

Finitely Generated Idempotent-free Semilattice-Indecomposable Semigroups with Relations I

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A semigroup S is called S -indecomposable if S has no semilattice homomorphic image except the trivial semilattice. We assume $S \neq S^2$, $|S \setminus S^2| < \infty$ and S is generated by $S \setminus S^2$. Let $B = S \setminus S^2 = \{a_1, \dots, a_k\}$. The purpose of this paper is to report the structure of idempotent-free S -indecomposable semigroup S generated by B with relation as defined below. Let Z_+ be the set of all positive integers. We assume

$$(1) \quad a_1^{m_1} = \dots = a_k^{m_k} \quad \text{for some } m_1, \dots, m_k \in Z_+.$$

In particular we study here the free semigroup satisfying (1), that is, every such semigroup is a homomorphic image of the free one. The condition (1) is so strong that the property of S -indecomposability is derived from (1).

Lemma 1. *If S is a semigroup generated by B and satisfies (1), then S is S -indecomposable.*

Since $|B| < \infty$ the condition (1) is equivalent to (1') below.

(1') For each pair $a_i, a_j \in B$ there exist $n_i, n_j \in Z_+$ such that

$$a_i^{n_i} = a_j^{n_j}.$$

If B satisfies (1'), equivalently (1), we say S is *power jointly generated* by B .

Let S be an idempotent-free semigroup which is power-jointly generated by B with (1), and let F be the free semigroup over B . There is a homomorphism $f : F \rightarrow S$ which satisfies the following conditions

- i) $X \in F, a \in B, f(X) = f\{(a)\} \Rightarrow X = \{a\}$.
- ii) $f(a_1)^{m_1} = \dots = f(a_k)^{m_k}$.

Let ρ denote the congruence on F generated by the set of binary relations

$$\{(a_i^{m_i}, a_j^{m_j}) : a_i, a_j \in B\}.$$

Then S is a homomorphic image of F/ρ keeping every element of B fixed. In this paper we study F/ρ . For simplicity of notation, let $S = F/\rho$, so $X\rho Y$ in F if and only if $X = Y$ in S .

From (1) we immediately have

Lemma 2. $a_i^{\lambda m_i} a_j^x = a_j^x a_i^{\lambda m_i}$ for $i, j = 1, \dots, k$, for any $\lambda \in Z_+$.

Let $X \in S$. X has the form $X = a_{i_1}^{x_1'} \dots a_{i_s}^{x_s'}$ where $a_{i_j} \in B$, $(j = 1, \dots, s)$ $x_i' \in Z_+$,

(2) $a_{i_j} \neq a_{i_{j+1}}$, $(j = 1, \dots, s-1)$.

We rewrite $x_i' = x_i + \lambda_i m_i$ where $0 < x_i \leq m_i$, $\lambda_i \in Z_+^0 = Z_+ \cup \{0\}$. Let $M = a_1^{m_1} = \dots = a_k^{m_k}$. By using Lemma 2 repeatedly we have

$$(3) \quad X = a_{i_1}^{x_1} \dots a_{i_s}^{x_s} a_{i_1}^{\lambda_1 m_{i_1}} \dots a_{i_s}^{\lambda_s m_{i_s}} = a_{i_1}^{x_1} \dots a_{i_s}^{x_s} M^\lambda \quad \text{where } \lambda = \lambda_1 + \dots + \lambda_s.$$

Likewise $Y = a_{j_1}^{y_1} \dots a_{j_t}^{y_t} a_{j_1}^{\mu_1 m_{j_1}} \dots a_{j_t}^{\mu_t m_{j_t}} = a_{j_1}^{y_1} \dots a_{j_t}^{y_t} M^\mu$ where $\mu = \mu_1 + \dots + \mu_t$.

Consider the product XY . Again by using Lemma 2 we have:

$$\text{If } i_s \neq j_1 \quad XY = a_{i_1}^{x_1} \dots a_{i_s}^{x_s} a_{j_1}^{y_1} \dots a_{j_t}^{y_t} M^{\lambda+\mu}.$$

$$\text{If } i_s = j_1 \text{ and } x_s + y_1 \leq 2m_{i_s}, \text{ then } XY = a_{i_1}^{x_1} \dots a_{i_{s-1}}^{x_{s-1}} a_{i_s}^{z_s} a_{j_2}^{y_2} \dots a_{j_t}^{y_t} M^{\lambda+\mu}$$

where $0 < z_s \leq m_{i_s}$ and $z_s \equiv x_s + y_1 \pmod{m_{i_s}}$.

$$\text{If } i_s = j_1 \text{ and } x_s + y_1 > 2m_{i_s}, \text{ then } XY = a_{i_1}^{x_1} \dots a_{i_{s-1}}^{x_{s-1}} a_{i_s}^{z_s} a_{j_2}^{y_2} \dots a_{j_t}^{y_t} M^{\lambda+\mu+1}$$

where $0 < z_s \leq m_{i_s}$ and $z_s \equiv x_s + y_1 \pmod{m_{i_s}}$.

