

# Port Reduction of Multiport Cauer Ladder Network

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The multiport Cauer ladder network (CLN) method is an efficient and accurate model order reduction (MOR) method for multi-input/multi-output systems. This manuscript discusses a theoretical aspect of port reduction in multiport CLN to generate a further reduced system. An intuitively derived method is found to be inaccurate. A recurrence formula and its practical implementation are presented to generate an accurate reduced system, which is proven by computational examples.

**Index Terms**— Cauer ladder network, port reduction.

## I. INTRODUCTION

THE full finite element (FE) magnetic field analysis of an electric machine is often computationally expensive, particularly when the analysis is coupled with a control circuit under many operating conditions. Model order reduction (MOR) is a useful tool for accurately reconstructing the response of an electric machine even with a small number of unknowns. Several MOR methods have been developed. For example, proper orthogonal decomposition (POD) [1] and Padé via the Lanczos (PVL) [2] methods are often used in their multiport versions [3][4] to handle multi-input/multi-output (MIMO) systems.

The Cauer ladder network (CLN) method [5] was recently developed as an efficient and accurate MOR method for eddy current (EC) fields based on the FE method. Its multiport version [6] has been applied to the MOR of induction motors [7].

When the number of input ports is  $M$ , the computational cost of solving the network equations of the multiport CLN increases with  $M^2$  because the network consists of conductance and inductance matrices of size  $M \times M$ .

In some cases, the multi-port input is approximately decomposed into several dominant modes, such as the parallel or series connection mode and the positive and negative phase modes. If the number of dominant modes  $p$  is far less than  $M$ , the number of ports can be reduced substantially, which means that further order reduction is possible. However, the theory of port reduction is not known generally.

This manuscript discusses a theoretical aspect of port reduction in multiport CLN because its simple procedure provides a simple derivation of the relevant formulae. First, an intuitive method is derived; however, it yields an inaccurate reduced representation. A recurrence formula is derived to generate an accurate reduction system.

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## II. PORT REDUCTION OF MULTIPORT CLN

### A. Multiport CLN

The EC field is described in a FE space as

$$\mathbf{C}^T \mathbf{v} \mathbf{C} \mathbf{a} = \mathbf{K} \mathbf{a} = \boldsymbol{\sigma} \mathbf{e} = \boldsymbol{\sigma} (\mathbf{e}_s - \mathbf{s} \mathbf{a}), \quad \mathbf{C} \mathbf{e} = -\mathbf{s} \mathbf{C} \mathbf{a} \quad (1)$$

where  $\mathbf{a}$  and  $\mathbf{e}$  are the unknown vectors consisting of the line integrals of the vector potential and electric field,  $\mathbf{s}$  is the time-derivative operator,  $\mathbf{e}_s$  is the electric field due to the source voltage,  $\mathbf{v}$  is the reluctivity matrix and  $\boldsymbol{\sigma}$  is the conductivity matrix;  $\mathbf{K} = \mathbf{C}^T \mathbf{v} \mathbf{C}$  is the stiffness matrix for FE EC analysis where  $\mathbf{C}$  is the face-edge incidence matrix. While (1) supposes the use of edge elements, the discussion below is applicable to nodal elements used in the 2D analysis.

Let the number of ports be  $M$ . The multiport CLN [6] is constructed using the following recurrence relations:

$$\mathbf{K}(\mathbf{a}_{2n+1} - \mathbf{a}_{2n-1}) = \boldsymbol{\sigma} \mathbf{e}_{2n} \mathbf{G}_{2n}^{-1} \quad (2)$$

$$\mathbf{e}_{2n+2} - \mathbf{e}_{2n} = -\mathbf{a}_{2n+1} \mathbf{L}_{2n+1}^{-1} \quad (3)$$

$$\mathbf{G}_{2n} = \mathbf{e}_{2n}^T \boldsymbol{\sigma} \mathbf{e}_{2n}, \quad \mathbf{L}_{2n+1} = \mathbf{a}_{2n+1}^T \mathbf{K} \mathbf{a}_{2n+1} \quad (4)$$

