

Asymptotic behavior of radially symmetric solutions for the Burgers equation in several space dimensions

大阪市立大学数学研究所/関西大学システム理工学部 橋本 伊都子

Itsuko Hashimoto

OCAMI, Faculty of Engineering Science Kansai university
(joint work with Akitaka Matsumura)

1 Introduction

In the present article, we consider the asymptotic behavior of radially symmetric solutions of the multi-dimensional Burgers equation. Burgers equation in multi-dimensional space is written as

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \mu \Delta u, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (1.1)$$

where $u = (u_1(t, x), \dots, u_n(t, x))$ is a vector valued unknown function of $t > 0$ and $x = (x_1, \dots, x_n)$, and μ is a given positive constant.

In previous papers [1, 2], we investigated an initial boundary value problem for radially symmetric solutions of (1.1) on the exterior domain $|x| > r_0$ for some positive constant r_0 , where the initial data, and the boundary and far-field conditions are prescribed. Introducing a new unknown variable $v(t, r)$ by $u = (x/r)v(t, r)$ with $r = |x|$, our problem for (1.1) can be rewritten with respect to r as follows:

$$\begin{cases} v_t + vv_r = \mu(v_{rr} + (n-1)(\frac{v}{r})_r), & r > r_0, \quad t > 0, \\ v(t, r_0) = v_-, \quad \lim_{r \rightarrow +\infty} v(t, r) = v_+, & t > 0, \\ v(0, r) = v_0(r), & r > r_0, \end{cases} \quad (1.2)$$

where the initial data v_0 is assumed to satisfy the compatibility conditions $v_0(r_0) = v_-$ and $\lim_{r \rightarrow \infty} v_0(r) = v_+$.

In a previous article [1], we considered the initial boundary value problem (1.2) for $n \geq 2$ for the cases in which the boundary and far field conditions satisfy (a): $v_- < v_+ = 0$, (b): $v_- < 0 < v_+$, and (c): $0 = v_- < v_+$, and we showed that the asymptotic states of the time-global solution are given by a monotonically increasing stationary wave in case (a), a superposition of a monotonically increasing

stationary wave and a rarefaction wave in case (b), and a rarefaction wave in case (c). These results are similar to those for the 1-D Burgers equation investigated by Liu-Matsumura-Nishihara [5]. Note that in all cases the corresponding 1-D Riemann problem admits a single rarefaction wave. Here the monotonically increasing property of the stationary wave played an important role.

In [2], we considered case (d), in which $0 < v_- < v_+$, which was excluded in [1]. We first showed that if and only if $0 < v_- \leq 2(n-2)\mu/r_0$ and $v_+ = 0$, there exists a stationary wave, “that decreases monotonically” to zero as $r \rightarrow \infty$, which never occurs for the 1-D Burgers equation. Based on this result, we considered the initial boundary value problem (1.2) for $n \geq 3$ in case (d), and showed that the asymptotic state of the time-global solution is given by a superposition of a monotonically decreasing stationary wave and a rarefaction wave under the condition $0 < v_- < \mu/(2r_0)$. Note that in this case the corresponding 1-D Riemann problem still admits a single rarefaction wave, and the asymptotic state for the solution of 1-D Burgers equation is proved to be the rarefaction wave by Liu-Matsumura-Nishihara [5] and Nakamura [8].

In [3], we further investigate cases (e): $0 = v_+ < v_-$, (f): $0 < v_+ \leq v_-$ and (g): $v_- \leq v_+ < 0$. We show that the asymptotic states of the time-global solutions are still given by a monotonically decreasing stationary wave in case (e), and a linear superposition of a monotonically decreasing stationary wave and a rarefaction wave in case (f) under the assumption $0 < v_- < 2\mu/(r_0(1 + \sqrt{(n-3)/(n-1)}))$. For case (g), we first show the existence of a non-monotonic stationary wave, which neither increases nor decreases monotonically. This never occurs in the case of the 1-D Burgers equation, and we show that this non-monotonic stationary wave is asymptotically stable. Here, we use the spatial weighted energy method because of the difficulty arising from the non-monotonic property of the stationary wave. Note that in case (g), the asymptotic state for the solution of the 1-D Burgers equation is known to be a monotonically increasing stationary wave (see Liu-Matsumura-Nishihara [5]). This suggests that, the 1-D Riemann problem for the non-viscous part can never classify all of the asymptotic states of the multi-dimensional Burgers equation. In particular, the remaining case (h): $0 > v_+ < v_-$ is open in general. Nevertheless, for the case in which $n = 3$, due to the specific structure of the 3-D equation, we can reduce the problem (1.2) to that for the plain 1-D Burgers equation, and we eventually succeed in obtaining the complete classification of asymptotic states, including a linear superposition of stationary wave and a viscous shock wave. Thus, we can exactly clarify the multi-dimensional effects on the asymptotic behaviors for the case in which $n = 3$.

