

Fourier transform of holomorphic discrete series
— the case of tube domains —

by

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§1. Preliminaries.

Let G be a non-compact connected linear simple Lie group and K a maximal compact subgroup of G . We assume throughout this note that G/K carries a structure of hermitian symmetric space and that G/K is holomorphically equivalent to a tube domain. The Lie algebras of G and K are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition with the associated Cartan involution θ . Since G/K is a hermitian symmetric space, there is a linear operator J on $\mathfrak{p}$ such that J commutes with $(\text{Ad } k)|_{\mathfrak{p}}$ ($k \in K$) and $J^2 = -1_{\mathfrak{p}}$. One knows that J is written as $J = (\text{ad } Z_0)|_{\mathfrak{p}}$ for some element Z_0 in the center $\mathfrak{c}$ of $\mathfrak{k}$. Note that since G is assumed to be simple, $\mathfrak{c}$ is necessarily of one dimension.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then one can prove that $\mathfrak{t}$ is a (compact) Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system with respect to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and we denote by $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ ($\alpha \in \Delta$) the root subspace corresponding to the root $\alpha \in \Delta$. Then,

$\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{t}_{\mathbb{C}}$ or $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$. Let $\Delta_{\mathbb{C}}$ (resp. Δ_n) be the set of all roots $\alpha \in \Delta$ such that $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{t}_{\mathbb{C}}$ (resp. $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subset \mathfrak{p}_{\mathbb{C}}$). A root α in $\Delta_{\mathbb{C}}$ (resp. in Δ_n) is said to be compact (resp. non-compact). We introduce an order in Δ compatible with the complex structure of G/K so that the $+i$ (resp. $-i$)-eigenspace $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) of the J extended to $\mathfrak{p}_{\mathbb{C}}$ by complex linearity coincides with the sum of all root subspaces corresponding to non-compact positive (resp. negative) roots. The set of all positive roots is denoted by Δ^+ and $\Delta_{\mathbb{C}}^+$ (resp. Δ_n^+) stands for the set of all compact (resp. non-compact) positive roots. Both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$ normalized by $K_{\mathbb{C}}$.

Let $\gamma_1, \dots, \gamma_{\ell}$ be a maximal system of strongly orthogonal non-compact positive roots constructed as follows: for each j , γ_j is the largest positive non-compact root strongly orthogonal to $\gamma_{j+1}, \dots, \gamma_{\ell}$. Let B be the Killing form of $\mathfrak{g}_{\mathbb{C}}$. For every $\alpha \in \Delta$, we choose $H_{\alpha} \in \mathfrak{t}_{\mathbb{C}}$ and $X_{\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ so that

$$B(H_{\alpha}, H) = \alpha(H) \quad (\forall H \in \mathfrak{t}_{\mathbb{C}}), \quad X_{\alpha} - X_{-\alpha} \in \mathfrak{t} + i\mathfrak{p}, \quad (1.1)$$

$$i(X_{\alpha} + X_{-\alpha}) \in \mathfrak{t} + i\mathfrak{p}, \quad [X_{\alpha}, X_{-\alpha}] = \frac{2H_{\alpha}}{\alpha(H_{\alpha})} =: H'_{\alpha}.$$

Then $H_{\alpha} \in i\mathfrak{t}$ and one can prove that $\mathfrak{a} := \sum_{1 \leq i \leq \ell} \mathbb{R}(X_{\gamma_i} + X_{-\gamma_i})$ is a maximal abelian subspace of $\mathfrak{p}$. Hence ℓ is equal to the real rank of G .

Let $G_{\mathbb{C}}$ be the complexification of G . We denote by $K_{\mathbb{C}}$ and $P_{\pm}$ the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\pm}$ respectively. Then, every element x in $P_+ K_{\mathbb{C}} P_-$ can be expressed in a unique way as

$$x = \exp \xi_+(x) \cdot k(x) \cdot \exp \xi_-(x)$$

with $\xi_{\pm}(x) \in \mathfrak{p}_{\pm}$ and $k(x) \in K_{\mathbb{C}}$. Furthermore $G \subset P_+ K_{\mathbb{C}} P_-$. Since $\xi_+(xk) = \xi_+(x)$ ($k \in K_{\mathbb{C}}$), we have a mapping $\psi: G \rightarrow \mathfrak{p}_+$ defined by $\psi(gK) = \xi_+(g)$ ($g \in G$). The ψ is a holomorphic diffeomorphism onto a bounded symmetric domain $\mathcal{D}$ in $\mathfrak{p}_+$. The image $\mathcal{D} = \psi(G/K) \subset \mathfrak{p}_+$ is called the Harish-Chandra realization of G/K . Let $q \in \mathcal{D}$. If $g \in G_{\mathbb{C}}$ satisfies $g \exp q \in P_+ K_{\mathbb{C}} P_-$, then $g \cdot q \in \mathfrak{p}_+$ is well-defined and is given by $\xi_+(g \exp q)$. Moreover, for $g \in G$ and $x \in G/K$, $g \cdot \psi(x)$ is always well-defined and we have $g \cdot \psi(x) = \psi(gx)$. Let

$$(1.2) \quad m_* = \exp \pi Z_0.$$

It is clear that m_* lies in the center of K and $\theta = \text{Ad } m_*$. Thus $m_* \in N_K(A)$, the normalizer of $A := \exp a$ in K . Now it is easily seen that $m_* \cdot q = -q$ ($\forall q \in \mathcal{D}$), so m_* gives the symmetry of $\mathcal{D}$ at the origin $0 \in \mathcal{D}$.