where  $\mathbf{G}_{2n}$  and  $\mathbf{L}_{2n+1}$  ( $n = 0, 1, \dots$ ) are  $M \times M$  conductance and inductance matrices, respectively, and  $\mathbf{e}_{2n} = (\mathbf{e}_{1,2n}, \dots, \mathbf{e}_{M,2n})$  and  $\mathbf{a}_{2n+1} = (\mathbf{a}_{1,2n+1}, \dots, \mathbf{a}_{M,2n+1})$  are the  $n$ -th basis vector matrices. The  $m$ -th columns  $\mathbf{e}_{m,2n}$  and  $\mathbf{a}_{m,2n+1}$  are field-basis vectors when source  $m$  has a unit voltage and the other voltages are zero.

The electromagnetic fields are expanded as

$$\mathbf{e} = \sum_{n=0}^{N-1} \mathbf{e}_{2n} \mathbf{V}_{2n}, \quad \mathbf{a} = \sum_{n=0}^{N-1} \mathbf{a}_{2n+1} \mathbf{I}_{2n+1} \quad (5)$$

where  $N$  is the number of ladder stages and  $\mathbf{V}_{2n} = (\mathbf{V}_{1,2n}, \dots, \mathbf{V}_{M,2n})^T$  and  $\mathbf{I}_{2n+1} = (\mathbf{I}_{1,2n+1}, \dots, \mathbf{I}_{M,2n+1})^T$  are the coefficient vectors. These are determined from the multiport CLN [Fig. 1] equations, as follows:

$$\mathbf{V}_0 = \mathbf{V}_s - \mathbf{s} \mathbf{L}_1 \mathbf{I}_1,$$

$$\mathbf{V}_{2k} = \mathbf{s}(\mathbf{L}_{2k-1} \mathbf{I}_{2k-1} - \mathbf{L}_{2k+1} \mathbf{I}_{2k+1}) \quad (k = 1, 2, \dots) \quad (6)$$

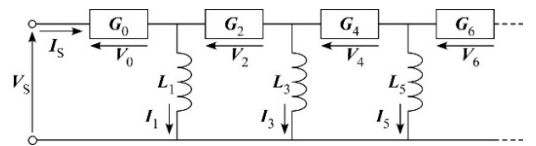


Fig. 1. Multiport CLN, where  $\mathbf{V}_s$ ,  $\mathbf{I}_s$ ,  $\mathbf{V}_{2k}$ , and  $\mathbf{I}_{2k+1}$  are vectors, and  $\mathbf{G}_{2k}$  and  $\mathbf{L}_{2k+1}$  are matrices ( $k = 0, 1, \dots$ ).

$$I_{2k+1} = G_{2k} V_{2k} - G_{2k+2} V_{2k+2} . \quad (7)$$

### B. Simple Port Reduction

Suppose that a multiport power source has  $p$  ( $< M$ ) dominant modes. Then we can reduce the number of ports from  $M$  to  $p$ . Voltage source  $V_S$  is decomposed into  $p$  modes as follows:

$$V_S = v_{S1} u_1 + \dots + v_{Sp} u_p = u v_S, \quad v_S = [v_{S1}, \dots, v_{Sp}]^T \quad (8)$$

where

$$u = [u_1, \dots, u_p], \quad u^T u = \mathbf{1} \quad (9)$$

is the set of orthonormal basis vectors, and  $\mathbf{1}$  is a unit matrix. The components of  $u_j$  means the ratios of input voltages. For instance, if all the ports are connected in parallel to one voltage source,  $u_j = [1, 1, \dots]/\sqrt{M}$ ; In the case of balanced three-phase operation, the dominant modes are  $u_1 = [2, -1, -1]^T/\sqrt{6}$  and  $u_2 = [0, 1, -1]^T/\sqrt{2}$  with  $M = 3$  given by the Clarke transformation.