Some Notation. We denote the usual Lebesgue space of square integrable functions over (r_0, ∞) by $L^2 = L^2((r_0, \infty))$, and denote the corresponding k th-order Sobolev space by $H^k, k = 1, 2, \dots$. Further, we denote the space of functions $f \in H^1$ with $f(r_0) = 0$ by $H_0^1 = H_0^1((r_0, \infty))$. For $\beta > 0$, we also denote the first-order weighted Sobolev space, that is, the space of functions $(1+r)^{\beta/2}f \in H^k$,

by $H^{k,\beta} = H^{k,\beta}((r_0, \infty))$. For an interval $I \subset \mathbb{R}^1$ and a Banach space X , $C^k(I; X)$ denotes the space of k -times continuously differentiable X -valued functions on I , and $L^2(I; X)$ denotes the space of square integrable X -valued functions on I .

2 Main theorems

Before we state the main theorems, let us recall the stationary and rarefaction waves of (1.2). We call $\phi(r)$ the stationary wave of (1.2) if ϕ satisfies the stationary problem corresponding to (1.2):

$$\begin{cases} (\frac{1}{2}\phi^2)_r = \mu (\phi_{rr} + (n-1) (\frac{\phi}{r})_r), & r > r_0, \\ \phi(r_0) = v_-, \quad \lim_{r \rightarrow +\infty} \phi(r) = v_+. \end{cases} \quad (2.1)$$

In what follows, we write the solution of (2.1) as $\phi_{v_-, v_+}(r)$ when we emphasize the boundary value of the stationary solution v_- and the far-field state v_+ . The basic properties of the stationary wave are given as follows.

Proposition 2.1 (Case $v_+ = 0$). *Suppose $n \geq 3$, $0 < v_- \leq 2\mu(n-2)/r_0$, and $v_+ = 0$. Then the stationary problem (2.1) has a unique smooth solution $\phi(r)$ satisfying the following.*

- (i) *If $v_- = 2\mu(n-2)/r_0$, then $\phi(r) = 2\mu(n-2)/r$, $r \geq r_0$.*
- (ii) *If $0 < v_- < 2\mu(n-2)/r_0$, then $0 < \phi(r) \leq v_-$ and $\phi_r(r) < 0$, $r \geq r_0$. Moreover, ϕ satisfies $|\phi(r)| \sim (r+1)^{-n+1}$, $r \rightarrow \infty$.*

The proof of Proposition 2.1 is clear because the solution of (2.1) is exactly given by the formula

$$\phi(r) = \phi_{v_-, 0}(r) = \frac{v_-}{(1 - \frac{r_0 v_-}{2\mu(n-2)})(r/r_0)^{n-1} + \frac{r_0 v_-}{2\mu(n-2)}(r/r_0)}. \quad (2.2)$$

Next, we state the non-monotonic stationary wave which is used in Section 3.

Proposition 2.2 (Case $v_+ < 0$). *Suppose $n \geq 3$, $v_- \leq v_+ < 0$. Then the stationary problem (2.1) has a unique smooth solution ϕ satisfying the following.*

- (i) *There exists a negative constant $\nu_0 \in (v_+, 0)$ such that*

$$\phi(r) < \nu_0, \quad r \geq r_0. \quad (2.3)$$

- (ii) *It holds that*

$$\phi(r) - \frac{\mu(n-1)}{r} < v_+, \quad r \geq r_0, \quad (2.4)$$

and $\phi - \mu(n-1)^2/(2r)$ is monotonically increasing, that is,

$$\phi_r(r) > -\frac{\mu(n-1)^2}{2r^2}, \quad r \geq r_0. \quad (2.5)$$

(iii) It holds that

$$|\phi(r) - v_+ - \frac{\mu(n-1)}{r}| \leq O(r^{-2}), \quad r \rightarrow \infty. \quad (2.6)$$

Next, for $0 \leq v_- < v_+$, we define the rarefaction wave ψ_{v_-,v_+} of (1.2) which connects constant states v_- to v_+ by $\psi_{v_-,v_+} = \hat{\psi}_{v_-,v_+}((r-r_0)/t)$ for $t > 0$, where

$$\hat{\psi}_{v_-,v_+}(\xi) := \begin{cases} v_-, & \xi \leq v_-, \\ \xi, & v_- \leq \xi \leq v_+, \\ v_+, & v_+ \leq \xi. \end{cases} \quad (2.7)$$

Now we are ready to state our first main theorem.