Put

$$(1.3) \quad c := \exp \frac{\pi}{4} \sum_{j=1}^{\ell} (X_{\gamma_j} - X_{-\gamma_j}) \in G_{\mathbb{C}}.$$

Then $c \in P_+ K_{\mathbb{C}} P_-$. Setting

$$(1.4) \quad X_0 := \sum_{i=1}^{\ell} X_{\gamma_i}, \quad H'_0 := \sum_{i=1}^{\ell} H'_{\gamma_i}, \quad Y_0 := \sum_{i=1}^{\ell} X_{-\gamma_i},$$

we have

$$(1.5) \quad \xi_+(c) = X_0, \quad k(c) = \exp(\log \sqrt{2}) H'_0, \quad \xi_-(c) = -Y_0.$$

Let τ be the conjugation in $\mathfrak{g}_{\mathbb{C}}$ relative to the compact real form $\mathfrak{t} + i\mathfrak{p}$. One knows that $(x, y) := -B(x, \tau y)$ ($x, y \in \mathfrak{g}_{\mathbb{C}}$) defines a hermitian inner product on $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{t}^- := \sum_{j=1}^{\ell} \mathbb{R} H_{\gamma_j} \subset \mathfrak{t}$ and $\mathfrak{t}^+$ be the orthogonal complement to $\mathfrak{t}^-$ in $\mathfrak{t}$. Set $\nu = \text{Ad}_{G_{\mathbb{C}}} c$, where c is the element in $G_{\mathbb{C}}$ defined by (1.3). Then ν is an isometry of $\mathfrak{g}_{\mathbb{C}}$ and we have

$$(1.6) \quad \begin{aligned} \nu(X_{\gamma_j} + X_{-\gamma_j}) &= H'_{\gamma_j}, & \nu(X_{\gamma_j} - X_{-\gamma_j}) &= X_{\gamma_j} - X_{-\gamma_j}, \\ \nu(H'_{\gamma_j}) &= -(X_{\gamma_j} + X_{-\gamma_j}). \end{aligned}$$

Hence $\mathfrak{t}^- = \nu(\mathfrak{a})$, and $\mathfrak{t}_{\mathbb{C}}^+ + \mathfrak{a}_{\mathbb{C}} = \nu^{-1}(\mathfrak{t}_{\mathbb{C}})$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. For every $\alpha \in \Delta$, $\text{res}_{\mathfrak{t}^-} \alpha$ will stand for the restriction of α to $\mathfrak{t}^-$. We denote still by γ_j the restriction $\text{res}_{\mathfrak{t}^-} \gamma_j$.

Let $\alpha_j := \gamma_j \circ \nu$ ($j = 1, 2, \dots, \ell$). Since we are assuming that G/K is holomorphically equivalent to a tube domain, the restricted root theorem due to Moore [6] can be stated as follows.

Theorem 1.1 (Moore). *Let $\Delta(\mathfrak{a})$ be the $\mathfrak{a}$ -root system. Then, the positive system $\Delta(\mathfrak{a})^+$ of $\Delta(\mathfrak{a})$ is described as*

$$\Delta(\mathfrak{a})^+ = \left\{ \frac{1}{2}(\alpha_m + \alpha_k); 1 \leq k \leq m \leq \ell \right\} \cup \left\{ \frac{1}{2}(\alpha_m - \alpha_k); 1 \leq k < m \leq \ell \right\}.$$

For any $\alpha \in \Delta(\mathfrak{a})$, we denote by $\mathfrak{g}_{\alpha}$ the corresponding $\mathfrak{a}$ -root subspace. Put

$$(1.7) \quad u_k := \frac{i}{2} (H'_{\gamma_k} - X_{\gamma_k} + X_{-\gamma_k}) \quad (k = 1, 2, \dots, \ell).$$

Since $H'_{\gamma_k} \in \mathfrak{if}$ and $X_{\gamma_k} - X_{-\gamma_k} \in \mathfrak{ip}$ (cf. (1.1)), we see that $u_k \in \mathfrak{g}$. Moreover, (1.6) leads us to $\nu^{-1}(X_{\gamma_k}) = iu_k$, so that $u_k \in \mathfrak{g}_{\alpha_k}$.

Let

$$(1.8) \quad s := \sum_{k=1}^{\ell} u_k \in \mathfrak{g}(1), \quad a_0 := \sum_{k=1}^{\ell} \frac{1}{2} (X_{\gamma_k} + X_{-\gamma_k}) \in \mathfrak{a}.$$

Then, $\text{ad } a_0$ is semisimple. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{f}$ and put

$$\mathfrak{g}(0) = \mathfrak{m} + \mathfrak{a} + \sum_{k < m} (\mathfrak{g}_{(\alpha_m - \alpha_k)/2} + \mathfrak{g}_{-(\alpha_m - \alpha_k)/2}),$$

$$\mathfrak{g}(1) = \sum_{k \leq m} \mathfrak{g}_{(\alpha_m + \alpha_k)/2}, \quad \mathfrak{g}(-1) = \sum_{k \leq m} \mathfrak{g}_{-(\alpha_m + \alpha_k)/2}.$$