Using  $u$ ,  $e_0$  is also represented in reduced form as

$$e'_0 = [e'_{0,1}, \dots, e'_{0,p}] = e_0 u \quad (10)$$

$$e'_{0,j} = e_0 u_j . \quad (11)$$

The network variable vectors are accordingly reduced as

$$V_{2n} = v_{2n,1} u_1 + \dots + v_{2n,p} u_p = u v_{2n}, \quad v_{2n} = [v_{2n,1}, \dots, v_{2n,p}]^T \quad (12)$$

$$I_{2n+1} = i_{2n+1,1} u_1 + \dots + i_{2n+1,p} u_p = u i_{2n+1}, \quad i_{2n+1} = [i_{2n+1,1}, \dots, i_{2n+1,p}]^T . \quad (13)$$

Substituting (12) and (13) into (6) and (7) yields

$$u v_0 = u v_S - s L_1 u i_1, \quad u v_{2k} = s(L'_{2k-1} u i_{2k-1} - L'_{2k+1} u i_{2k+1}) \quad (k = 1, 2, \dots) \quad (14)$$

$$u i_{2k+1} = G_{2k} u v_{2k} - G_{2k+2} u v_{2k+2} . \quad (15)$$

By multiplying (14) and (15) by  $u^T$ , the reduced CLN equations are obtained as follows:

$$v_0 = v_S - s L'_1 i_1, \quad v_{2k} = s(L'_{2k-1} i_{2k-1} - L'_{2k+1} i_{2k+1}) \quad (k = 1, 2, \dots) \quad (16)$$

$$i_{2k+1} = G'_{2k} v_{2k} - G'_{2k+2} v_{2k+2} \quad (17)$$

$$G'_{2k} = u^T G_{2k} u, \quad L'_{2k+1} = u^T L_{2k+1} u . \quad (18)$$

### C. Port reduction using the recurrence formula

When the first basis vector  $e_0$  is replaced by  $e'_0$ , the first reduced conductance matrix  $G'_0$  is given by (4) as

$$G'_0 = e'^0_T \sigma e'_0 = u^T e_0^T \sigma e_0 u = u^T G_0 u \quad (19)$$

which agrees with (18). The reduced basis  $a'_1$  is generated from

$$K a'_1 = \sigma e'_0 G'_0{}^{-1} = \sigma e_0 u G'_0{}^{-1} \quad (20)$$

which results in

$$a'_1 = K^{-1} \sigma e_0 u G'_0{}^{-1} = a_1 G_0 u G'_0{}^{-1} . \quad (21)$$

Consequently, the reduced inductance matrix  $L'_1$  is given as

$$L'_1 = a'^1_T K a'_1 = G'_0{}^{-1} u^T G_0 a_1^T K a_1 G_0 u G'_0{}^{-1} = G'_0{}^{-1} u^T G_0 L_1 G_0 u G'_0{}^{-1} \quad (22)$$

which does not agree with (18).

Suppose that the reduced bases are given by the linear combination of original bases as

$$e'_{2n} = \sum_{k=0}^n e_{2k} P_{2n,2k}, \quad a'_{2n+1} = \sum_{k=0}^n a_{2k+1} P_{2n+1,2k+1} . \quad (23)$$

where  $P_{2n,2k}$  and  $P_{2n+1,2k+1}$  are  $M \times p$  coefficient matrices. From (10) and (19), we obtain

$$e'_0 = e_0 u, \quad P_{0,0} = u, \quad G'_0 = u^T G_0 u . \quad (24)$$

As is derived in Appendix, the reduced CLN is generated using the recurrence formulas starting from (24) as below:

$$P_{2n+1,2n+1} = G_{2n} P_{2n,2n} G'_{2n}{}^{-1}, \quad P_{2n+1,2k+1} = P_{2n-1,2k+1} + (G_{2k} P_{2n,2k} - G_{2k+2} P_{2n,2k+2}) G'_{2n}{}^{-1} \quad (k = 0, \dots, n-1), \quad (25)$$

