

Instantons and representations of an associative algebra

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In this note we show that instantons on S^4 can be identified with some representations of an associative algebra.

Let A be the free algebra over $\mathbb{C}$ generated by two elements q, p . We define a new product $*$ in A as follows:

$$f_1 * f_2 = f_1(pq - qp)f_2, \quad f_1, f_2 \in A.$$

Then $(A, *)$ is an associative algebra (with no unit), which is an extension of the Weyl algebra $A/(pq - qp - 1)$. We consider finite dimensional representations of $(A, *)$. Let W be the complex vector space of dimension l , and h be a linear map from A to $\text{End } W$. Then h induces a linear map

$$\tilde{h}: A \otimes W \rightarrow A^* \otimes W$$

defined by

$$\langle \tilde{h}(f_1 \otimes w), f_2 \rangle = h(f_2 f_1)w, \quad f_1, f_2 \in A, w \in W.$$

We denote by $H(l, k)$ the set of all algebra homomorphisms $h: (A, *) \rightarrow \text{End } W$ such that the rank of $\tilde{h}$ is k .

Let P be the principal $SU(l)$ bundle over $S^4 = \mathbb{R}^4 \cup \infty$ with $c_2 = k$, and $\widetilde{M}(SU(l), k)$ be the framed moduli space for anti-self-dual (ASD) connections on P : $\{ \text{ASD connections on } P \} / \mathcal{G}_\infty$, where $\mathcal{G}_\infty$ stands for the group of all gauge transformations on P fixing the points in the fiber over ∞ . $\widetilde{M}(SU(l), k)$ is a $4kl$ -dimensional smooth manifold.

Our main result is the following:

THEOREM 1. *The framed moduli space $\widetilde{M}(SU(l), k)$ is diffeomorphic to $H(l, k)$.*

§1. Some remarks on a theorem of Donaldson.

Let $X = \text{Mat}(k, k; \mathbb{C}) \times \text{Mat}(k, k; \mathbb{C}) \times \text{Mat}(l, k; \mathbb{C}) \times \text{Mat}(k, l; \mathbb{C})$. We define the action of $G = GL(k, \mathbb{C})$ on X as follows:

$$p \cdot (\alpha_1, \alpha_2, a, b) = (p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb)$$

for $p \in G$, $(\alpha_1, \alpha_2, a, b) \in X$. We call a point x in X *stable* when the map $G \ni p \mapsto p \cdot x \in X$ is proper. We denote by X^s the set of all stable points in X . Let

$$\begin{aligned}\omega(\alpha_1, \alpha_2, a, b) &= \text{tr}(d\alpha_1 \wedge d\alpha_2 + db \wedge da), \\ \mu &= \alpha_1 \alpha_2 - \alpha_2 \alpha_1 + ba.\end{aligned}$$

We can show by easy computation that

$$\begin{aligned}\omega(p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb) \\ = \omega(\alpha_1, \alpha_2, a, b) + \text{tr}(p^{-1} dp \wedge d\mu) + \text{tr}(p^{-1} dp \wedge p^{-1} dp \cdot \mu).\end{aligned}$$

(This is suggested to the author by H. Nakajima from the viewpoint of hyperkähler structure.)

THEOREM (DONALDSON [1]). *The framed moduli space $\widetilde{M}(SU(l), k)$ is diffeomorphic to $G \backslash \mu^{-1}(0) \cap X^s$.*

So we deduce from geometric invariant theory [4] that $\widetilde{M}(SU(l), k)$ is an open dense nonsingular subset of an affine algebraic variety.

Next we seek a criterion for the stability in this case. Let $A^m \in \text{Mat}(2^m l, k; \mathbb{C})$ be the matrix which is the column of matrices $a\alpha_{i_1} \cdots \alpha_{i_m}$, $i_j = 0, 1$, and $B^m \in \text{Mat}(k, 2^m l; \mathbb{C})$ be the matrix which is the row of matrices $\alpha_{i_1} \cdots \alpha_{i_m} b$, i. e.

$$\begin{aligned}A^0 &= a, A^1 = \begin{pmatrix} a\alpha_1 \\ a\alpha_2 \end{pmatrix}, A^2 = \begin{pmatrix} a\alpha_1\alpha_1 \\ a\alpha_2\alpha_1 \\ a\alpha_1\alpha_2 \\ a\alpha_2\alpha_2 \end{pmatrix}, \dots, A^m = \begin{pmatrix} A^{m-1}\alpha_1 \\ A^{m-1}\alpha_2 \end{pmatrix}, \\ B^0 &= b, B^1 = (\alpha_1 b \quad \alpha_2 b), B^2 = (\alpha_1\alpha_1 b \quad \alpha_1\alpha_2 b \quad \alpha_2\alpha_1 b \quad \alpha_2\alpha_2 b), \\ \dots, B^m &= (\alpha_1 B^{m-1} \quad \alpha_2 B^{m-1}).\end{aligned}$$