Then, $\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(1)$, an orthogonal direct sum of vector subspaces. It is easy to see that $\mathfrak{g}(k)$ is the k -eigenspace of $\text{ad } a_0$. Letting $\mathfrak{g}(k) = \{0\}$ for $|k| > 1$, we have

$$(1.9) \quad [\mathfrak{g}(k), \mathfrak{g}(m)] \subset \mathfrak{g}(k+m).$$

We also have

$$(1.10) \quad \begin{aligned} & \text{(i) } \dim \mathfrak{g}_{\alpha_k} = 1 \text{ for all } 1 \leq k \leq \ell, \\ & \text{(ii) } a := \dim \mathfrak{g}_{(\alpha_m - \alpha_k)/2} = \dim \mathfrak{g}_{(\alpha_m + \alpha_k)/2} \text{ is independent} \\ & \quad \text{of } m, k \text{ (} m > k \text{)}. \end{aligned}$$

§2. Realization of G/K as a tube domain.

2.1. **Basic facts about Jordan algebras.** We begin this section with the definition of Jordan algebra. Our reference is the book [1]. Let $\mathfrak{U}$ be a finite dimensional vector space over K ($K = \mathbb{R}$ or $\mathbb{C}$). A product $x, y \mapsto xy$ in $\mathfrak{U}$ is, by definition, a bilinear mapping $\mathfrak{U} \times \mathfrak{U} \rightarrow \mathfrak{U}$. The associative law is not assumed here. The vector space $\mathfrak{U}$, equipped with a product, is called a *Jordan algebra* if

$$(J-1) \quad xy = yx,$$

$$(J-2) \quad x^2(xy) = x(x^2y)$$

hold for all $x, y \in \mathfrak{U}$. Now let $\mathfrak{U}$ be a Jordan algebra with the unit element e . For $x \in \mathfrak{U}$, we define a linear operator $L(x)$ on $\mathfrak{U}$ by

$$(2.1) \quad L(x)y = xy.$$

Then we have $L(x)y = L(y)x$ and the assignment $x \mapsto L(x)$ is clearly linear. In terms of these operators, (J-2) is rewritten as

$$[L(x), L(x^2)] = 0.$$

We know that any Jordan algebra is *power-associative*, that is, defining the power x^n of an element $x \in \mathfrak{U}$ by $x^n = xx^{n-1}$ inductively, we have $x^m x^n = x^{m+n}$. Therefore the subalgebra $K[x]$

generated by e and x is associative. Set

$$(2.2) \quad P(x) = 2L(x)^2 - L(x^2) \quad (x \in \mathcal{U}).$$

The mapping $x \mapsto P(x)$ is called the *quadratic representation* of $\mathcal{U}$.

It is well-known that $P(x^n) = P(x)^n$ ($n = 1, 2, \dots$). Furthermore we have the following formula named as the *fundamental formula*:

$$(2.3) \quad P(P(x)y) = P(x)P(y)P(x) \quad (\forall x, y \in \mathcal{U}).$$

An element $x \in \mathcal{U}$ is said to be *invertible* if one of the following three mutually equivalent conditions holds:

- (i) The operator $P(x)$ is invertible, that is, $\det P(x) \neq 0$.
- (ii) There is $y \in \mathbb{K}[x]$ such that $xy = e$.
- (iii) There is $y \in \mathcal{U}$ such that $[L(x), L(y)] = 0$ and $xy = e$.

Then, if x is invertible, the y in (ii) or (iii) is uniquely given by $y = P(x)^{-1}x$, and will be written as x^{-1} . The set of all invertible elements of $\mathcal{U}$ is denoted by $\mathcal{U}^\times$. Moreover $P(x^{-1}) = P(x)^{-1}$ holds for any $x \in \mathcal{U}^\times$.

Now let $\mathcal{U}$ be a real Jordan algebra. $\mathcal{U}$ is said to be *formally real* if

$$(FR-1) \quad x^2 + y^2 = 0 \text{ implies } x = y = 0.$$

It is known that (FR-1) is equivalent to the following (FR-2):

$$(FR-2) \quad \text{the symmetric bilinear form } x, y \mapsto \operatorname{tr} L(xy) \text{ is positive definite.}$$

We remark here that the linear form $\mathfrak{U} \ni x \mapsto \text{tr } L(x)$ is associative in the sense that

$$(2.4) \quad \text{tr } L((xy)z) = \text{tr } L(x(yz)) \quad (\forall x, y, z \in \mathfrak{U}).$$

In particular, the operators $L(x)$ (hence $P(x)$, too) are symmetric with respect to the bilinear form $\text{tr } L(xy)$.

We assume now that $\mathfrak{U}$ is a formally real Jordan algebra. Then $\mathfrak{U}$ has the unit element e . The *positive cone* Ω is, by definition, the interior of the squares, i.e., $\Omega = \text{Int}\{x^2; x \in \mathfrak{U}\}$. Ω is an open convex cone in $\mathfrak{U}$ and selfdual with respect to the inner product $\text{tr } L(xy)$:

$$\Omega = \{y \in \mathfrak{U}; \text{tr } L(xy) > 0 \text{ for all } x \in (C\Omega) \setminus \{0\}\}.$$

We note:

- (i) Ω coincides with the connected component of $\mathfrak{U}^\times$ containing e .
- (ii) $x \in \Omega$ if and only if $L(x)$ is positive definite.
- (iii) If $x \in \Omega$, then $P(x)$ is positive definite.