$$L'_{2n+1} = \sum_{k=0}^n P_{2n+1,2k+1}^T L_{2k+1} P_{2n+1,2k+1} . \quad (26)$$

$$P_{2n+2,2n+2} = L_{2n+1} P_{2n+1,2n+1} L'_{2n+1}{}^{-1}, \quad P_{2n+2,2k} = P_{2n,2k} + (L_{2k-1} P_{2n+1,2k-1} - L_{2k+1} P_{2n+1,2k+1}) L'_{2n+1}{}^{-1} \quad (k = 1, \dots, n) \quad (27)$$

$$P_{2n+2,0} = P_{2n,0} - L_1 P_{2n+1,1} L'_{2n+1}{}^{-1} \quad (28)$$

### D. Practical implementation of the recurrence formula

The generation of a reduced system based on the derivation in the preceding subsection may be cumbersome. This can be obtained by applying the CLN procedure again to the multiport CLN represented by (6), (7), and Fig. 1 as follows:

The voltages, currents and voltage source in Fig. 1 are denoted as

$$V = \begin{pmatrix} V_0 \\ V_2 \\ \vdots \end{pmatrix}, \quad I = \begin{pmatrix} I_1 \\ I_3 \\ \vdots \end{pmatrix}, \quad E_S = \begin{pmatrix} V_S \\ 0 \\ \vdots \end{pmatrix} \quad (29)$$

The currents and magnetic flux corresponding to  $V$  and  $I$  are denoted as

$$I_R = G V, \quad \Phi = L I \quad (30)$$

where

$$G = \text{blockdiag}(G_0, G_2, \dots), \quad L = \text{blockdiag}(L_1, L_3, \dots). \quad (31)$$

Kirchhoff's current and voltage laws are written as

$$I = D I_R, \quad V = E_S - s D^T \Phi \quad (32)$$

where

$$D = \begin{pmatrix} 1 & -1 & 0 & \dots \\ 0 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (33)$$

From (30) and (32), the network equation is given as

$$\mathbf{L}^{-1}\Phi = \mathbf{D}\mathbf{I}_R = \mathbf{D}\mathbf{G}(\mathbf{E}_S - \mathbf{s}\mathbf{D}^T\Phi). \quad (34)$$

By setting

$$\Phi_R = \mathbf{D}^T\Phi \quad (35)$$

(34) is rewritten as

$$\mathbf{D}^{-1}\mathbf{L}^{-1}\mathbf{D}^{-T}\Phi_R = \mathbf{K}^r\Phi_R = \mathbf{G}(\mathbf{E}_S - \mathbf{s}\Phi_R) \quad (36)$$

where

$$\mathbf{K}^r = (\mathbf{D}^T\mathbf{L}\mathbf{D})^{-1}. \quad (37)$$

The similarity between (36) and (1) allows port reduction by applying the CLN procedure to system (39). Starting from

$$\alpha'_{-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \varepsilon'_0 = \begin{pmatrix} \mathbf{u} \\ 0 \\ \vdots \end{pmatrix}, \quad \mathbf{G}'_0 = \varepsilon'^T_0 \mathbf{G} \varepsilon'_0 = \mathbf{u}^T \mathbf{G}_0 \mathbf{u} \quad (38)$$

the reduced network elements:  $\mathbf{G}'_{2n}$ ,  $\mathbf{L}'_{2n+1}$  and basis vectors:  $\varepsilon'_{2n}$ ,  $\alpha'_{2n+1}$  are generated from (39)-(42):

$$\mathbf{K}^r(\alpha'_{2n+1} - \alpha'_{2n-1}) = \mathbf{G}\varepsilon'_{2n}\mathbf{G}'_{2n-1}, \quad (39)$$

$$\mathbf{L}'_{2n+1} = \alpha'^T_{2n+1} \mathbf{K}^r \alpha'_{2n+1}, \quad (40)$$

$$\varepsilon'_{2n+2} - \varepsilon'_{2n} = -\alpha'_{2n+1} \mathbf{L}'_{2n+1}{}^{-1}, \quad (41)$$

$$\mathbf{G}'_{2n+2} = \varepsilon'^T_{2n+2} \mathbf{G} \varepsilon'_{2n+2}. \quad (42)$$

