

On the weak simultaneous resolution of a negligible
truncation of the Newton boundary

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§1. Introduction

Let $f(z_1, \dots, z_n, t)$ be an analytic function defined in a open set $U \times D$ where U is a neighbourhood of $\vec{0} \in \mathbb{C}^n$ and D is a open disk in $\mathbb{C}$ containing the unit interval I and let $V = \{ (z, t) \in U \times D : f(z, t) = 0 \}$ and let $\pi : V \rightarrow D$ be the projection. We also use the notation $f_t(z) = f(z, t)$. We assume that for each $t \in D$, $V_t = \pi^{-1}(t) = f_t^{-1}(0)$ has an isolated singularities at the origin and that the Milnor number $\mu(f_t)$ is constant ($= \mu(f_0)$). The question which we are interested in this paper is the following.

Are V_t ($t \in D$) topologically equivalent to V_0 ?

The assertion is true for $n \neq 3$ by Lê and Ramanujam [8]. Thus the question is open only for $n = 3$. If $\mu^*(f_t)$ ($t \in D$) are constant, the assertion is true by Teissier [6]. In particular, if the Newton boundary $\Gamma(f_t)$ is non-degenerate

in the sense of Kouchnirenko [1] and if $\Gamma(f_t) = \Gamma(f_0)$, the assertion is true by [6]. See also [3].

$\Psi : \tilde{V} \rightarrow V$ is called a weak simultaneous resolution of $\pi : V \rightarrow D$ if the following conditions are satisfied.

- (i) Ψ is a proper modification.
- (ii) $\pi \circ \Psi : \tilde{V} \rightarrow D$ is a flat map.
- (iii) $\Psi : \tilde{V}_t \rightarrow V_t$ is a resolution of V_t .
- (iv) Let $E = \Psi^{-1}(\vec{0} \times D)$. Then $\pi \circ \Psi : E \rightarrow D$ is simple.

See Teissier [7] and Laufer [2] for further detail.

In our case, the existence of a weak resolution is equivalent to the topological stability of $\{V_t\}$ by Theorem 6.4 of [2].

Briançon and Speder gave the following example of μ -constant family which is not a μ^* -constant family.

$$(1.1) \quad z_1^5 + tz_1z_3^6 + z_2z_3^7 + z_2^{15} = 0.$$

This has a weak simultaneous resolution by [5, 10]. The purpose of this paper is to generalize this in the following case.

Let $f_1(z) = \sum_{\nu} b_{\nu} z^{\nu}$ be an analytic function defined in a neighbourhood of the origin. We assume that f_1 has a non-degenerate Newton boundary which is convenient and let A be a vertex of $\Gamma(f_1)$. Let $f_t(z) = f_1(z) - (1-t) b_A z^A$. We say that $f_t(z)$ a negligible truncation, if the following conditions are satisfied.

- (i) There exists an open disk D in $\mathbb{C}$ containing the unit

interval $[0,1]$ such that $f_t(z)$ has a non-degenerate Newton boundary $\Gamma(f_t)$ for each $t \in D$.

(ii) f_0 is convenient and $\Gamma_-(f_1)$ is a proper subset of $\Gamma_-(f_0)$.

(iii) $\nu(\Gamma_-(f_1)) = \nu(\Gamma_-(f_0))$.

Here $\nu(W)$ is the Newton number of the polyhedron W . See §2.

The example of Biançon-Speder does not satisfy the convenience condition in (ii). But this can be modified by adding y^N for a sufficiently large N for which the isomorphism class of f_t does not change.

Let $f_t(z)$ be a negligible truncation and let $\pi : V \rightarrow D$ be as above. Then the following is the result.

Theorem (1.2) $\pi : V \rightarrow D$ has a weak simultaneous resolution.

In general, the family of a negligible truncation is not μ^* -constant.