Finally, since the mapping $\Omega \ni x \mapsto x^2 \in \Omega$ is a diffeomorphism (its tangent mapping at $x_0 \in \Omega$ is $2L(x_0)$), its inverse mapping will be denoted by $\Omega \ni y \mapsto y^{1/2} \in \Omega$.

2.2. Jordan algebra structure on $\mathfrak{g}(1)$. We retain the notation of §1 and recall the element s defined by (1.8).

Lemma 2.1. (i) The real vector space $\mathfrak{g}(1)$ has a structure of Jordan algebra by $x \cdot y = -\frac{1}{2} [[x, \theta s], y]$ ($x, y \in \mathfrak{g}(1)$). The unit element is s .

(ii) Let $L(x)$ be the operator defined by $L(x)y = x \cdot y$ ($x, y \in \mathfrak{g}(1)$). Then, $\text{tr}_{\mathfrak{g}(1)} L(x \cdot y) = -2B(x, \theta y)$, so that $\mathfrak{g}(1)$ with the product in (i) is a formally real Jordan algebra.

Henceforth we denote by $\mathfrak{U}$ the formally real Jordan algebra described in Lemma 2.1. Now consider the complexification $\mathfrak{g}(1)_{\mathbb{C}}$. The product $x \cdot y$ in $\mathfrak{g}(1)$, which is a real bilinear mapping, is naturally extended to a complex bilinear mapping $\mathfrak{g}(1)_{\mathbb{C}} \times \mathfrak{g}(1)_{\mathbb{C}} \rightarrow \mathfrak{g}(1)_{\mathbb{C}}$. It is easy to see that the complex vector space $\mathfrak{g}(1)_{\mathbb{C}}$ with this complex bilinear product becomes a Jordan algebra. We denote by $\mathfrak{U}_{\mathbb{C}}$ the complex Jordan algebra thus obtained. The multiplication operators, the quadratic representation of $\mathfrak{U}_{\mathbb{C}}$ are still denoted by $L(x)$, $P(x)$ respectively.

Consider the tube domain $T_{\Omega} := \mathfrak{U} + i\Omega \subset \mathfrak{U}_{\mathbb{C}}$.

Lemma 2.2. (i) One has $T_{\Omega} \subset (\mathfrak{U}_{\mathbb{C}})^{\times}$, that is, every $z \in T_{\Omega}$ is invertible in the Jordan algebra $\mathfrak{U}_{\mathbb{C}}$.

(ii) If $z \in T_{\Omega}$, then $-z^{-1} \in T_{\Omega}$. Moreover the mapping $T_{\Omega} \ni z \mapsto -z^{-1} = -P(z)^{-1}z \in T_{\Omega}$ is holomorphic and has the unique fixed point is , where s is the unit element of $\mathfrak{U}_{\mathbb{C}}$ defined by (1.8).

Sketch. Let $z \in T_{\Omega}$ and put $z = x + iy$ with $x \in \mathfrak{U}$ and $y \in \Omega$. (i) Set $u = y^{1/2} \in \Omega$. Then,

$$(2.5) \quad x + iy = P(u)(P(u)^{-1}x + is).$$

Thus it suffices to consider the elements of the form $x + is$

$(x \in \mathfrak{U})$. But the following formula shows that $x + is$ is invertible:

$$P(x+is)P(x-is) = P(x^2+s),$$

because $x^2 + s \in \Omega$. (ii) Since $-(x+is)^{-1} = -(x^2+s)^{-1}(x-is)$ (this computation is done in the associative algebra $\mathbb{C}[x]$), we see immediately $-(x+is)^{-1} \in T_\Omega$. Thus by (2.5), $-z^{-1} \in T_\Omega$ for any $z \in T_\Omega$. For the rest, it suffices to solve the equations $x^2 - y^2 + s = 0$, $x \cdot y = 0$. Q.E.D.

On the other hand, one knows that $c \cdot \mathfrak{Q} \subset \mathfrak{p}_+$ and that $\nu^{-1} \cdot c(\mathfrak{Q}) = T_\Omega$ (note $\nu(g(1)_\mathbb{C}) = \mathfrak{p}_+$). Thus T_Ω realizes G/K and G acts on T_Ω by

$$(2.6) \quad g \cdot z = \nu^{-1}(c \cdot (g \cdot q)) \quad (g \in G, z \in T_\Omega),$$

where $q = c^{-1} \cdot (\nu(z)) \in \mathfrak{Q}$. We will make (2.6) more explicit for some elements of G .