If the procedure above is executed without port reduction using

$$\alpha_{-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \varepsilon_0 = \begin{pmatrix} \mathbf{1} \\ 0 \\ \vdots \end{pmatrix} \quad (43)$$

instead of (38), procedures (39)–(42) yield the same network elements as  $\mathbf{G}_{2n}$  and  $\mathbf{L}_{2n+1}$ . Let  $\varepsilon_{2n}$  and  $\alpha_{2n+1}$  denote the basis vectors in this case without port reduction. Using  $\varepsilon_{2n}$  and  $\alpha_{2n+1}$ , we can derive the reduced basis vectors as follows:

$$\varepsilon'_{2n} = \sum_{k=0}^n \varepsilon_{2k} \mathbf{P}_{2n,2k}, \quad \alpha'_{2n+1} = \sum_{k=0}^n \alpha_{2k+1} \mathbf{P}_{2n+1,2k+1}. \quad (44)$$

where coefficient matrices  $\mathbf{P}_{2n,2k}$  and  $\mathbf{P}_{2n+1,2k+1}$  are given by (25) and (27), respectively. Thus, CLN procedures (39)–(42) yield the same reduced system as that derived in Subsection II.C.

### III. COMPUTATIONAL RESULTS

A bulk core with 1st and 2nd solid conductors [Fig. 2(a)] was analyzed. The conductivities of the core and conductors were  $1 \times 10^6$  and  $6 \times 10^7$  S/m, respectively, and their relative permeabilities were 1000 and 1. The 1st-order triangular nodal element is used for the 2D FE analysis.

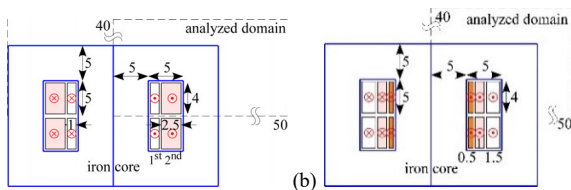


Fig. 2. A bulk core with solid conductors (unit: mm): (a) 2 ports and (b) 3 ports.

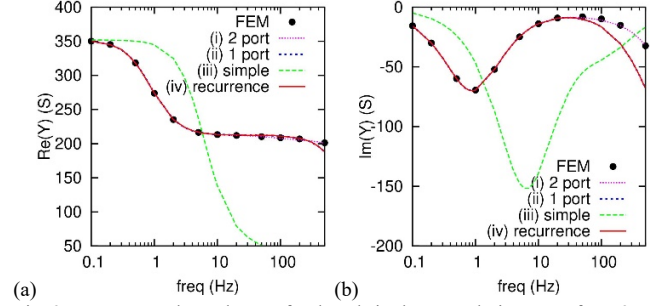


Fig. 3. Frequency dependence of reduced single-port admittance  $Y$  from 2 ports: (a)  $\text{Re}(Y)$  and (b)  $-\text{Im}(Y)$ .

The dominant mode was assumed to be  $p = 1$  and

$$\mathbf{u}_1 = [1/\sqrt{5}, -2/\sqrt{5}]^T. \quad (45)$$

Fig. 3 shows the frequency dependence of admittance given by the 2-stage CLN as follows: (i) the 2-port CLN, (ii) the single-port CLN with input voltage  $\mathbf{V}_s = \mathbf{u}_1$ , (iii) the simple reduction to a single port derived in Subsection III.B, and (iv) the recurrence formula derived in Subsection III.C. The 2-port admittance matrix  $\mathbf{Y}$  is reduced to a single-port admittance by  $\mathbf{u}_1^T \mathbf{Y} \mathbf{u}_1$  in case (i). For comparison, the exact solution was obtained via FE analysis. Simple reduction (iii) results in an inaccurate admittance. In contrast, the recurrence formula (iv) provides the same admittance as the single-port CLN (ii) because they are theoretically equivalent. Two-port CLN (i) is the most accurate because it uses twice as many basis vectors per ladder stage as single-port CLN.