§2. Positivity of the Newton numbers

Let W be a polyhedron in $R_+^n = \{ (x_i) \in R^n ; x_i \geq 0 \}$. Recall that the Newton number $\nu(W)$ is defined by

$$\sum_I (-1)^{n-|I|} |I|! \dim \text{volume}(W^I)$$

where the sum is taken for every subset I of $\{1, \dots, n\}$ and $W_I = \{ (x_i) ; x_i = 0 \text{ for } i \notin I \}$. The corresponding term for $I = \emptyset$, is $(-1)^n$ or 0 according to $\vec{0} \in W$ or not. Let

$W = W_1 \cup W_2$ be a polyhedral decomposition of W . Then we have

$$(2.1) \quad \nu(W) = \nu(W_1) + \nu(W_2) - \nu(W_1 \cap W_2).$$

Now we consider the case that $n = 3$ and W is a three dimensional simplex with integral vertices A, B, C and D . Let $A = (a_1, a_2, a_3), \dots, D = (d_1, d_2, d_3)$. We assume that $\vec{0} \notin W$. Let h be the number of I 's such that $|I| = 2$ and $\dim W^I = 2$. Let ℓ the number of i 's such that $\dim W^{\{i\}} = 1$.

Lemma 2.2. Assume that $\vec{0}$ is not in W . Then $\nu(W) \geq 0$. The equality holds if and only if one of the following conditions are satisfied (up to a permutation of the coordinates).

- (i) $h = 2, \ell = 1$. We assume that A, C be in $R^{\{3\}}$, B be in $R^{\{1,3\}}$ and D be in $R^{\{2,3\}}$. Then either $b_1 = 1$ or $d_2 = 1$.
- (ii) $h = 1$ and $\ell = 0$. Assume that A, B and C be in $R^{\{1,3\}}$. Then $d_2 = 1$.

Proof. If $h = 0$, it is obvious that $\nu(W) > 0$. Assume that $h = 1$ and A, B and C be in $R^{\{1,3\}}$. Then we have

$$\nu(W) \geq s d_2 - s = s (d_2 - 1) \geq 0$$

where $s = 2 \text{ volume } W^{\{1,3\}}$. The first equality holds if and only if $\ell = 0$. Thus $\nu(W) = 0$ if and only if $d_2 = 1$ and $\ell = 0$. Assume that $h = 2$ and $b_2 = d_1 = 0$ and $a_i = c_i = 0$ for $i = 1, 2$ and that $c_3 > a_3$. Then we have

$$\nu(W) = (c_3 - a_3)(d_2 b_1 - b_1 - d_2 + 1) = (c_3 - a_3)(d_2 - 1)(b_1 - 1).$$

Thus $\nu(W) = 0$ if and only if b_1 or d_2 is 1. The case $h = 3$ is eliminated by the assumption on W . This completes the proof.

Remark 2.3. The analogous assertion in Lemma 2.2 is not true for the higher dimensions. For example, let $n = 4$ and let W be the simplex spun by $A = (1+t, 0, 0, 0)$, $B = (1, 0, 0, 0)$, $C = (1, 2, 3, 0)$, $D = (1, 3, 2, 0)$ and $E = (1, 1, 1, 1)$. Then $\nu(W)$ is $-t$.

§3. Negligible truncations

Let $f_1(z_1, z_2, z_3) = \sum_{\nu} b_{\nu} z^{\nu}$ be an analytic function defined in a neighbourhood of the origin and assume that $f_1(z)$ has a non-degenerate Newton boundary $\Gamma(f_1)$. We also assume that f_1 is convenient in the sense of Kouchnirenko [1] where $\mu(f_1)$ is the Milnor number of f_1 at $\vec{0}$. Namely $\Gamma(f_1)^{(i)}$ is non-empty for each i . Let $\Gamma_-(f_1)$ be the cone of $\Gamma(f_1)$ with the origin. Recall that $\nu(\Gamma_-(f_1)) = \mu(f_1)$ by Kouchnirenko [1]. Let $A = (a_1, a_2, a_3)$ be a vertex of $\Gamma(f)$ and let $f_t(z) = f(z) - (1-t) b_A z^A$ be a negligible truncation. Let W be the closure of $\Gamma(f_0) - \Gamma(f_1)$ and let $\hat{W}$ be the cone of W with A . Then it is easy to see that $\Gamma_-(f_0) = \Gamma_-(f_1) \cup \hat{W}$. Thus the assumption that $\nu(\Gamma_-(f_1)) = \nu(\Gamma_-(f_0))$ and (2.1) implies $\nu(\hat{W}) = 0$. We can triangulate W in a finite two-simplices so that any vertex on a coordinate axis is contained in a unique two-simplex.