Let $G(0) = Z_G(a_0)$, the centralizer in G of the $a_0 \in \mathfrak{a}$ defined by (1.8). Then, $\mathfrak{g}(0) = \text{Lie } G(0)$ and $G(0)$ is reductive. Let $G(1) = \exp \mathfrak{g}(1)$ and $P_0 := G(1)G(0)$. Then, P_0 is a maximal parabolic subgroup of G with $G(0)$ a Levi part, $G(1)$ the unipotent radical. We have $cP_0c^{-1} \subset P_+K_\mathbb{C}$, so that

$$(2.7) \quad g \cdot z = \nu^{-1} \xi_+(cg c^{-1} \exp \nu(z)) \quad (g \in P_0, z \in T_\Omega).$$

Now let $g_0 \in G(0)$. Then $cg_0c^{-1} \in K_\mathbb{C}$, and so

$$(2.8) \quad g_0 \cdot z = (\text{Ad } g_0)z \quad (g_0 \in G(0), z \in T_\Omega).$$

For $a \in \mathfrak{g}(1)$, recalling $\nu(\mathfrak{g}(1)_{\mathbb{C}}) = \mathfrak{p}_+$, we see easily that

$$(2.9) \quad (\exp a) \cdot z = z + a \quad (a \in \mathfrak{g}(1), z \in T_{\Omega}).$$

Finally for the element m_* defined by (1.2), we get

$$(2.10) \quad m_* \cdot z = -z^{-1}.$$

Since P_0 and the element m_* generate G , the formulas (2.8) ~ (2.10) describe the G -action on T_{Ω} .

§3. Holomorphic discrete series.

3.1. **Realization on D .** We retain the notation in the preceding sections. Let Λ be a K -dominant K -integral form on $\mathfrak{t}_{\mathbb{C}}$: thus

- (i) $\Lambda(H_{\alpha}) \geq 0$ for all $\alpha \in \Delta_{\mathbb{C}}^+$,
- (ii) $\xi_{\Lambda}(h) := \exp \Lambda(\log h)$ is a character of $T := \exp \mathfrak{t}$.

We denote by τ_{Λ} the irreducible unitary representation of K on a finite dimensional Hilbert space E_{Λ} with highest weight Λ . The inner product on E_{Λ} is written as $(\cdot, \cdot)_{\Lambda}$. We describe here a realization of holomorphic discrete series of G following Vergne-Rossi [9]. Let $U(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Every element of $U(\mathfrak{g}_{\mathbb{C}})$ is canonically considered as a left invariant differential operator on G . Let $\mathcal{O}(\Lambda)$ be the space of all E_{Λ} -valued C^{∞} -functions on G such that

- (i) $\varphi(xk) = \tau_{\Lambda}(k)^{-1} \varphi(x) \quad (x \in G, k \in K),$
- (ii) $X\varphi = 0$ for all $X \in \mathfrak{p}_-,$
- (iii) $\int_G |\varphi(x)|^2 dx < \infty,$

where dx is the Haar measure on G . We define an inner product on $\mathcal{O}(\Lambda)$ by

$$(\varphi_1, \varphi_2) := \int_G (\varphi_1(x), \varphi_2(x))_{\Lambda} dx.$$

Then, one knows that $\mathcal{O}(\Lambda)$ with this inner product is a Hilbert space. G acts on $\mathcal{O}(\Lambda)$ by left translations: $L_{\Lambda}(g)\varphi(x) = \varphi(g^{-1}x)$. The representation $(U_{\Lambda}, \mathcal{O}_{\Lambda})$ of G is irreducible and belongs to holomorphic discrete series of G if $\dim \mathcal{O}(\Lambda) > 0$. By Harish-Chandra [4], the condition $\dim \mathcal{O}(\Lambda) > 0$ is equivalent to

$$(3.1) \quad (\Lambda + \rho)(H_{\beta}) < 0 \quad \text{for all } \beta \in \Delta_n^+,$$

where $2\rho = \sum_{\alpha \in \Delta^+} \alpha$.

We will assume from now on that Λ satisfies (3.1), so that $\mathcal{O}(\Lambda) \neq \{0\}$. In order to get a realization of holomorphic discrete series on a function space on D , one needs a map $\Phi: G \rightarrow GL(E_{\Lambda})$ such that

$$\begin{aligned} \Phi(gk) &= \Phi(g)\tau_{\Lambda}(k) & (g \in G, k \in K), \\ X\Phi &= 0 & \text{for all } X \in \mathfrak{p}_-. \end{aligned}$$

We note that since $P_0 = G(0)G(1)$ is a parabolic subgroup of G , one has $G = P_0K$. Thus recalling $cP_0c^{-1} \subset P_+K_{\mathbb{C}}$, we get

$$(3.2) \quad cG \subset cP_0K \subset (cP_0c^{-1})cK \subset P_+K_{\mathbb{C}}P_+K_{\mathbb{C}}P_-K \subset P_+K_{\mathbb{C}}P_-,$$

so that $k(cg) \in K_{\mathbb{C}}$ is well-defined for any $g \in G$. Extending τ_{Λ} to a holomorphic representation of $K_{\mathbb{C}}$, we now set, after Vergne-Rossi [9, p.18],

$$(3.3) \quad \Phi_{\Lambda}(g) = \tau_{\Lambda}(k(c))^{-1} \tau_{\Lambda}(k(cg)).$$

Then we have immediately that $\Phi_{\Lambda}(e) = 1_{E_{\Lambda}}$ and

$$(3.4) \quad \begin{aligned} \Phi_{\Lambda}(gk) &= \Phi_{\Lambda}(g) \tau_{\Lambda}(k) & (g \in G, k \in K), \\ X\Phi_{\Lambda} &= 0 & \text{for all } X \in \mathfrak{p}_-. \end{aligned}$$

We further extend τ_{Λ} to a representation of the semidirect product $P_+K_{\mathbb{C}}$ by defining $\tau_{\Lambda}(p) = 1_{E_{\Lambda}}$ for all $p \in P_+$. Noting that for $g \in P_0$, we have $k(cg) = k(cgc^{-1})k(c)$ by (3.2), we see

$$\Phi_{\Lambda}(g) = \tau_{\Lambda}(k(c))^{-1} \tau_{\Lambda}(cgc^{-1}) \tau_{\Lambda}(k(c)) \quad (g \in P_0).$$

Thus $\Phi_{\Lambda}|_{P_0}$ is a representation of the parabolic subgroup P_0 .