We set

$$A_m = \begin{pmatrix} A^0 \\ \vdots \\ A^m \end{pmatrix}, \quad B_m = (B^0 \quad \dots \quad B^m).$$

LEMMA 2. *The point $x = (\alpha_1, \alpha_2, a, b) \in X$ is stable if and only if $\text{rank } A_{k-1} B_{k-1} = k$.*

LEMMA 2'. *The point $x = (\alpha_1, \alpha_2, a, b) \in X$ is stable if and only if $\text{rank } A_m B_n = k$ for some m, n .*

PROOF: We can test the stability of a point by the following:

HILBERT CRITERION ([1,4]). The point $x \in X$ is stable for G if and only if for all $g \in G$ and integers $(w_1, \dots, w_k) \neq (0, \dots, 0)$:

$$g \begin{pmatrix} t^{w_1} & & \\ & \ddots & \\ & & t^{w_k} \end{pmatrix} g^{-1} \cdot x \rightarrow \infty \quad \text{as } t \rightarrow \infty.$$

CLAIM: If $\text{rank } A_{m+1} = \text{rank } A_m$, then $\text{rank } A_{m'} = \text{rank } A_m$ for all $m' \geq m$. Similarly, if $\text{rank } B_{m+1} = \text{rank } B_m$, then $\text{rank } B_{m'} = \text{rank } B_m$ for all $m' \geq m$.

PROOF: Assume that $\text{rank } A_{m+1} = \text{rank } A_m$. Then the row vectors in A^{m+1} can be written by the linear combinations of the row vectors in A_m . So the row vectors in $A^{m+2} = \begin{pmatrix} A^{m+1}\alpha_1 \\ A^{m+1}\alpha_2 \end{pmatrix}$ are the linear combinations of the row vectors in $A_m\alpha_1, A_m\alpha_2$, which are the row vectors in A_{m+1} . So $\text{rank } A_{m+2} = \text{rank } A_{m+1}$. The claim follows by induction. ■

Now we go back to the proof of Lemma 2, 2'. First we assume that $\text{rank } A_{k-1} = k' < k$. If $k = 1$, then $a = 0$ and

$$t^{-1} \cdot (\alpha_1, \alpha_2, a, b) = (\alpha_1, \alpha_2, 0, t^{-1}b) \rightarrow (\alpha_1, \alpha_2, 0, 0) \quad \text{as } t \rightarrow \infty.$$

This implies that $(\alpha_1, \alpha_2, a, b)$ is not stable.

If $k > 1$, we deduce from the Claim that $\text{rank } A_{k-2} = k'$. So

$$A_{k-1}g = \begin{pmatrix} A_{k-2}g \\ A^{k-1}g \end{pmatrix} = \begin{pmatrix} A' & 0 \\ * & 0 \end{pmatrix},$$

for some $g \in G$, where the column vectors in A' are linearly independent. Particularly, $ag = (* \ 0)$. Since the row vectors in $A_{k-2}\alpha_1$ are the ones in A_{k-1} ,

$$(A' \ 0)g^{-1}\alpha_1g = (* \ 0).$$

This implies that $g^{-1}\alpha_1g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Similarly we get $g^{-1}\alpha_2g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. So

$$\begin{pmatrix} 1_{k'} & \\ & t^{-1}1_{k-k'} \end{pmatrix} \cdot (g^{-1}\alpha_1g, g^{-1}\alpha_2g, ag, g^{-1}b)$$

converges as $t \rightarrow \infty$. Therefore if $\text{rank } A_{k-1} < k$, then $x = (\alpha_1, \alpha_2, a, b)$ is not stable. Similarly, if $\text{rank } B_{k-1} < k$, x is not stable.

Next we assume that $(\alpha_1, \alpha_2, a, b)$ is not stable. From the Hilbert Criterion we get some $g \in G, (w_1, \dots, w_k)$ such that

$$\begin{pmatrix} t^{w_1} & & \\ & \ddots & \\ & & t^{w_k} \end{pmatrix} \cdot (g^{-1}\alpha_1 g, g^{-1}\alpha_2 g, ag, g^{-1}b)$$

converges as $t \rightarrow \infty$. We may assume that $w_1 \geq \dots \geq w_k$. If $w_{k'} \geq 0 > w_{k'+1}$, we deduce that

$$ag = \begin{pmatrix} * & 0 \end{pmatrix}, g^{-1}\alpha_1 g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, g^{-1}\alpha_2 g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

This implies that $A_m g = \begin{pmatrix} * & 0 \end{pmatrix}$. Similarly, if $w_{k'} > 0 \geq w_{k'+1}$, then $g^{-1}B_n = \begin{pmatrix} 0 \\ * \end{pmatrix}$. Therefore if $(\alpha_1, \alpha_2, a, b)$ is not stable, then $\text{rank } A_m B_n < k$ for all m, n . ■

§2 The proof of Theorem 1.