Then by the equality $\nu(\hat{W}) = 0$ and (2.1) and Lemma 2.2, we conclude that W is a simplex spun by three integral vertices B, C, D and $\hat{W}$ satisfies one of the conditions in Lemma 2.2. We say that $f_t(z)$ a negligible truncation of type (i) or of (ii) when $\hat{W}$ is of type (i) or of (ii) of Lemma 2.2 respectively. Before we proceed to prove Theorem 1.2, we give some examples.

Example 3.1 (E_7 , type (i)) Let $f_t(x, y, z) = x^2 + y^3 + yz^3 + tz^5 + z^k$ for $k > 5$. Then f_t is a μ^* -constant family.

More generally, assume that $f_t(z)$ be a negligible truncation of type (i). Then one can show that $f_t(z)$ is a μ^* -constant family.

Example 3.2 (Type (ii), Briançon-Speder) Let

$$f_t(x, y, z) = z^5 + t y^6 z + y^7 x + y^k + x^{15} \quad (k \geq 8)$$

This is not a μ^* -constant family.

Example 3.3. (Type (ii)) Let

$$f_t(x, y, z) = x^8 + y^l + z^l + t x^5 z^2 + x^3 y z^3$$

where $l \geq 16$. Then μ^* is not constant. In fact, $\mu(f_t) = 2l^2 + 18l + 7$ and the Milnor numbers of the generic hyperplane sections of f_1 and f_0 are $2l + 23$ and $2l + 24$ respectively. These examples show that negligible truncations of type (ii) are not generally μ^* -constant.

§4. Resolution by Toroidal embedding

In this section, we recall the resolution of $V_1 = f_1^{-1}(0)$ briefly. The dual vector space of $\mathbb{R}^3$ can be identified canonically with itself. To distinguish vectors in $\mathbb{R}^3$ and in the dual space, we write dual vectors by column vectors. Let N^+ be the subset of the dual vectors which are non-negative. Let P be a vector in N^+ . We denote by $\Delta(P)$ the face of $\Gamma_+(f_1)$ where P takes its minimal value $d(P)$ as a function on $\Gamma_+(f)$. Here $\Gamma_+(f_1)$ is the upper half space in $(\mathbb{R}^+)^3$ with boundary $\Gamma(f_1)$. We introduce an equivalence relation $\sim$ in N^+ by $P \sim Q$ if and only if $\Delta(P) = \Delta(Q)$. This gives a conical subdivision of N^+ which we call the dual Newton diagram and denote it by $\Gamma^*(f_1)$. We can subdivide $\Gamma^*(f_1)$ into a cone over a simplicial complex Σ^* such that each three simplex $\sigma = (P_1, P_2, P_3)$ of Σ^* is a unimodular matrix. We call Σ^* a unimodular simplicial subdivision of $\Gamma^*(f_1)$. See Varchenko [9] and Oka [4] for detail. Let Σ^* be a unimodular simplicial subdivision. For each three simplex $\sigma = (P_1, P_2, P_3) = (p_{ij})$ of Σ^* , we associate a tree space C_σ^3 with coordinates $y_\sigma = (y_{\sigma 1}, y_{\sigma 2}, y_{\sigma 3})$ and the birational morphism $\pi_\sigma : C_\sigma^3 \rightarrow C^3$ which is defined by $z_i = y_{\sigma 1}^{p_{i1}} y_{\sigma 2}^{p_{i2}} y_{\sigma 3}^{p_{i3}}$ ($i = 1, 2, 3$). Then we glue C_σ^3 's in a canonical way to get a smooth complex manifold X and proper birational morphism $\hat{\pi} : X \rightarrow C^3$. Let $\tilde{V}_1$ be the proper transform of V_1 and let $\pi : \tilde{V}_1 \rightarrow V_1$. Then $\pi : \tilde{V}_1 \rightarrow V_1$ is a resolution of the singularity $\vec{0}$ of V_1 . In C_σ^3 , $\tilde{V}_1$ is defined by $f_{1,\sigma}(y_\sigma) = 0$ where

$$f_{1,\sigma}(y_\sigma) = f_1(\pi_\sigma(y_\sigma)) / \prod_{i=1}^3 y_{\sigma i}^{d(P_i)}.$$

For each vertex P such that $\dim \Delta(P) \geq 1$, there is a corresponding exceptional divisor $E(P)$. Suppose that $P = P_1$. Then $E(P)$ is defined in $\mathbb{C}_\sigma^3$ by

$$h_\sigma(y_\sigma) = f_{1,\Delta(P)}(y_\sigma) / \prod_{i=1}^3 y_{\sigma i}^{d(P_i)}.$$

For a further detail, we refer Oka [4].

