

Perfectness and Multicoloring of Unit Disk Graphs on Triangular Lattice Points

上智大学・理工学部 宮本 裕一郎 (Yuichiro MIYAMOTO)
Faculty of Science and Technology,
Sophia University

東京大学大学院・情報理工学系研究科 松井 知己 (Tomomi MATSUI)
Graduate School of Information Science and Technology,
University of Tokyo

概要

Given a pair of non-negative integers m and n , $P(m, n)$ denotes a subset of 2-dimensional triangular lattice points defined by $P(m, n) \stackrel{\text{def.}}{=} \{(xe_1 + ye_2) \mid x \in \{0, 1, \dots, m-1\}, y \in \{0, 1, \dots, n-1\}\}$ where $e_1 \stackrel{\text{def.}}{=} (1, 0)$, $e_2 \stackrel{\text{def.}}{=} (1/2, \sqrt{3}/2)$. Let $T_{m,n}(d)$ be an undirected graph defined on vertex set $P(m, n)$ satisfying that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d . In this paper, we discuss a necessary and sufficient condition that $T_{m,n}(d)$ is perfect. More precisely, we show that $[\forall m \in \mathbb{Z}_+, T_{m,n}(d) \text{ is perfect}]$ if and only if $d \geq \sqrt{n^2 - 3n + 3}$.

Given a non-negative vertex weight vector $w \in \mathbb{Z}_+^{P(m,n)}$, a multicoloring of $(T_{m,n}(d), w)$ is an assignment of colors to $P(m, n)$ such that each vertex $v \in P(m, n)$ admits $w(v)$ colors and every adjacent pair of two vertices does not share a common color. We also give an efficient algorithm for multicoloring $(T_{m,n}(d), w)$ when $P(m, n)$ is perfect.

In general case, our results on the perfectness of $P(m, n)$ implies a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d), w)$. Our algorithm finds a multicoloring which uses at most $\alpha(d)\omega + O(d^3)$ colors, where ω denotes the weighted clique number. When $d = 1, \sqrt{3}, 2, \sqrt{7}, 3$, the approximation ratio $\alpha(d) = (4/3), (5/3), (5/3), (7/4), (7/4)$, respectively. When $d > 1$, we showed that $\alpha(d) \leq \left(1 + \frac{2}{\sqrt{3} + \frac{2\sqrt{3}-3}{d}}\right)$.

We also showed the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d), w)$ with strictly less than $(4/3)\omega$ colors.

1 Introduction

Given a pair of non-negative integers m and n , $P(m, n)$ denotes the subset of 2-dimensional integer triangular lattice points defined by

$$P(m, n) \stackrel{\text{def.}}{=} \{(xe_1 + ye_2) \mid x \in \{0, 1, 2, \dots, m-1\}, y \in \{0, 1, 2, \dots, n-1\}\}$$

where $e_1 \stackrel{\text{def.}}{=} (1, 0)$, $e_2 \stackrel{\text{def.}}{=} (1/2, \sqrt{3}/2)$. Given a finite set of 2-dimensional points $P \subseteq \mathbb{R}^2$ and a positive real d , a *unit disk graph*, denoted by (P, d) , is an undirected graph with vertex set

P such that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d . We denote the unit disk graph $(P(m, n), d)$ by $T_{m,n}(d)$.

Given an undirected graph H and a non-negative integer vertex weight w' of H , a *multicoloring* of (H, w') is an assignment of colors to vertices of H such that each vertex v admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A *multicoloring problem* on (H, w') finds a multicoloring of (H, w') which minimizes the required number of colors. The multicoloring problem is also known as weighted coloring [4], minimum integer weighted coloring [15] or w -coloring [12].

In this paper, we study weighted unit disk graphs on triangular lattice points $(T_{m,n}(d), w)$. First, we show a necessary and sufficient condition that $T_{m,n}(d)$ is a perfect graph. If the graph is perfect, we can solve the multicoloring problem easily. Next, we propose a polynomial time approximation algorithm for multicoloring $(T_{m,n}(d), w)$. Our algorithm is based on the well-solvable case that the given graph is perfect. For any $d \geq 1$, our algorithm finds a multicoloring which uses at most

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}}d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right) \omega + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) [d+1]^2$$

colors, where ω denotes the weighted clique number. Table 1 shows the values of the above approximation ratio in case that d is small.