Theorem 2.3. Suppose $n \geq 3$, $0 < v_- < 2\mu/(r_0(1 + \sqrt{(n-3)/(n-1)}))$, and $0 \leq v_+$. Then we have the following results.

(1) (Asymptotic stability) Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^1), \quad v_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\limsup_{t \rightarrow \infty} \sup_{r > r_0} \left| v(t, r) - \phi_{v_-,0}(r) - \hat{\psi}_{0,v_+} \left(\frac{r-r_0}{t} \right) \right| = 0.$$

(2) (Decay rate) Further assume that $v_0 - v_+ \in H^1 \cap L^1$. Then the solution v satisfies the following decay rate estimates: if $v_+ > 0$, it holds that

$$\|(v - \phi_{v_-,0} - \psi_{0,v_+})(t)\|_{H^1} \leq C(1+t)^{-\frac{1}{4}} \log^2(2+t), \quad t \geq 1, \quad (2.8)$$

and if $v_+ = 0$, it holds that

$$\|(v - \phi_{v_-,0})(t)\|_{H^1} \leq C(1+t)^{-\frac{1}{4}}, \quad t \geq 0. \quad (2.9)$$

As for the asymptotic stability of the non-monotonic stationary solution ϕ , we have the following:

Theorem 2.4. Suppose that $n \geq 3$, $v_- \leq v_+ < 0$ and $v_0 - \phi \in H^{1, \frac{n-1}{2}}$. Then there exists a positive constant ϵ_0 such that if $\|v_0 - \phi\|_{H^{1, \frac{n-1}{2}}} \leq \epsilon_0$, then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v \in C^0([0, \infty); H^1), \quad v_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\limsup_{t \rightarrow \infty} \sup_{r > r_0} |v(t, r) - \phi(r)| = 0.$$

Next, we state the theorem for the complete classification of asymptotic states for the space dimension $n = 3$. In this case, if we introduce a new unknown valuable V by

$$v(t, r) = \frac{2\mu}{r} + V(t, r), \quad (2.10)$$

then our original problem (1.2) for the 3-D Burgers equation is surprisingly reduced to the 1-D Burgers equation:

$$\begin{cases} V_t + VV_r = \mu V_{rr}, & t > 0, r > r_0, \\ V(t, r_0) = v_- - 2\mu/r_0 =: V_-, & t > 0, \\ \lim_{r \rightarrow +\infty} V(t, r) = v_+ =: V_+, & t > 0, \\ V(0, r) = V_0(r) := v_0(r) - 2\mu/r, & r > r_0. \end{cases} \quad (2.11)$$

Once the problem is reduced to the 1-D Burgers equation, all of the asymptotic behaviors have been classified in terms of the boundary and far field values $V_{\pm}$ by Liu-Matsumura-Nishihara [5], Nakamura [8], Liu-Nishihara [6], Nishihara [9], and Liu-Yu [7]. To state the results precisely, let us first recall the stationary wave solution Φ to the problem (2.11):

$$\begin{cases} \Phi \Phi_r = \mu \Phi_{rr}, & r > r_0, \\ \Phi(r_0) = V_-, \quad \lim_{r \rightarrow +\infty} \Phi(r) = V_+. \end{cases} \quad (2.12)$$

An elementary calculation shows the following properties.

Proposition 2.5. *If and only if $V_+ \leq 0$ and $V_- < |V_+|$, a solution of (2.12), except the trivial solution $\Phi \equiv 0$, uniquely exists and satisfies the following.*

- (i) *For $V_- = V_+$, the solution is given by the constant state $\Phi(r) = V_- = V_+$.*
- (ii) *For $V_- < V_+ = 0$, the solution is monotonically increasing and, given by*

$$\Phi(r) = \frac{V_-}{1 - \frac{V_-}{2\mu}(r - r_0)}. \quad (2.13)$$

- (iii) *For $V_+ < 0$ and $V_- < V_+$ (resp. $V_+ < V_- < |V_+|$), the solution is monotonically increasing (resp. decreasing), and is given in both cases by*

$$\Phi(r) = \frac{V_+ \left(1 - \frac{V_+ - V_-}{V_- + V_+} e^{V_+(r-r_0)/\mu} \right)}{1 + \frac{V_+ - V_-}{V_- + V_+} e^{V_+(r-r_0)/\mu}}. \quad (2.14)$$