A bulk core with three solid conductors [Fig. 2(b)] was analyzed. The dominant mode was assumed to be  $p = 2$  and

$$\mathbf{u}_1 = [1/\sqrt{2}, 1/\sqrt{2}, 0]^T, \quad \mathbf{u}_2 = [-1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}]^T. \quad (46)$$

The frequency dependence of admittance given by the 4-stage CLN is shown in Fig. 4 as follows: (i) the 3-port CLN, (ii) the 2-port CLN with input voltages  $\mathbf{V}_s = \mathbf{u}_1$  and  $\mathbf{u}_2$ , (iii) the simple reduction to a 2-port CLN derived in Subsection III.B, and (iv) the recurrence formula derived in Subsection III.C. The 3-port admittance matrix  $\mathbf{Y}$  is reduced to a 2-port admittance by  $\mathbf{u}^T \mathbf{Y} \mathbf{u}$ . For comparison, the exact solution was obtained via FE analysis. Simple reduction (iii) gives an inaccurate admittance, whereas the recurrence formula (iv) provides the same admittance as the 2-port CLN (ii), which is slightly less accurate than the 3-port CLN (i). However, the computational cost is the 3-port CLN uses  $\mathbf{L}$  and  $\mathbf{G}$  matrices of size  $3 \times 3$ , which require 9/4 times the computational cost of the 2-port CLN. Fig. 5 shows the frequency plot of  $Y_{11}$  given by (i) the 3-port CLN with  $N = 2$ , (ii) the 3-port CLN with  $N = 4$ , (iii) the recurrence formula with  $N = 4$ , and (iv) the recurrence formula with  $N = 9$ . The accuracy improves when  $N$  is large. The frequency responses above are computed by a laptop PC using GNU Octave, which required (i) 12, (ii) 30, (iii) 9 and (iv) 31 msec. Even though the computational cost in case (iv) is approximately the same as that in case (ii), the recurrence formula (iv) provides a more accurate admittance than the 3-port CLN (ii). The recurrence formula (iii) provides a more accurate  $\text{Im}(Y_{11})$  than the 3-port CLN (i) in the low-frequency range. Thus, the recurrence formula can reduce the

computational cost of the reduced system.

#### IV. CONCLUDING REMARKS

The port reduction of multiport CLNs was discussed theoretically. A simple and direct method yields an inaccurate, reduced representation. A recurrence formula was derived to generate an accurate reduced system that can reduce the computational cost of the reduced system by decreasing the size of the network element matrices. A port reduction similar to that of the PVL method is expected because of its similarity to that of the CLN method.

#### ACKNOWLEDGMENT

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#### APPENDIX

Suppose that  $\mathbf{e}'_0, \dots, \mathbf{e}'_{2n}$  and  $\mathbf{a}'_1, \dots, \mathbf{a}'_{2n-1}$  are already been obtained. Then  $\mathbf{G}'_{2n}$  is given as

$$\begin{aligned} \mathbf{G}'_{2n} &= \mathbf{e}'_{2n}^T \boldsymbol{\sigma} \mathbf{e}'_{2n} = (\sum_{k=0}^n \mathbf{P}_{2n,2k}^T \mathbf{e}_{2k}^T) \boldsymbol{\sigma} (\sum_{j=0}^n \mathbf{e}_{2j} \mathbf{P}_{2n,2j}) \\ &= \sum_{k=0}^n \mathbf{P}_{2n,2k}^T \mathbf{G}_{2k} \mathbf{P}_{2n,2k}. \end{aligned} \quad (47)$$