表 1: Approximation ratio

d	$1 \dots$	$\sqrt{3} \dots$	$\sqrt{7} \dots$	$2\sqrt{3} \dots$	$\sqrt{13} \dots$	$\sqrt{19} \dots$
ratio	$4/3$	$5/3$	$7/4$	2	$9/5$	2
d	$\sqrt{21} \dots$	$3\sqrt{3} \dots$	$\sqrt{31} \dots$	$\dots$		∞
ratio	$11/6$	2	$13/7$	$\dots$		$1 + 2/\sqrt{3}$

We also show the NP-completeness of the problem to determine the existence of a multicoloring of $(T_{m,n}(d), w)$ which uses strictly less than $(4/3)\omega$ colors.

The multicoloring problem has been studied in several context. When a given graph is the triangular lattice graph $T_{m,n}(1)$, the problem is related to the radio channel (frequency) assignment problem. McDiarmid and Reed [9] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [9, 12] independently gave $(4/3)$ -approximation algorithms for this problem. In case that a given graph H is a square lattice graph or a hexagonal lattice graph, the graph H becomes bipartite and so we can obtain an optimal multicoloring of (H, w') in polynomial time (see [9] for example). Halldórsson and Kortsarz [5] studied planar graphs and partial k -trees. For both classes, they gave a polynomial time approximation scheme (PTAS) for variations of multicoloring problem with min-sum objectives. These objectives appear in the context of multiprocessor task scheduling. For coloring (general) unit disk graphs, there exists a 3-approximation algorithm [6, 8, 14]. Here we note that the approximation ratio of our algorithm is less than $1 + 2/\sqrt{3} < 2.155$ for any $d \geq 1$.

2 Well-Solvable Cases and Perfectness

In this section, we discuss some well-solvable cases such that the multicoloring number is equivalent to the weighted clique number.

An undirected graph G is *perfect* if for each induced subgraph H of G , the chromatic number of H , denoted by $\chi(H)$, is equal to its clique number $\omega(H)$. The following theorem is a main result of this paper.

Theorem 1 *When $n \geq 1$ and $d \geq 1$, we have the following;*
 $[\forall m \in \mathbb{Z}_+, T_{m,n}(d) \text{ is perfect}]$ *if and only if* $d \geq \sqrt{n^2 - 3n + 3}$.

Table 2 shows the perfectness and imperfectness of $T_{m,n}(d)$ for small n and d .

表 2: Perfectness and imperfectness				
$n \backslash d$	$1 \dots \sqrt{3} \dots 2 \dots$	$\sqrt{7} \dots 3 \dots 2\sqrt{3} \dots$	$\sqrt{13} \dots 4 \dots \sqrt{19} \dots$	$\sqrt{21} \dots$
1				
2				
3				
4				Perfect
5				
6	Imperfect			
$\vdots$				

To show the above theorem, we introduce some definitions. We say that an undirected graph has a transitive orientation property, if each edge can be assigned a one-way direction in such a way that the resulting directed graph (V, F) satisfies that $[(a, b) \in F \text{ and } (b, c) \in F \text{ imply } (a, c) \in F]$. An undirected graph which is transitively orientable is called *comparability graph*. The complement of a comparability graph is called *co-comparability graph*. It is well-known that every co-comparability graph is perfect.

Lemma 1 *For any integer $n \geq 1$, if $d \geq \sqrt{n^2 - 3n + 3}$, then $T_{m,n}(d)$ is a co-comparability graph.*

Proof: omitted.

The following lemma deals with the special case that $n = 3$, $d = 1$.

Lemma 2 *For any $m \in \mathbb{Z}_+$ and $1 \leq \forall d < \sqrt{3}$, the graph $T_{m,3}(d)$ is perfect.*

Proof: We only need to consider the case that $d = 1$, since $T_{m,n}(d) = T_{m,n}(1)$ when $1 \leq d < \sqrt{3}$. Let H be an induced subgraph of $T_{m,3}(1)$. When $\omega(H) \leq 2$, H has no 3-cycle. Then it is easy to show that H has no odd cycle and thus $\chi(H) = \omega(H)$, since H is bipartite. If $\omega(H) \geq 3$, then it is clear that $\omega(H) = 3$ and $\chi(H) \leq 3$, since $\omega(T_{m,3}(1)) = 3$ and $T_{m,3}(1)$ has a trivial 3-coloring. ■

Note that though the graph $T_{m,3}(1)$ is perfect, the graph $T_{m,3}(1)$ is not co-comparability graph.

From the above, the perfectness of a graph satisfying the conditions of Theorem 1 is clear. In the following, we discuss the inverse implication. We say that an undirected graph G has an *odd-hole*, if G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5. It is obvious that if a graph has an odd-hole, the graph is not perfect. In the following, we denote a point $(xe_1 + ye_2) \in P(m, n)$ by $\langle x, y \rangle$.