We write the stationary wave Φ of (2.12) also as Φ_{V_-, V_+} when the boundary and the far field states are emphasized. Based on the result of Proposition 2.5, it is easy to see that, for the original stationary problem (2.1) with $n = 3$, except the trivial solution $\phi = 0$, the necessary and sufficient condition for the existence of the

nontrivial stationary solution is $v_+ \leq 0$ and $v_- - 2\mu/r_0 < |v_+|$, and then the solution is given by the formula

$$\phi_{v_-, v_+}(r) = \frac{2\mu}{r} + \Phi_{v_- - 2\mu/r_0, v_+}(r). \quad (2.15)$$

Next, let us recall the viscous shock wave for the 1-D Burgers equation on the whole space $\mathbb{R}^1$ with the far field states $V_\pm$:

$$\begin{cases} V_t + VV_r - \mu V_{rr}, & t > 0, r \in \mathbb{R}^1, \\ \lim_{r \rightarrow \pm\infty} V(t, r) = V_\pm, & t > 0. \end{cases} \quad (2.16)$$

We refer to a traveling wave solution of (2.16) with the form $V = \tilde{V}(\xi)$, $\xi = r - st$, as a viscous shock wave of (2.16), where $s \in \mathbb{R}^1$ is the shock speed. The viscous shock wave is known to exist uniquely up to the shift under the entropy condition $V_- > V_+$ and the Rankine-Hugoniot condition $-s(V_+ - V_-) + (V_+^2/2 - V_-^2/2) = 0$, that is, $s = (V_- + V_+)/2$, and has the properties $V_+ < \tilde{V}(\xi) < V_-$, $\tilde{V}_\xi(\xi) < 0$, $\xi \in \mathbb{R}^1$. In fact, the viscous shock wave with $\tilde{V}(0) = (V_- + V_+)/2$ is concretely given by

$$V = \tilde{V}(r - st) = \frac{V_+ + V_-}{2} + \frac{V_+ - V_-}{2} \tanh\left(-\frac{(V_- - V_+)}{2\mu}(r - st)\right). \quad (2.17)$$

In the case $s > 0$, that is $V_- + V_+ > 0$, note that the viscous shock wave $\tilde{V}$ is expected to be a good approximation of (2.11) because $\tilde{V}(r_0 - st)$ exponentially tends toward V_- as $t \rightarrow \infty$. We write the viscous shock wave $\tilde{V}$ in (2.17) also as $\tilde{V}_{V_-, V_+}$ when the far field states are emphasized

Now we are ready to state our second main theorem.

Theorem 2.6. *Suppose that $n = 3$. Then we have the following classification of the asymptotic states.*

(I) **Case** $v_- - 2\mu/r_0 < v_+ \leq 0$:

Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^1), \quad v_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} |v(t, r) - \phi_{v_-, v_+}(r)| = 0.$$

(II) **Case** $v_- - 2\mu/r_0 < 0 < v_+$:

Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^1), \quad v_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} \left| v(t, r) - \phi_{v_-, 0}(r) - \hat{\psi}_{0, v_+} \left(\frac{r - r_0}{t} \right) \right| = 0.$$

Here note that $\hat{\psi}_{0, v_+} = 0$ for $v_+ = 0$.

(III) **Case** $0 \leq v_- - 2\mu/r_0 < v_+$:

Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^1), \quad v_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} \left| v(t, r) - \frac{2\mu}{r} - \hat{\psi}_{v_- - \frac{2\mu}{r_0}, v_+} \left(\frac{r - r_0}{t} \right) \right| = 0.$$

(IV) **Case** $v_- - 2\mu/r_0 > v_+$, $v_- + v_+ < 2\mu/r_0$:

Assume that $v_0 - 2\mu/r - v_+ \in H^1 \cap L^1$. Then there exists a positive constant ϵ_0 such that if $\|\chi_0\|_{H^2} \leq \epsilon_0$ then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^1), \quad v_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} |v(t, r) - \phi_{v_-, v_+}(r)| = 0,$$

where the function χ_0 is defined by

$$\chi_0(r) = \int_r^\infty (v_0(y) - 2\mu/y - \phi_{v_-, v_+}(y)) dy, \quad r \geq r_0.$$