§5. Proof of Theorem 1.2

Let $f_1(z) = \sum_{\nu} b_{\nu} z^{\nu}$ and let $f(z,t) = f_1(z) - (1-t) b_A z^A$ be a negligible truncation as in Theorem 1.2 of §1. Let $V = \{(z,t) \in \mathbb{C}^3 \times D : f_t(z) = 0\}$. Let Σ^* be a unimodular simplicial subdivision. We apply the mapping $\pi : X \rightarrow \mathbb{C}^3$ simultaneously to f_t to resolve the singularities of V . Namely let $\hat{\pi} : X \times D \rightarrow \mathbb{C}^3 \times D$ be the projection defined by $\hat{\pi}(y,t) = (\hat{\pi}(y), t)$ and let $\tilde{V}$ be the proper transform of V . Let $\omega : V \rightarrow D$ be the projection into D . We denote the restriction of $\hat{\pi}$ to $\tilde{V}$ by Π . Let $\tilde{V}_t$ be $\Pi^{-1}(\omega^{-1}(t))$. We are going to show that $\Pi : \tilde{V} \rightarrow V$ is a simultaneous resolution of V if we choose a suitable unimodular simplicial subdivision Σ^* .

Let $\hat{W}$ and A, B, C and D be as in §3. We first prove Theorem 1.2 assuming $\hat{W}$ is of type (ii) of Lemma 2.2. The case of (i) can be proved in a similar way. Let $A = (a_1, 0, a_3)$, $B = (b_1, 0, b_3)$, $C = (c_1, 0, c_3)$ and

$D = (d_1, 1, d_3)$. Let Δ_1 and Δ_2 be the faces of $\Gamma(f_1)$ which are spun by A, B, D and A, C, D respectively. Let $P = {}^t(p_1, p_2, p_3)$ and $Q = {}^t(q_1, q_2, q_3)$ be the respective weight vector of Δ_1 and Δ_2 . By the definition, $\Delta(P) = \Delta_1$ and $\Delta(Q) = \Delta_2$. (Strictly speaking, Δ_1 may be a subset of $\Delta(P)$ and Δ_2 may be a subset of $\Delta(Q)$.) P and Q must satisfy the following equalities and inequalities.

$$(5.1) \quad p_1 a_1 + p_3 a_3 = p_1 b_1 + p_3 b_3 = p_1 d_1 + p_2 + p_3 d_3 \\ < p_1 c_1 + p_3 c_3$$

$$(5.2) \quad q_1 a_1 + q_3 a_3 = q_1 c_1 + q_3 c_3 = q_1 d_1 + q_2 + q_3 d_3 \\ < q_1 b_1 + q_3 b_3.$$

From (5.1) and (5.2), we have

$$(5.3) \quad p_2 = p_1(a_1 - d_1) + p_3(a_3 - d_3) \quad \text{and}$$

$$(5.4) \quad q_2 = q_1(a_1 - d_1) + q_3(a_3 - d_3).$$

As P and Q are primitive vectors, (5.3) and (5.4) implies

$$(5.6) \quad \text{G.C.D.}(p_1, p_3) = \text{G.C.D.}(q_1, q_3) = 1$$

We may assume that $b_1 > a_1 > c_1$, or equivalently $b_3 < a_3 < c_3$. From (5.1), (5.2) and (5.6), we have