Let P denote the set of finite sequences V of elements of B , $V = a_{i_1} \dots a_{i_s}$ satisfying $a_{i_j} \neq a_{i_{j-1}}$, $j = 1, \dots, s-1$.

The binary operation on P is defined by

$$(a_{i_1} \dots a_{i_s}) * (a_{j_1} \dots a_{j_t}) = \begin{cases} a_{i_1} \dots a_{i_s} a_{j_1} \dots a_{j_t} & \text{if } i_s \neq j_1 \\ a_{i_1} \dots a_{i_s} a_{j_2} \dots a_{j_t} & \text{if } i_s = j_1 \end{cases}$$

that is, if $i_s \neq j_1$, the product is juxtaposition, if $i_s = j_1$ then one of a_{i_s} and a_{j_1} is omitted.

Proposition 1. P is a semigroup and S is homomorphic onto P under the mapping $a_{i_1}^{x_1} \dots a_{i_s}^{x_s} \rightarrow a_{i_1} \dots a_{i_s}$.

P is regarded as the set of finite sequences $i_1 \dots i_s$ of elements of $B = \{1, \dots, k\}$ subject to $i_j \neq i_{j+1}$, $j = 1, \dots, s-1$, $s \geq 1$. In the form (3): $X = a_{i_1}^{x_1} \dots a_{i_s}^{x_s} M^\lambda$, we

rewrite x_j by x_{i_j} ($j = 1, \dots, s$)

$$(3') \quad X = a_{i_1}^{x_{i_1}} \dots a_{i_s}^{x_{i_s}} M^\lambda.$$

The sequence $x_{i_1} \dots x_{i_s}$ is regarded as a mapping from a sequence $i_1 \dots i_s$ of elements of $\{1, \dots, k\}$ to a sequence $x_{i_1} \dots x_{i_s}$ such that $x_{i_j} \in Z_{m_{i_j}}$ (i.e. an element modulo m_{i_j}) and $0 < x_{i_j} \leq m_{i_j}$, $j = 1, \dots, s$, $s = 1, \dots, k$. Let $\varphi : i_1 \dots i_s \rightarrow x_{i_1} \dots x_{i_s}$, $\psi : j_1 \dots j_s \rightarrow y_{j_1} \dots y_{j_s}$ and let Φ denote the set of all such φ 's and define the binary operation $\varphi\psi$ on Φ as follows:

$$\text{If } i_s \neq j_1, \quad (i_1 \dots i_s) * (j_1 \dots j_t) = i_1 \dots i_s j_1 \dots j_t \rightarrow x_{i_1} \dots x_{i_s} y_1 \dots y_{j_t}.$$

$$\text{If } i_s = j_1, \quad (i_1 \dots i_s) * (j_1 \dots j_t) = i_1 \dots i_{s-1} i_s j_2 \dots j_t \rightarrow x_{i_1} \dots x_{i_{s-1}} z_{i_s} y_{j_2} \dots y_{j_t}, \text{ where } z_{i_s} \equiv x_{i_s} + y_{j_1} \pmod{m_{i_s}}, \quad 0 < z_{i_s} \leq m_{i_s}.$$

Proposition 2. Φ is a semigroup, and S is homomorphic onto Φ under the mapping $X = a_{i_1}^{x_{i_1}} \dots a_{i_s}^{x_{i_s}} M^\lambda \rightarrow \varphi$ where $\varphi : i_1 \dots i_s \rightarrow x_{i_1} \dots x_{i_s}$.

Define a mapping $g : \Phi \times \Phi \rightarrow Z_+^0$ as follows:

$$g(\varphi, \psi) = \begin{cases} 1 & \text{if } i_s = j_1 \text{ and } x_{i_s} + y_{j_1} > m_{i_s} \\ 0 & \text{otherwise} \end{cases}.$$

Let $\Gamma = \{(\varphi, \lambda) : \varphi \in \Phi, \lambda \in Z_+^0\}$ and define the binary operation on Γ as follows:

$$(\varphi, \lambda)(\psi, \mu) = (\varphi\psi, \lambda + \mu + g(\varphi, \psi)).$$

Note that g satisfies the condition:

$$g(\varphi, \psi) + g(\varphi\psi, \xi) = g(\varphi, \psi\xi) + g(\psi, \xi) \quad \text{for all } \varphi, \psi, \xi \in \Phi.$$

Now we have the main theorem

Theorem . Γ is a semigroup and S is isomorphic onto Γ under the mapping $X = a_{i_1}^{x_{i_1}} \dots a_{i_s}^{x_{i_s}} M^\lambda \rightarrow (\varphi, \lambda)$ where $\varphi : i_1 \dots i_s \rightarrow x_{i_1} \dots x_{i_s}$.

The idea of constructing S -indecomposable semigroups from a certain free semigroup was initiated by the author in case of finite nil semigroups 1958 [2], and also the idea was used in case of finitely generated Z -semigroups [3].

The representation of S by means of Γ is similar to N -semigroups (i.e. idempotent-free cancellative commutative archimedean semigroups) [1].

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