(V) **Case** $v_- - 2\mu/r_0 > v_+$, $v_- + v_+ > 2\mu/r_0$:

Assume that $v_0 - 2\mu/r - v_+ \in H^2 \cap L^1$ and $\int_{r_0}^\infty (v_0(r) - 2\mu/r - v_+) dr > 0$. Then there exist positive constants ϵ_0 and β_0 such that if $(1/d_0) + \|W_0\|_{H^{3, \beta_0}} < \epsilon_0$, then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^2), \quad v_r \in L^2(0, T; H^2), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} \left| v(t, r) - \left(2\mu/r + \tilde{V}_{v_- - 2\mu/r_0, v_+}(r - st - d(t)) \right) \right| = 0,$$

for a function $d(t) \in C^0([0, \infty))$ which has a finite limit $d_\infty = \lim_{t \rightarrow \infty} d(t)$. Here $s = (v_+ + v_- - 2\mu/r_0)/2 > 0$, and d_0 and W_0 are defined as

$$\begin{aligned} \int_{r_0}^{\infty} (v_0(r) - 2\mu/r - \tilde{V}_{v_- - 2\mu/r_0, v_+}(r - d_0)) dr &= 0, \\ W_0(r) &= \int_r^{\infty} (v_0(y) - 2\mu/y - \tilde{V}_{v_- - 2\mu/r_0, v_+}(y - d_0)) dy, \quad r \geq r_0. \end{aligned}$$

(VI) **Case** $v_- - 2\mu/r_0 > v_+$, $v_- + v_+ = 2\mu/r_0$:

Assume that $v_0 - 2\mu/r - v_+ \in H^2 \cap L^1$ and $\int_{r_0}^{\infty} (v_0(r) - 2\mu/r - v_+) dr > 0$. Then there exist positive constants ϵ_0 and β_0 such that if $(1/d_0) + \|W_0\|_{H^{3, \beta_0}} < \epsilon_0$, then the initial-boundary value problem (1.2) has a unique time-global solution v satisfying

$$v - v_+ \in C^0([0, \infty); H^2), \quad v_r \in L^2(0, T; H^2), \quad T > 0,$$

and the asymptotic behavior

$$\limsup_{t \rightarrow \infty} \sup_{r > r_0} \left| v(t, r) - \left(2\mu/r + \tilde{V}_{-v_+, v_+}(r - d(t)) \right) \right| = 0,$$

for a function $d(t) \in C^0([0, \infty))$ which has the property $d(t) \sim \log t$, $t \rightarrow \infty$.

The proofs for cases (I) and (II) are based on the arguments in Liu-Matsumura-Nishihara [5]. The proof for case (III) is based on the arguments in Nakamura [8], and the proofs for cases (IV) and (V) are based on the arguments in Liu-Nishihara [6]. The proof for case (VI), which is the most subtle case, is based on the arguments in Nishihara [9] and Liu-Yu [7]. Note that the arguments in [7] are made in a classical function space by using the maximum principle.

Because Theorem 2.3 is proved by combining the results of [1] and [4], we only show the rough sketch of the proof of Theorem 2.4.

3 Asymptotic stability of stationary wave in the case $v_+ < 0$

3.1 Reformulation of the problem

Recall the non-monotonic stationary wave which satisfies (2.1) with $v_- \leq v_+ < 0$. Integrating the equation in (2.1) with respect to r once, we have

$$\begin{cases} \mu\phi_r + \frac{\mu(n-1)}{r}\phi = \frac{1}{2}(\phi^2 - v_+^2), & r > r_0, \\ \phi(r_0) = v_-, \quad \phi(\infty) = v_+. \end{cases} \quad (3.1)$$

Letting us introduce the perturbation $w(t, r)$ from $\phi(r)$ by

$$v(t, r) = \phi(r) + w(t, r),$$

we rewrite our original problem (1.2) in terms of w as

$$\begin{cases} w_t + \frac{1}{2}(w^2 + 2\phi w)_r = \mu(w_{rr} + (n-1)\left(\frac{w}{r}\right)_r), & r > r_0, \quad t > 0, \\ w(t, r_0) = 0, & t > 0, \\ w(0, r) = w_0(r) := v_0(r) - \phi(r), & r > r_0. \end{cases} \quad (3.2)$$

