

Note on Galois extensions

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Throughout, A will represent a ring with 1, and B a subring of A containing 1. Let V be the centralizer $V_A(B)$ of B in A , $H = V_A(V)$, and C the center of A .

The present note contains several results concerning Galois extensions, where an (Artinian) simple ring extension A/B is called a [finite] Galois extension if $[_B A]$ is finitely generated and V is a simple ring and B coincides with the fixring of the group G of all B -ring automorphisms in A (cf. [7]). A Galois extension A/B is said to be outer Galois or inner Galois according as $V = C$ or $H = B$. If A/B is finite outer Galois, then it is known that A/B is $\tilde{G}$ -Galois (cf. [7, Proposition 9.6]). As to other terminologies and notations used in this note, we follow [2] and [7].

First, we shall prove the following which will enrich the substance of the theory of separable extensions:

Theorem 1. If A/B is finite Galois then it is a separable extension.

Proof. Since A/H is finite inner Galois, we have $\text{End } A_H = A_L \cdot V_R$. Noting that $[\text{Hom}(A_H, H):H]_R = [A:H]$, we readily see that

$$\begin{aligned} {}_A A \otimes_H {}_H A &\cong {}_A \text{Hom}(H_H, A_H) \otimes_H {}_H A \cong {}_A \text{Hom}(\text{Hom}(A_H, H_H), A_H) \\ &\cong {}_A (\text{End } A_H)_A \cong {}_A (A \oplus \dots \oplus A)_A. \end{aligned}$$

Hence, ${}_A A \otimes_H {}_H A$ is homogeneously completely reducible and the A - A -homomorphism $A \otimes_H A \longrightarrow A$ ($a \otimes a' \longmapsto aa'$) splits. On the other hand, H/B being finite outer Galois, by [1, Proposition 3.3] the

H - H -homomorphism $H \otimes_B H \longrightarrow H$ ($h \otimes h' \longmapsto hh'$) splits, whence it follows that the A - A -homomorphism $A \otimes_B A \longrightarrow A \otimes_H A$ ($a \otimes a' \longmapsto a \otimes a'$) splits. Combining those above, we readily see that the A - A -homomorphism $A \otimes_B A \longrightarrow A$ ($a \otimes a' \longmapsto aa'$) splits.

If A/B is H -separable, i.e., if $A \otimes_B A \leq \bigoplus_A (A \oplus \dots \oplus A)_A$, then there exist some $v_i \in V$ and $\sum_j x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A = \{u \in A \otimes_B A \mid au = ua \text{ for all } a \in A\}$ such that $\sum_{i,j} x_{ij} \otimes y_{ij} v_i = 1 \otimes 1$. For $g \in \text{End } A_B$, $\sum_{i,j} x_{ij} \otimes y_{ij} a v_i = a \otimes 1$ implies then $\sum_{i,j} g(x_{ij}) \otimes y_{ij} a v_i = g(a) \otimes 1$, whence it follows $\sum_{i,j} g(x_{ij}) y_{ij} a v_i = g(a)$ (cf. [3]). Especially, we have

Proposition 1. If A/B is H -separable, then $\text{End } A_B = A_L \cdot V_R$.

As was shown in [8, Proposition 1.1], if A is a separable R -algebra and a projective R -module then A is a finitely generated R -module. We state here an analogue of the above for H -separable extensions.

Proposition 2 (cf. [6]). If A/B is H -separable and A_B is projective then A_B is finitely generated.

Proof. Let $\{a_\kappa; f_\kappa\}_{\kappa \in K}$ ($a_\kappa \in A$, $f_\kappa \in \text{Hom}(A_B, B_B)$) be a projective coordinate system for A_B . Then, f_κ extends naturally to $f_\kappa^* \in \text{Hom}(A \otimes_B A, A)$ and $\{a_\kappa \otimes 1, f_\kappa^*\}_{\kappa \in K}$ is a projective coordinate system for $A \otimes_B A$. Since $A \otimes_B A$ is finitely generated, we can find a finite subset K' of K such that $A \otimes_B A = \sum_{\kappa \in K'} (a_\kappa \otimes 1)A$. We consider here the set $K'' = \{\kappa \in K \mid f_\kappa(a_{\kappa'}) \neq 0 \text{ for some } \kappa' \in K'\}$, which is obviously a finite subset of K .

If a is an arbitrary element of A then $\{\kappa \in K \mid f_{\kappa}^*(a \otimes 1) \neq 0\} \subseteq K''$ and we have

$$\begin{aligned} a \otimes 1 &= \sum_{\kappa \in K} (a_{\kappa} \otimes 1) f_{\kappa}^*(a \otimes 1) = \sum_{\kappa \in K''} (a_{\kappa} \otimes 1) f_{\kappa}^*(a \otimes 1) \\ &= \sum_{\kappa \in K''} a_{\kappa} f_{\kappa}(a) \otimes 1, \end{aligned}$$

which implies $a \in \sum_{\kappa \in K''} a_{\kappa} A$.