Lemma 3 *If $1 \leq d < \sqrt{7}$, then $\forall m \geq 5$, $T_{m,4}(d)$ has at least one odd-hole.*

Proof: If $1 \leq d < \sqrt{3}$, then a subgraph induced by $\{ \langle 2, 0 \rangle, \langle 1, 1 \rangle, \langle 0, 2 \rangle, \langle 0, 3 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 1 \rangle, \langle 3, 0 \rangle \}$ is a 9-hole. If $\sqrt{3} \leq d < 2$, then a subgraph induced by $\{ \langle 3, 0 \rangle, \langle 1, 1 \rangle, \langle 0, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle, \langle 4, 2 \rangle, \langle 4, 1 \rangle \}$ is a 7-hole. If $2 \leq d < \sqrt{7}$, then a subgraph induced by $\{ \langle 2, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 3 \rangle, \langle 3, 2 \rangle, \langle 3, 0 \rangle \}$ is a 5-hole. When $1 \leq d < \sqrt{7}$, $T_{5,4}(d)$ has at least one odd-hole, and hence the proof is completed. ■

Lemma 4 *If $1 \leq d < \sqrt{13}$, then $\forall m \geq 6$, $T_{m,5}(d)$ has at least one odd-hole.*

Proof: If $1 \leq d < \sqrt{7}$, then odd-holes in the proof of Lemma 3 are induced subgraph of $T_{6,5}(d)$. If $\sqrt{7} \leq d < 3$, then a subgraph induced by $\{ \langle 2, 0 \rangle, \langle 0, 2 \rangle, \langle 1, 4 \rangle, \langle 4, 2 \rangle, \langle 4, 0 \rangle \}$ is a 5-hole. If $3 \leq d < \sqrt{13}$, then a subgraph induced by $\{ \langle 3, 0 \rangle, \langle 0, 3 \rangle, \langle 2, 4 \rangle, \langle 5, 3 \rangle, \langle 5, 0 \rangle \}$ is a 5-hole. When $1 \leq d < \sqrt{13}$, $T_{6,5}(d)$ has at least one odd-hole, and hence the proof is completed. ■

Lemma 5 *For any integer $n \geq 4$, if $1 \leq d < \sqrt{n^2 - 3n + 3}$, then $\exists m \in \mathbb{Z}_+$, $T_{m,n}(d)$ is imperfect.*

Proof: In the following, we show that $\forall n \geq 4$, if $1 \leq d < \sqrt{n^2 - 3n + 3}$, then $\exists m \in \mathbb{Z}_+$, $T_{m,n}(d)$ has at least one odd-hole, by induction on n . When $n = 4, 5$, it is clear from Lemmas 3 and 4, respectively.

Now we consider the case that $n = n' \geq 6$ under the assumption that if $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3}$, then $\exists m' \in \mathbb{Z}_+$, $T_{m',n'-1}(d)$ has at least one odd-hole. If $1 \leq d < \sqrt{(n'-1)^2 - 3(n'-1) + 3} = \sqrt{n'^2 - 5n' + 7}$, then $T_{m',n'}(d)$ has at least one odd-hole, since $T_{m',n'-1}(d)$ is an induced subgraph of $T_{m',n'}(d)$. In the remained case that $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$, the set of points $\{ \langle n' - 3, 0 \rangle, \langle 0, n' - 2 \rangle, \langle n' - 4, n' - 1 \rangle, \langle 2n' - 7, n' - 2 \rangle, \langle 2n' - 6, 0 \rangle \}$ is contained in $P(m'', n')$, if $m'' = 2n' - 5$. It is easy to see that the above five vertices induces a 5-hole of $T_{m'',n'}(d)$, when $n' \geq 6$ and $\sqrt{n'^2 - 5n' + 7} \leq d < \sqrt{n'^2 - 3n' + 3}$. ■

Lemma 5 shows the imperfectness of every graph which violates a condition of Theorem 1. Thus, we completed a proof of Theorem 1. From the above lemmas, the following is immediate.