After obtaining  $\mathbf{G}'_{2n}$ ,  $\mathbf{a}'_{2n+1}$  is generated from

$$\begin{aligned} \mathbf{K}(\mathbf{a}'_{2n+1} - \mathbf{a}'_{2n-1}) &= \boldsymbol{\sigma} \mathbf{e}'_{2n} \mathbf{G}'_{2n}^{-1} = \boldsymbol{\sigma} \sum_{k=0}^n \mathbf{e}_{2k} \mathbf{P}_{2n,2k} \mathbf{G}'_{2n}^{-1} \\ &= \mathbf{K} \sum_{k=0}^n (\mathbf{a}_{2k+1} - \mathbf{a}_{2k-1}) \mathbf{G}_{2k} \mathbf{P}_{2n,2k} \mathbf{G}'_{2n}^{-1}. \end{aligned} \quad (48)$$

which results in

$$\begin{aligned} \mathbf{a}'_{2n+1} &= \mathbf{a}'_{2n-1} + \sum_{k=0}^n (\mathbf{a}_{2k+1} - \mathbf{a}_{2k-1}) \mathbf{G}_{2k} \mathbf{P}_{2n,2k} \mathbf{G}'_{2n}^{-1} \\ &= \sum_{k=0}^{n-1} \mathbf{a}_{2k+1} \mathbf{P}_{2n-1,2k+1} + \sum_{k=0}^n \mathbf{a}_{2k+1} \mathbf{G}_{2k} \mathbf{P}_{2n,2k} \mathbf{G}'_{2n}^{-1} \\ &\quad - \sum_{k=0}^{n-1} \mathbf{a}_{2k+1} \mathbf{G}_{2k+2} \mathbf{P}_{2n,2k+2} \mathbf{G}'_{2n}^{-1}. \end{aligned} \quad (49)$$

The comparison of (23) with (49) gives

$$\begin{aligned} \mathbf{P}_{2n+1,2n+1} &= \mathbf{G}_{2n} \mathbf{P}_{2n,2n} \mathbf{G}'_{2n}^{-1}, \\ \mathbf{P}_{2n+1,2k+1} &= \mathbf{P}_{2n-1,2k+1} + (\mathbf{G}_{2k} \mathbf{P}_{2n,2k} - \mathbf{G}_{2k+2} \mathbf{P}_{2n,2k+2}) \mathbf{G}'_{2n}^{-1} \\ &\quad (k = 0, \dots, n-1). \end{aligned} \quad (50)$$

Next,  $\mathbf{L}'_{2n+1}$  is obtained as

$$\begin{aligned} \mathbf{L}'_{2n+1} &= \mathbf{a}'_{2n+1}^T \mathbf{K} \mathbf{a}'_{2n+1} \\ &= (\sum_{k=0}^n \mathbf{P}_{2n+1,2k+1}^T \mathbf{a}_{2k+1}^T) \mathbf{K} (\sum_{j=0}^n \mathbf{a}_{2j+1} \mathbf{P}_{2n+1,2j+1}) \\ &= \sum_{k=0}^n \mathbf{P}_{2n+1,2k+1}^T \mathbf{L}_{2k+1} \mathbf{P}_{2n+1,2k+1}. \end{aligned} \quad (51)$$

Similarly, reduced basis  $\mathbf{e}'_{2n}$  is generated as

$$\begin{aligned} \mathbf{e}'_{2n+2} &= \sum_{k=0}^n \mathbf{e}_{2k} \mathbf{P}_{2n,2k} + \sum_{k=1}^{n+1} \mathbf{e}_{2k} \mathbf{L}_{2k-1} \mathbf{P}_{2n+1,2k-1} \mathbf{L}'_{2n+1}^{-1} \\ &\quad - \sum_{k=0}^n \mathbf{e}_{2k} \mathbf{L}_{2k+1} \mathbf{P}_{2n+1,2k+1} \mathbf{L}'_{2n+1}^{-1}. \end{aligned} \quad (52)$$