First we give the map φ from $\widetilde{M}(SU(l), k)$ to $H(l, k)$. Let

$$h(f) = \varphi(\alpha_1, \alpha_2, a, b)(f) = af(\alpha_1, \alpha_2)b$$

for $(\alpha_1, \alpha_2, a, b) \in \mu^{-1}(0) \cap X^s$. φ is G -invariant. Since $\mu(\alpha_1, \alpha_2, a, b) = 0$,

$$\begin{aligned} h(f_1 * f_2) &= h(f_1(pq - qp)f_2) \\ &= af_1(\alpha_1, \alpha_2)(\alpha_2\alpha_1 - \alpha_1\alpha_2)f_2(\alpha_1, \alpha_2)b \\ &= af_1(\alpha_1, \alpha_2)ba f_2(\alpha_1, \alpha_2)b \\ &= h(f_1)h(f_2). \end{aligned}$$

We give $i: \mathbb{C}^k \rightarrow A^* \otimes \mathbb{C}^l, j: A \otimes \mathbb{C}^l \rightarrow \mathbb{C}^k$ by

$$\begin{aligned} \langle i(v), f \rangle &= af(\alpha_1, \alpha_2)v \\ j(f \otimes w) &= f(\alpha_1, \alpha_2)bw \end{aligned}$$

for $f \in A, v \in V, w \in W$. Then we have $\widetilde{h} = i \circ j$. Lemma 2' implies that i is injective and that j is surjective, so $\text{rank } \widetilde{h} = k$. Therefore $h \in H(l, k)$.

On the other hand, the inverse $\psi: H(l, k) \rightarrow \widetilde{M}(SU(l), k)$ is defined as follows. For $h' \in H(l, k)$, we set $V = \text{Coim } \widetilde{h}' \cong \text{Im } \widetilde{h}' \cong \mathbb{C}^k$. Let

$$\begin{aligned} \widetilde{h}' &= i' \circ j', & i': V &\rightarrow A^* \otimes W, \\ & & j': A \otimes W &\rightarrow V. \end{aligned}$$

For $f \in A$ we define $\langle f| \in \text{Hom}(V, W)$, $|f\rangle \in \text{Hom}(W, V)$ by

$$\begin{aligned}\langle f|(v) &= \langle i'(v), f \rangle, \quad v \in V, \\ |f\rangle(w) &= j'(f \otimes w), \quad w \in W.\end{aligned}$$

We set $a' = \langle 1|$, $b' = |1\rangle$. The multiplications by q, p in A induce linear maps $\alpha'_1, \alpha'_2 \in \text{End } V$ respectively:

$$\alpha'_1|f\rangle = |qf\rangle, \quad \alpha'_2|f\rangle = |pf\rangle$$

for $f \in A$. If $|f\rangle = 0$, then $h(f'f) = 0$ for all $f' \in A$. So $\alpha'_1, \alpha'_2 \in \text{End } V$ are well-defined. We get

$$\psi(h') = (\alpha'_1, \alpha'_2, a', b') \in X$$

by fixing the basis of V, W . Since

$$\begin{aligned}\bigcap_{f \in A} \text{Ker } a'f(\alpha'_1, \alpha'_2) &= \bigcap_{f \in A} \text{Ker } \langle f| = 0, \\ \sum_{f \in A} \text{Im } f(\alpha'_1, \alpha'_2)b' &= \sum_{f \in A} \text{Im } |f\rangle = V,\end{aligned}$$

we deduce from Lemma 2' that $\psi(h')$ is stable. Since $h': (A, *) \rightarrow \text{End } W$ is an algebra homomorphism, we have

$$\begin{aligned}\langle f_1|\alpha'_1\alpha'_2 - \alpha'_2\alpha'_1 + b'a'|f_2\rangle \\ = h'(f_1(qp - pq)f_2) + \langle f_1|1\rangle\langle 1|f_2\rangle \\ = -h'(f_1 * f_2) + h'(f_1)h'(f_2) \\ = 0.\end{aligned}$$

Therefore $\psi(h') \in G \setminus \mu^{-1}(0) \cap X^s$.

If $(\alpha'_1, \alpha'_2, a', b') = \psi(h')$,

$$\begin{aligned}a'f(\alpha'_1, \alpha'_2)b' &= \langle 1|f(\alpha'_1, \alpha'_2)|1\rangle \\ &= \langle 1|f\rangle \\ &= h'(f).\end{aligned}$$

Hence $\varphi \circ \psi(h') = h'$.

If $h' = \varphi(\alpha_1, \alpha_2, a, b)$, we can take $i' = i, j' = j$ by the stability. Then

$$\langle f| = af(\alpha_1, \alpha_2), \quad |f\rangle = f(\alpha_1, \alpha_2)b.$$

This implies that

$$\begin{aligned}\langle 1| &= a, \quad |1\rangle = b, \\ |qf\rangle &= \alpha_1 f(\alpha_1, \alpha_2)b = \alpha_1|f\rangle, \\ |pf\rangle &= \alpha_2 f(\alpha_1, \alpha_2)b = \alpha_2|f\rangle.\end{aligned}$$

Hence $\psi \circ \varphi = \text{id}$. ■

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