Corollary 1 *Let $d > 1$ be a real number. Then, $T_{m,n}(d)$ is a co-comparability graph, if and only if $n \leq \frac{3 + \sqrt{4d^2 - 3}}{2}$.*

Lastly, we discuss some algorithmic aspects. Assume that we have a co-comparability graph G and related digraph H which gives a transitive orientation of the complement of G . Then each independent set of G corresponds to a chain (directed path) of H . The multicoloring

problem on G is essentially equivalent to the minimum size chain cover problem on H . Every clique of G corresponds to an anti-chain of H . Thus the equality $\omega(G) = \chi(G)$ is obtained from Dilworth's decomposition theorem [2]. It is well-known that the minimum size chain cover problem on an acyclic graph is solvable in polynomial time by using an algorithm for minimum-cost circulation flow problem (see [13] for example).

Though an weighted graph $(T_{m,3}(1), \mathbf{w})$ is not a co-comparability graph, we can construct exact multicoloring algorithm for the graph. Here we omit the detail.

3 Approximation Algorithm

In this section, we propose an approximation algorithm for multicoloring the graph $(T_{m,n}(d), \mathbf{w})$. When $d = 1$, McDiarmid and Reed [9] proposed an approximation algorithm for $(T_{m,n}(1), \mathbf{w})$, which finds a multicoloring with at most $(4/3)\omega(T_{m,n}(1), \mathbf{w}) + 1/3$ colors.

In the following, we propose an approximation algorithm for $(T_{m,n}(d), \mathbf{w})$ when $d > 1$. The basic idea of our algorithm is similar to the shifting strategy [7].

Theorem 2 *When $d > 1$, there exists a polynomial time algorithm for multicoloring $(T_{m,n}(d), \mathbf{w})$ such that the number of required colors is bounded by*

$$\left(1 + \frac{\left\lfloor \frac{2}{\sqrt{3}}d \right\rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right) \omega(T_{m,n}(d), \mathbf{w}) + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) \chi(T_{m,n}(d)).$$

Proof: We describe an outline of the algorithm. For simplicity, we define $K_1 = \left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor$ and $K_2 = \left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor + \left\lfloor \frac{2}{\sqrt{3}}d \right\rfloor$.

First, we construct K_2 vertex weights $\mathbf{w}'_k$ for $k \in \{0, 1, \dots, K_2 - 1\}$ by setting

$$w'_k(x, y) = \begin{cases} 0, & y \in \left\{k, k+1, \dots, k + \left\lfloor \frac{2}{\sqrt{3}}d \right\rfloor - 1\right\} \pmod{K_2}, \\ \left\lfloor \frac{w(x, y)}{K_1} \right\rfloor, & \text{otherwise.} \end{cases}$$

Next, we exactly solve K_2 multicoloring problems defined by K_2 pairs $(T_{m,n}(d), \mathbf{w}'_k)$, $k \in \{0, 1, \dots, K_2 - 1\}$ and obtain K_2 multicolorings. We can solve each problem exactly in polynomial time, since every connected component of the graph induced by the set of vertices with positive weight is a perfect graph discussed in the previous section. Thus $\chi(T_{m,n}(d), \mathbf{w}'_k) = \omega(T_{m,n}(d), \mathbf{w}'_k)$ for any $k \in \{0, 1, \dots, K_2 - 1\}$. Put $\mathbf{w}'' = \mathbf{w} - \sum_{k=0}^{K_2-1} \mathbf{w}'_k$. Then each element of $\mathbf{w}''$ is less than or equal to $K_1 - 1$. Thus we can find a multicoloring of $(T_{m,n}(d), \mathbf{w}'')$ from the direct sum of $K_1 - 1$ trivial colorings of $T_{m,n}(d)$. The obtained multicoloring uses at most $(K_1 - 1)\chi(T_{m,n}(d))$ colors. Lastly, we output the direct sum of $K_2 + 1$ multicolorings obtained above. The definition of the weight vector $\mathbf{w}'_k$ implies that $\forall k \in \{0, 1, \dots, K_2 - 1\}$, $K_1 \omega(T_{m,n}(d), \mathbf{w}'_k) \leq \omega(T_{m,n}(d), \mathbf{w})$. Thus, the obtained multicoloring uses at most $(K_2/K_1)\omega(T_{m,n}(d), \mathbf{w}) + (K_1 - 1)\chi(T_{m,n}(d))$ colors. ■

The following lemma gives the chromatic number of $T_{m,n}(d)$.

表 3: Unit disk graphs on square lattice points

$n \backslash d$	$1 \dots$	$\sqrt{2} \dots$	$2 \dots$	$\sqrt{5} \dots 2\sqrt{2} \dots$
1				
2				
3				
4				

Perfect

Imperfect

Lemma 6 If m, n are sufficiently large, then $\chi(T_{m,n}(d)) = \hat{d}^2$ where $\hat{d}$ is the minimum Euclidean distance between two points in $P(m, n)$ subject to that distance being greater than d . Clearly, $d < \hat{d} \leq \lfloor d + 1 \rfloor$.