The next is only a combination of [4, Theorem 1.5] and [5, Theorem 2.1]. However, Propositions 1 and 2 and the proof of Theorem 1 enable us to obtain a shorter proof.

Theorem 2. If B is simple, then the following conditions are equivalent:

- (1) A/B is finite inner Galois.
- (2) ${}_A A \otimes_B A \cong_A (A \oplus \dots \oplus A)_A$.
- (3) A/B is H-separable.

Proof. As $(1) \Rightarrow (2)$ is obvious by the proof of Theorem 1 and $(2) \Rightarrow (3)$ is trivial, it remains only to prove $(3) \Rightarrow (1)$. By Proposition 2, A_B is finitely generated projective. Hence, there exist $a_1, \dots, a_n \in A$ and $f_1, \dots, f_n \in \text{Hom}(A_B, B_B) \subseteq \text{End } A_B = A_L \cdot V_R$ (Proposition 1) such that $a = \sum_i a_i f_i(a)$ for any $a \in A$. If I is an arbitrary non-zero ideal of A then $f_i(I) \subseteq AIV \cap B = I \cap B$, and so $I \subseteq \sum_i A f_i(I) \subseteq A(I \cap B)$, namely, $I = A(I \cap B) = AB = A$. This means that A is simple. Moreover, the simplicity of $\text{End } A_B = A_L \cdot V_R (\cong A \otimes_C V^0)$ yields the simplicity of V and $[V:C] = [A:B]_R$. Hence, A/B is finite inner Galois.

Remark 1. Let A be the 2×2 -matrix ring over a division ring D ; $A = \sum_{i,j=1}^2 De_{ij}$, and $B = De_{11} \oplus De_{22}$. Then, A/B is a free

H-separable extension (but A is not centrally projective over B , i.e., ${}_B A_B$ can not be a direct summand of ${}_B (B \oplus \dots \oplus B)_B$). In fact, $\{1, t = e_{12} + e_{21}\}$ is a right [left] free B -basis of A and $\{1 \otimes e_{11} + t \otimes e_{12}, 1 \otimes e_{22} + t \otimes e_{21}\}$ is a free A -basis of $A \otimes_B A$ contained in $(A \otimes_B A)^A$. This will show that for a projective H-separable extension A/B the simplicity of A need not imply that of B .

Finally, we shall prove a slight improvement of [2, Theorem 3].

Theorem 3. Let A be Galois and left algebraic over B . If $[V:C] < \infty$ then the following conditions are equivalent:

- (1) ${}_B A_B$ is completely reducible.
- (2) ${}_H A_H$ and ${}_B H_B$ are completely reducible.
- (3) ${}_H H_H < \oplus {}_H A_H$ and ${}_B B_B < \oplus {}_B H_B$.
- (4) V/C is separable and ${}_B B_B < \oplus {}_B H_B$.

Proof. The equivalence of (1), (2) and (4) has been proved in [2, Theorem 3]. It suffices therefore to show that if ${}_H H_H < \oplus {}_H A_H$ then V/C is separable. To be easily seen, $\text{End } {}_H A_H = V_L \cdot V_R$ is canonically V - V -isomorphic to $V \otimes_C V$. Hence, there exists an element $\sum_i v_i \otimes v'_i \in V \otimes_C V$ such that $\sum_i v_i v'_i = 1$ and $\sum_i v_i a v'_i \in H$ for all $a \in A$. If v is in V then $\sum_i v v_i a v'_i = \sum_i v_i a v'_i v$, which means $\sum_i v_i \otimes v'_i \in (V \otimes_C V)^V$. Accordingly, V/C is separable.

As a special case of Theorem 3, we have the following which contains [5, Corollary 2 (2)]:

Corollary 1. Let A be finite inner Galois over B . If B' is

a simple intermediate ring of A/B , then the following conditions are equivalent:

- (1) ${}_B A_{B'}$ is completely reducible.
- (2) ${}_B B'_{B'} < \oplus {}_B A_{B'}$.
- (3) $V_A(B')/C$ is separable.

Remark 2. The following example will show that, under the hypothesis of Corollary 1, ${}_B B'_{B'} < \oplus {}_B A_{B'}$ need not imply ${}_B B'_{B'} < \oplus {}_B A_{B'}$: Let P/Φ be a two dimensional purely inseparable field extension. If we set $A = (\Phi)_2$ and $B = \Phi$, then A/B is finite inner Galois and there exists an intermediate ring B' of A/B which is B -ring isomorphic to P . By Corollary 1, ${}_B A_{B'}$ is not completely reducible, but ${}_B A_B$ is obviously completely reducible.

Remark 3. Theorem 3 will form a contrast with [2, Theorem 4]: Let A be Galois and left algebraic over B . If $[V:C] < \infty$ then the following conditions are equivalent:

- (1) ${}_B A_B$ is completely indecomposable.
- (2) ${}_H A_H$ and ${}_B H_B$ are completely indecomposable.
- (3) i) $V = V_B(B)$ and is purely inseparable over C ;
 ii) either A/B is inner Galois or the Galois group of H/B is a pro- p -group and A is of characteristic p .

References

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