Moreover by (1.5) and (1.6), we have

$$(3.5) \quad \begin{aligned} \Phi_{\Lambda}(g_0) &= \tau_{\Lambda}(cg_0c^{-1}) & (g_0 \in G(0)), \\ \Phi_{\Lambda}(\exp x) &= 1_{E_{\Lambda}} & (x \in \mathfrak{g}(1)). \end{aligned}$$

We remark here that $[\det P(t)]^{-1/2} dt$, dt being the Lebesgue measure on $\mathfrak{g}(1)$, is a P_0 -invariant measure on Ω , where $t \mapsto P(t)$ is the quadratic representation of the Jordan algebra $\mathfrak{U}$ described in Lemma 2.1 (recall that $P(t)$ is positive definite for $t \in \Omega$).

Let $S = (\exp \sum_{\alpha \in \Delta(\mathfrak{a})^+} s_\alpha)A$, the Iwasawa solvable subgroup of G . Put $S(0) = G(0) \cap S$. We denote by η_0 the diffeomorphism of Ω onto $S(0)$ such that $(\text{Ad } \eta_0(t))s = t$ ($t \in \Omega$).

With these preparations, we now introduce a Hilbert space $H(\Lambda)$ of E_Λ -valued holomorphic functions F on T_Ω such that

$$\|F\|^2 := \int_{T_\Omega} \|\Phi_\Lambda(\eta_0(y))^{-1}F(x+iy)\|_\Lambda^2 \frac{dx dy}{\det P(y)} < \infty.$$

Letting $\alpha(g) = g \cdot is \in T_\Omega$ ($g \in G$), we define

$$T_\Lambda F(g) := \Phi_\Lambda(g)^{-1}F(\alpha(g)) \quad (F \in H(\Lambda), g \in G).$$

Then T_Λ is a unitary mapping from $H(\Lambda)$ onto $\mathcal{O}(\Lambda)$. Let $\pi_\Lambda(g) := T_\Lambda^{-1}L_\Lambda(g)T_\Lambda$ ($g \in G$). To describe $\pi_\Lambda(g)$, we set

$$(3.6) \quad J_\Lambda(g, \alpha(h)) := \Phi_\Lambda(gh)\Phi_\Lambda(h)^{-1} \quad (g \in G, h \in S).$$

Then, one has

$$J_\Lambda(g_1 g_2, z) = J_\Lambda(g_1, g_2 \cdot z) J_\Lambda(g_2, z) \quad (g_1, g_2 \in G, z \in T_\Omega).$$

Now, a simple computation yields

$$(3.7) \quad \pi_\Lambda(g)F(z) = J_\Lambda(g^{-1}, z)^{-1}F(g^{-1} \cdot z) \quad (g \in G, z \in T_\Omega).$$

We note that since $\Phi_\Lambda|_{P_0}$ is a representation, we have $J_\Lambda(g, z) = \Phi_\Lambda(g)$ for all $g \in P_0$ and $z \in T_\Omega$. We also note that by (3.3),

$$(3.8) \quad J_\Lambda(k, is) = \tau_\Lambda(k) \quad \text{for all } k \in K.$$

3.2. **Some integrals over Ω .** Let us set $\langle x, y \rangle := -B(x, \theta y)$ for $x, y \in \mathfrak{g}(1)$. Then, $\langle \cdot, \cdot \rangle$ is an inner product of $\mathfrak{g}(1)$ relative to which Ω is selfdual (cf. Lemma 2.1 (ii)). For $\lambda \in \mathfrak{g}(1)$ and $u \in E_\Lambda$, define

$$\Gamma_\Lambda(\lambda; v) := \int_{\Omega} e^{-2\langle \lambda, t \rangle} \|\Phi_\Lambda(\eta_0(t))^{-1} v\|_\Lambda^2 \frac{dt}{\det P(t)} \quad (3.9)$$

$$E_\Lambda(\lambda) := \{v \in E_\Lambda; \Gamma_\Lambda(\lambda; v) < \infty\}.$$

It is an immediate consequence of the Minkowski's inequality that $E_\Lambda(\lambda)$ is a subspace of E_Λ . Moreover, we have

$$\Gamma_\Lambda(\lambda; v) = \infty \quad \text{for all } \lambda \notin \text{Cl } \Omega \text{ and non-zero } v \in E_\Lambda, \quad (3.10)$$

$$E_\Lambda(\lambda) = E_\Lambda \quad \text{for all } \lambda \in \Omega$$

(for a proof, see Rossi-Vergne [8, Lemmas 5.13 ~ 5.16]). Next we set for $\lambda \in \Omega$

$$\Gamma_\Lambda(\lambda) := \int_{\Omega} e^{-2\langle \lambda, t \rangle} \Phi_\Lambda(\eta_0(t))^{-1*} \Phi_\Lambda(\eta_0(t))^{-1} \frac{dt}{\det P(t)}, \quad (3.11)$$

where $\Phi_\Lambda(\eta_0(t))^{-1*}$ denotes the adjoint operator of $\Phi_\Lambda(\eta_0(t))^{-1}$.