Proof: See McDiarmid [9] for example. ■

When d is small, Table 1 shows the approximation ratio. The following corollary gives a simple upper bound of the approximation ratio.

Corollary 2 For any $d \geq 1$, we have $1 + \frac{\lfloor \frac{2}{\sqrt{3}}d \rfloor}{\frac{3 + \sqrt{4d^2 - 3}}{2}} \leq 1 + \frac{2}{\sqrt{3} + \frac{2\sqrt{3} - 3}{d}}$.

Here we note that if we apply our algorithm in the case that $d = 1$, then the algorithm finds a multicoloring which uses at most $(4/3)\omega(T_{m,n}(1), w) + 6$ colors.

4 Discussion

In this paper, we dealt with the triangular lattice. In the following, we discuss the square lattice. Given a pair of non-negative integers m and n , $Q(m, n) \stackrel{\text{def}}{=} \{0, 1, 2, \dots, m-1\} \times \{0, 1, 2, \dots, n-1\}$ denotes the subset of 2-dimensional integer square lattice points. We denote the unit disk graph $(Q(m, n), d)$ by $S_{m,n}(d)$. In case that $d < \sqrt{2}$, it is clear that $S_{m,n}(d) = S_{m,n}(1)$ and the graph is bipartite for any m and n . If $d = \sqrt{2}$, we proposed a $(4/3)$ -approximation algorithm for multicoloring $(S_{m,n}(\sqrt{2}), w)$ in our previous paper [11]. We also showed the NP-hardness of the problem.

Unfortunately, Theorem 1 is not extensible to the square lattice case. Table 3 shows the perfectness and imperfectness of unit disk graphs on the square lattice for small n and d . The perfectness of $T_{m,3}(\sqrt{2})$ was shown in [11]. The graph $S_{m,3}(2)$ contains a 5-hole: $\{(0, 0), (2, 0), (2, 1), (1, 2), (0, 2)\}$.

参考文献

- [1] M. Chudnovsky, N. Robertson, P. D. Seymour and R. Thomas: The strong perfect graph theorem, manuscript, <http://www.math.gatech.edu/~thomas/spgc.html>.

- [2] R. P. Dilworth: A decomposition theorem for partially ordered sets, *Annals of Mathematics*, **51** (1950) 161–166
- [3] M. C. Golumbic: *Algorithmic Graph Theory and Perfect Graphs* (Academic Press, 1980).
- [4] M. Grötschel, L. Lovász and A. Schrijver: *Geometric Algorithms and Combinatorial Optimization* (Springer-Verlag, 1988).
- [5] M. M. Halldórsson and G. Kortsarz: Tools for Multicoloring with Applications to Planar Graphs and Partial k -Trees. *Journal of Algorithms*, **42** (2002) 334–366.
- [6] D. S. Hochbaum: Efficient bounds for the stable set, vertex cover and set packing problems, *Discrete Applied Mathematics*, **6** (1983) 243–254.
- [7] D. S. Hochbaum: Various Notions of Approximations: Good, Better, Best and More. appears in *Approximation Algorithms for NP-hard Problems*, (PWS Publishing Company, 1997).
- [8] M. V. Marathe, H. Breu, H. B. Hunt, S. S. Ravi, and D. J. Rosenkrantz: Simple heuristics for unit disk graph, *Networks*, **25** (1995) 59–68.
- [9] C. McDiarmid and B. Reed: Channel Assignment and Weighted Coloring. *Networks*, **36** (2000) 114–117.
- [10] C. McDiarmid: Discrete Mathematics and Radio Channel Assignment. appears in *Recent Advances in Algorithms and Combinatorics* (Springer-Verlag, 2003).
- [11] Y. Miyamoto and T. Matsui: Linear Time Approximation Algorithm for Multicoloring Lattice Graphs with Diagonals. *Journal of the Operations Research Society of Japan*, (to appear).
- [12] L. Narayanan and S. M. Shende: Static Frequency Assignment in Cellular Networks. *Algorithmica*, **29** (2001) 396–409.
- [13] A. Schrijver: *Combinatorial Optimization* (Springer-Verlag, 2003).
- [14] D. de Werra: Heuristics for graph coloring, *Computing*, Suppl. **7** (1990) 191–208.
- [15] J. Xue: Solving the Minimum Weighted Integer Coloring Problem. *Combinatorial Optimization and Application*, **11** (1998) 53–64.