$$(5.7) \quad (b_1 - a_1) = rp_3, \quad (a_3 - b_3) = rp_1$$

$$(5.8) \quad (a_1 - c_1) = sq_3, \quad (c_3 - a_3) = sq_1$$

for some positive integers r and s . Thus by the inequality

of (5.1), we have

$$(5.9) \quad q_1 p_3 - q_3 p_1 > 0.$$

Let $R = {}^t(0,1,0)$. Then the interior T of the triangle $T(P, Q, R)$ is an equivalence class in $\Gamma^*(f_1)$. Namely for a dual vector S , $\Delta(S) = \{A\}$ if and only if $S \in T$. By (5.6), we have $\det(P, R) = \det(Q, R) = 1$. (For the definition $\det(P, Q)$, see Oka [4].) Thus we do not need any other vertex on the line segment $\overline{PR}$ and $\overline{QR}$. On the other hand, using (5.3) and (5.4), we easily see that

$$(5.10) \quad \det(P, Q) = \det(P, Q, R) = q_1 p_3 - q_3 p_1.$$

This implies the following. Let $T_1, \dots, T_k$ be the canonical primitive sequence of $\overline{PQ}$ in the sense of [4]. Then two-simplices (T_i, T_{i+1}, R) ($i = 0, \dots, k$) are already unimodular. Thus we do not need any new vertices in T to subdivide $\Gamma^*(f_1)$. This is the key to the proof. We take a unimodular simplicial subdivision Σ^* which is, restricted on two-simplex $T(P, Q, R)$, the one described above. Now we consider $\Pi : \tilde{V} \rightarrow V$ which is associated to Σ^* . It is easy to see that $\tilde{V} - \tilde{V}_0$ and $\tilde{V}_t$ ($t \neq 0$) are non-singular. Let $\sigma = (P_1, P_2, P_3)$. Then in the coordinate chart $C_\sigma^3 \times D$, $\tilde{V}$ is defined by

$$f_\sigma(y_\sigma, t) = f(\pi_\sigma(y_\sigma), t) / \prod_{i=1}^3 y_{\sigma i}^{d(P_i)} = 0.$$

and $E(P_1)$ is defined by

$$h_\sigma(y_\sigma, t) = f_{\Delta(P_1)}(\pi_\sigma(y_\sigma), t) / \prod_{i=1}^3 y_{\sigma i}^{d(P_i)} = 0.$$

This is a polynomial of $y_{\sigma 2}$, $y_{\sigma 3}$ and t and

$$f_{\sigma}(y_{\sigma}, t) \equiv h_{\sigma}(y_{\sigma}, t) \text{ modulo } (y_{\sigma 1}).$$

Let $\xi(\sigma) = \bigcap_{i=1}^3 \Delta(P_i)$. Then the constant term of $h_{\sigma}(y_{\sigma}, t)$ with respect to y_{σ} is $b_{\xi(\sigma)}$ if $\xi(\sigma) \neq A$. Thus in this case, $\tilde{V}$ and $\tilde{V}_0$ is non-singular and $\Pi : E \rightarrow D$ is simple in this chart where $E = \Pi^{-1}(\vec{0} \times D)$. Assume that $\sigma = (T_i, T_{i+1}, R)$ ($T_0 = P$ and $T_{k+1} = Q$). (This is the most essential chart to be studied carefully.) Then $E(T_i)$ is defined by

$$\begin{aligned} h_{\sigma}(y_{\sigma 2}, y_{\sigma 3}, t) &= b_A t + b_D y_{\sigma 3} = 0. \quad (0 < i < k-1) \\ &= b_A t + b_D y_{\sigma 3} + b_B y_{\sigma 2}^r = 0 \quad (i = 0) \end{aligned}$$

This is easy to see by a direct calculation using (5.3), ..., (5.9) and by the fact that

$$T_1 = (Q + \alpha P) / (q_1 p_3 - q_3 p_1)$$

for a (unique) integer α such that $0 < \alpha < \det(P, Q)$. See Oka [4]. Thus $E(T_i)$ is non-singular in any case by the existence of the linear term $b_D y_{\sigma 3}$. By the same reason, $\tilde{V}_0$ and V are non-singular and $\Pi : E \rightarrow D$ is simple over D . The smoothness of $E(Q)$ is proved in a similar way.

Now we consider the case of a negligible truncation of type (i). Then by the same notation as above, we have to replace $A = (0, 0, a_3)$, $B = (b_1, 0, b_3)$, $C = (0, 0, c_3)$ and $D = (0, 1, d_3)$. Also the weight vector Q is simply $t(1, 0, 0)$. The rest of the argument is completely parallel

to the above argument. This completes the proof of Theorem 1.2.

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