quad (81)$$

It follows that on the slow manifold ([1]):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -R/CR_* (1 - K^2 h'(-x)h'(-y)) \begin{bmatrix} 1 & Kh'(-y) \\ Kh'(-x) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (82)$$

$$f(v) = \beta (1 + h(v/\alpha)), \quad (83)$$

and  $h$  is an odd function satisfying  $\lim_{x \rightarrow \infty} h(x) = 1$  and  $h'(0) = 1$ . Then,

$$\partial \Sigma = \{(x, y) \in \mathbb{R}^2; K^2 h'(-x)h'(-y) = 1\}. \quad (84)$$

The stable manifold  $S$  and the unstable manifold  $T$  at the equilibrium point  $(x, y) = (0, 0)$  in (82) are given by the diagonal subspaces

$$S = \{(x, y) \in \mathbb{R}^2; x - y = 0\} \text{ and } T = \{(x, y) \in \mathbb{R}^2; x + y = 0\}. \quad (85)$$

Introducing a Nishino-Tchizawa quasi-potential  $F$  ([1]):

$$F(x, y, K) = R_* \beta x / K - R_* f(-\alpha y) - R_* \beta y / K + R_* f(-\alpha x), \quad (86)$$

where  $K \equiv R\beta/\alpha$  ( $K > 1$ ), we can describe the convergence of the trajectories in the system to an attractor-inf limit cycle.

### Theorem.

The slow orbit of the dimensionless equations (82) has a limit cycle  $L$  such that  $L \subset T$  as an attractor.

(proof)

As  $x_1 > x_*$  is shown by Lemma 2., from the form of the quasi-potential  $F$ , we can conclude the values of  $F$  on the cross section  $\partial \Sigma$  are monotone. Therefore, the values are convergent to the minimum where  $(x, y) = (-\alpha, \alpha)$ , or the maximum where  $(x, y) = (\alpha, \alpha)$  on the compact subset  $\partial \Sigma$  ( $-\alpha \leq x \leq \alpha$ ). Furthermore, as  $F(p_0) = F(p_3)$ ,  $F(p_5) = F(p_8)$ , ..., we can similarly see the convergence to the accumulation point  $p_*(p^*) \in T \cap \Gamma$  on the other cross section  $\Gamma$ , which is the set of the arrival points after jumping, where  $p_* \equiv \lim_{n \rightarrow \infty} (F(p_n) > F(p_{n+1}))$ ,  $p^* \equiv \lim_{m \rightarrow \infty} (F(p_m) < F(p_{m+1}))$ .

Thus, there is a limit cycle  $L: L = \{(x, y) \in T; F(p_*) \leq F(x, y) \leq F(p^*)\}$ . □

### Reference.

- [1]: "Simulation of an Electronic Multivibrator",  
P. Ashwin, G.P. King, J. Nijhof, G. Ikegami and K. Tchizawa (to appear) (1993)