The comparison of (23) with (52) gives

$$\begin{aligned} \mathbf{P}_{2n+2,2n+2} &= \mathbf{L}_{2n+1} \mathbf{P}_{2n+1,2n+1} \mathbf{L}'_{2n+1}^{-1}, \\ \mathbf{P}_{2n+2,2k} &= \mathbf{P}_{2n,2k} + (\mathbf{L}_{2k-1} \mathbf{P}_{2n+1,2k-1} - \mathbf{L}_{2k+1} \mathbf{P}_{2n+1,2k+1}) \mathbf{L}'_{2n+1}^{-1} \\ &\quad (k = 1, \dots, n) \end{aligned}$$

$$\mathbf{P}_{2n+2,0} = \mathbf{P}_{2n,0} - \mathbf{L}_1 \mathbf{P}_{2n+1,1} \mathbf{L}'_{2n+1}^{-1}. \quad (53)$$

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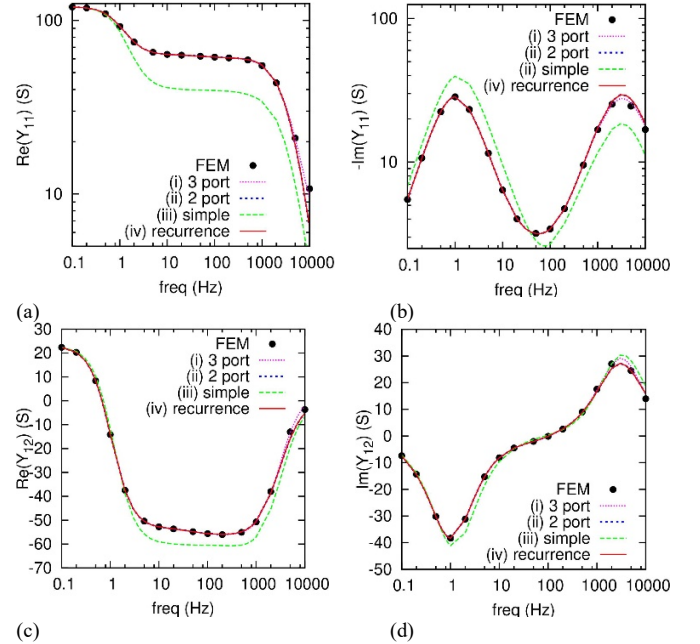


Fig. 4. Frequency dependence of reduced 2-port admittances  $Y_{11}$  and  $Y_{12}$  from 3ports: (a)  $\text{Re}(Y_{11})$ , (b)  $-\text{Im}(Y_{11})$ , (c)  $\text{Re}(Y_{12})$ , (d)  $\text{Im}(Y_{12})$ .

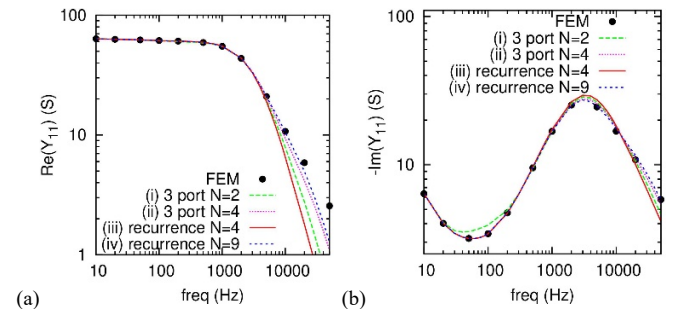


Fig. 5. Frequency plot of reduced admittance  $Y_{11}$  given by 3-port CLN with  $N = 2, 4$  and 2-port CLN with  $N=4, 9$ : (a)  $\text{Re}(Y_{11})$  and (b)  $-\text{Im}(Y_{11})$ .