Lemma 3.1. *The integral in (3.11) is absolutely convergent for any $\lambda \in \Omega$, so that $\Gamma_\Lambda(\lambda)$ is a positive definite hermitian operator.*

The following estimate of $\|\Gamma_\Lambda(\lambda)\|$ plays an important role in

the last part of Theorem 4.1 below.

Proposition 3.2. *There is a positive constant c_Λ such that*

$$\|\Gamma_\Lambda(\lambda)\| \geq c_\Lambda \|\lambda\|^{(\Lambda+\rho)(H'_0)} \quad \text{for all } \lambda \in \Omega.$$

§4. Fourier transform of holomorphic discrete series.

4.1. Paley-Wiener theorem. First of all, we note that if $F \in H(\Lambda)$, then for almost every $y \in \Omega$, the function

$$\mathcal{U} \ni x \mapsto \Phi_\Lambda(\eta_0(y))^{-1} F(x+iy) \in E_\Lambda$$

is square integrable by Fubini's theorem. Hence we can consider its Fourier transform ϕ_y : letting $\mathcal{U}_t := \{x \in \mathcal{U}; \|x\| < t\}$ ($t = 1, 2, \dots$), we set

$$(4.1) \quad \phi_y(\lambda) := \frac{1}{(2\pi)^{m/2}} \lim_{t \rightarrow \infty} \int_{\mathcal{U}_t} \Phi_\Lambda(\eta_0(y))^{-1} F(x+iy) e^{-i\langle \lambda, x \rangle} dx,$$

where $m = \dim \mathcal{U} = \dim \mathfrak{g}(1)$.

On the other hand, recall the operator $\Gamma_\Lambda(\lambda)$ ($\lambda \in \Omega$) defined by (3.11). We know by Lemma 3.1 that $\Gamma_\Lambda(\lambda)$ is positive definite hermitian. So, the positive definite square root $\Gamma_\Lambda(\lambda)^{1/2}$ is well-defined. We now introduce a Hilbert space $\hat{H}(\Lambda)$ of E_Λ -valued measurable functions ϕ on Ω such that

$$(4.2) \quad \|\phi\|^2 := \int_\Omega \|\Gamma_\Lambda(\lambda)^{1/2} \phi(\lambda)\|_\Lambda^2 d\lambda < \infty.$$

Theorem 4.1. Let $F \in H(\Lambda)$ and define ϕ_y by (4.1). Then, there is a measurable E_Λ -valued function ϕ on $\mathcal{U}$ with $\text{supp } \phi \subset C\ell \Omega$ such that

$$\phi_y(\lambda) = e^{-\langle \lambda, y \rangle} \theta_\Lambda(iy)^{-1} \phi(\lambda) \quad (\lambda \in \mathcal{U}).$$

Moreover, one has $\phi \in \hat{H}(\Lambda)$ and the correspondence $\mathcal{F}_\Lambda: H(\Lambda) \ni F \rightarrow \phi \in \hat{H}(\Lambda)$ is a unitary mapping. The inverse $\mathcal{F}_\Lambda^{-1}: \hat{H}(\Lambda) \ni \phi \rightarrow F \in H(\Lambda)$ is given by the absolutely convergent integral

$$(4.3) \quad F(z) = \frac{1}{(2\pi)^{m/2}} \int_{\Omega} \phi(\lambda) e^{i\langle \lambda, z \rangle} d\lambda.$$

The absolute convergence of (4.3) is a consequence of the Schwarz inequality and of Proposition 3.2 together with the fact $\|\Gamma_\Lambda(\lambda)^{-1/2}\| = \|\Gamma_\Lambda(\lambda)\|^{-1/2}$.

4.2. Holomorphic discrete series realized on $\hat{H}(\Lambda)$. With the unitary mapping $\mathcal{F}_\Lambda$ in Theorem 4.1 at hand let us set $\hat{\pi}_\Lambda(g) := \mathcal{F}_\Lambda \pi_\Lambda(g) \mathcal{F}_\Lambda^{-1}$ ($g \in G$). Then we get a holomorphic discrete series representation $\hat{\pi}$ of G on $\hat{H}(\Lambda)$. We will describe representation operators $\hat{\pi}_\Lambda(g)$ ($g \in G$). Recall the element $m_* \in N_K(A)$ defined by (1.2). Since G is generated by m_* and P_0 , it suffices to describe $\hat{\pi}_\Lambda(g)$ ($g \in P$) and $\hat{\pi}_\Lambda(m_*)$.

If $g \in P_0$, then since $J_\Lambda(g, z) = \Phi_\Lambda(g)$ for all $z \in T_\Omega$, we have

$$\pi_\Lambda(g)F(z) = \Phi_\Lambda(g)F(g^{-1} \cdot z) \quad (g \in P_0, F \in H(\Lambda)).$$

Suppose further $g = g_0 \in G(0)$. Then by (2.8), we have $g_0^{-1} \cdot z =$

$(\text{Ad } g_0)^{-1}z$. Therefore

$$\hat{\pi}_\Lambda(g_0)\phi(\lambda) = (\det_{\mathfrak{g}(1)} \text{Ad } g_0) \Phi_\Lambda(g_0)\phi((\text{Ad } g_0)^*\lambda) \quad (\phi \in \hat{H}(\lambda)),$$

where $(\text{Ad } g_0)^*$ is the adjoint to $(\text{Ad } g_0)$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Next let $g = \exp a$ ($a \in \mathfrak{g}(1)$). Then by (2.9), $g^{-1} \cdot z = z - a$, so that by virtue of (3.5)

$$\hat{\pi}_\Lambda(\exp a)\phi(\lambda) = e^{-i\langle \lambda, a \rangle} \phi(\lambda).$$

To describe $\hat{\pi}_\Lambda(m_*)$ we need the following lemma.