Now we further define a new unknown function z by

$$z(t, r) = r^{\frac{n-1}{2}} w(t, r).$$

Then the problem (3.2) is again rewritten in terms of z as in the form

$$\begin{cases} z_t + (\phi z)_r + \left(-\frac{n-1}{2r}\phi + \frac{\mu(n^2-1)}{4r^2}\right)z - \mu z_{rr} \\ \quad = R(z) := \frac{n-1}{2r^{\frac{n+1}{2}}}z^2 - \frac{1}{2r^{\frac{n-1}{2}}}(z^2)_r, & r > r_0, \quad t > 0, \\ z(t, r_0) = 0, & t > 0, \\ z(0, r) = z_0(r) := r^{\frac{n-1}{2}}(v_0(r) - \phi(r)), & r > r_0. \end{cases} \quad (3.3)$$

The theorem corresponding to Theorem 2.4 for the reformulated problem (3.3) is written as follows.

Theorem 3.1. *Suppose that $n \geq 3$, $v_- \leq v_+ < 0$ and $z_0 \in H^1$. Then there exists a positive constant ϵ_0 such that if $\|z_0\|_{H^1} \leq \epsilon_0$, then the initial-boundary value problem (3.3) has a unique time-global solution z satisfying*

$$z \in C^0([0, \infty); H^1), \quad z_r \in L^2(0, T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t \rightarrow \infty} \sup_{r > r_0} |z(r, t)| = 0.$$

Proof. We only show a desired *a priori* estimate for the solution. First, put

$$N(T) = \sup_{0 \leq t \leq T} \|v(t)\|_{H^1},$$

and then we suppose $N(T) \leq 1$ in what follows. Multiplying the equation in (3.3) by z and integrating the resultant equality in terms of r over $[r_0, \infty)$, we get

$$\frac{1}{2} \frac{d}{dt} \int_{r_0}^{\infty} z^2 dr + \frac{1}{2} \int_{r_0}^{\infty} \left(\phi_r - \frac{n-1}{r} \phi + \frac{\mu(n^2-1)}{2r^2} \right) z^2 dr$$

$$+\mu \int_{r_0}^{\infty} |z_r|^2 dr = \int_{r_0}^{\infty} zR(z) dr. \quad (3.4)$$

Hence, due to Proposition 2.2, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{r_0}^{\infty} z^2 dr + \frac{1}{2} \int_{r_0}^{\infty} \left(\frac{n-1}{r} \nu_0 + \frac{\mu(n-1)}{r^2} \right) z^2 dr \\ & + \mu \int_{r_0}^{\infty} |z_r|^2 dr \leq \int_{r_0}^{\infty} zR(z) dr, \end{aligned}$$

which implies

$$\begin{aligned} & \|z(t)\|_{L^2}^2 + \int_0^t \left(\int_{r_0}^{\infty} \frac{z^2(\tau, r)}{r} dr + \|z_r(\tau)\|^2 \right) d\tau \\ & \leq C(\|z_0\|_{L^2}^2 + N(t) \int_0^t \left(\int_{r_0}^{\infty} \frac{z^2(\tau, r)}{r} dr + \|z_r(\tau)\|^2 \right) d\tau). \end{aligned} \quad (3.5)$$

Next we proceed to the higher order estimate. Multiplying the equation in (3.3) by $-z_{rr}$ and integrating the resultant equality in terms of r and t over $[r_0, \infty) \times [0, t]$, we get

$$\begin{aligned} & \|z_r(t)\|_{L^2}^2 + \int_0^t \|z_{rr}(\tau)\|^2 d\tau \\ & \leq C(\|z_{0,r}\|_{L^2}^2 + N(t) \int_0^t \left(\int_{r_0}^{\infty} \frac{z^2(\tau, r)}{r} dr + \|z_r(\tau)\|^2 \right) d\tau), \end{aligned} \quad (3.6)$$

where we used the equation in (3.1) and basic estimate (3.5). Combining (3.5) and (3.6) and taking $N(t)$ suitably small, we have the desired estimate

$$\|z(t)\|_{H^1}^2 + \int_0^t \left(\left\| \frac{z}{r}(\tau) \right\|_{L^2}^2 + \|z_r(\tau)\|_{H^1}^2 \right) d\tau \leq C\|z_0\|_{H^1}^2. \quad (3.7)$$

Once the *a priori* estimate (3.7) is established, we can show the existence of time-global solution and its asymptotic behavior as in the same way as the previous papers. Thus the proof of the Theorem 3.1 is completed. $\square$

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