Lemma 4.2. (i) Let $r > 0$. Then

$$J_\Lambda(m_*^{-1}, rz)^{-1} = r^{\Lambda(H'_0)} J_\Lambda(m_*^{-1}, z)^{-1} \quad (z \in T_\Omega).$$

(ii) One has

$$\int_{\mathcal{U}} \|J_\Lambda(m_*^{-1}, z)^{-1}\| dx < \infty \quad (z = x+iy \text{ with } y \in \Omega).$$

We now define an operator valued function $\mathcal{J}_\Lambda$ on $\Omega \times \Omega$ by

$$(4.4) \quad \mathcal{J}_\Lambda(t, \lambda) := \frac{1}{(2\pi)^m} \int_{\mathcal{U}} J_\Lambda(m_*^{-1}, z)^{-1} \exp(-i(\langle \lambda, z^{-1} \rangle + \langle t, z \rangle)) dx \\ (z = x+iy, \quad y \in \Omega),$$

where $m = \dim \mathcal{U}$. We note that since $-z^{-1} \in T_\Omega$ if $z \in T_\Omega$, we have $\text{Im } \langle \lambda, z^{-1} \rangle < 0$ for $\lambda \in \Omega$. Therefore

$$|\exp(-i(\langle \lambda, z^{-1} \rangle + \langle t, z \rangle))| = \exp \text{Im}(\langle \lambda, z^{-1} \rangle + \langle t, z \rangle) \leq e^{t \text{Im } z}.$$

Thus the integral in (4.4) is absolutely convergent by Lemma 4.2.

We also note that since the integrand in (4.4) is holomorphic,

$\mathcal{J}_\Lambda(t, \lambda)$ is indeed independent of $y \in \Omega$. We call the function $\mathcal{J}_\Lambda(t, \lambda)$ the Bessel kernel associated to the holomorphic discrete series π_Λ .

Theorem 4.3. One has a realization $\hat{\pi}_\Lambda$ of holomorphic discrete series of G on $\hat{H}(\Lambda)$. The representation operators are given by

- (i) $\hat{\pi}_\Lambda(g_0)\phi(t) = (\det_{\mathfrak{g}(1)} \text{Ad } g_0) \Phi_\Lambda(g_0)\phi((\text{Ad } g_0)^* t) \quad (g_0 \in G(0)),$
- (ii) $\hat{\pi}_\Lambda(\exp a)\phi(t) = e^{-i\langle t, a \rangle} \phi(t) \quad (a \in \mathfrak{g}(1)),$
- (iii) $\hat{\pi}_\Lambda(m_*)\phi(t) = \int_\Omega \mathcal{J}_\Lambda(t, \lambda) \phi(\lambda) d\lambda \quad (\phi \in C_c^\infty(\Omega, E_\Lambda) \subset \hat{H}(\Lambda)).$

We close this note by showing that $\mathcal{J}_\Lambda(t, \lambda)$ is determined by $\mathcal{J}_\Lambda(t) := \mathcal{J}_\Lambda(t, s)$, where s is the unit element of the Jordan algebra $\mathcal{U}$, that is, the element given by (1.8).

Since Ω is diffeomorphic to $G(0) \cap \exp \mathfrak{p}$, there is, for each $t \in \Omega$, a unique element $p_0(t) \in G(0) \cap \exp \mathfrak{p}$ such that $(\text{Ad}_{\mathfrak{g}(1)} p_0(t))s = t$. Recall here the quadratic representation $P(\cdot)$ of the Jordan algebra $\mathcal{U}$. We have $P(t^{1/2})s = t$ for every $t \in \Omega$.

Lemma 4.4. $\text{Ad}_{\mathfrak{g}(1)} p_0(t) = P(t^{1/2})$ for all $t \in \Omega$.

Proposition 4.5. One has, for all $t, \lambda \in \Omega$

$$\mathcal{J}_\Lambda(t, \lambda) = (\det_{\mathfrak{g}(1)} \text{Ad } p_0(\lambda)) \Phi_\Lambda(p_0(\lambda)) \mathcal{J}_\Lambda(P(\lambda^{1/2})t) \Phi_\Lambda(p_0(\lambda)).$$

It would be interesting to study the operator valued function

$\mathcal{J}_\Lambda$ in detail. For $G = \mathrm{Sp}(\ell, \mathbb{R})$, $\mathrm{SU}(\ell, \ell)$ and $\mathrm{SO}^*(4\ell)$, $\mathcal{J}_\Lambda$ is essentially the reduced Bessel function investigated by Gross-Kunze [3]. For G equal to one of the above three groups or $\mathrm{SO}_0(\ell, 2)$ but with τ_Λ one dimensional, $\mathcal{J}_\Lambda$ is essentially the Bessel function studied by Faraut-Travaglini [2].

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