

一般化された skew information に関連した 不確定性関係

Uncertainty relations related to generalized skew information

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量子状態 ρ (密度作用素: $\rho^* = \rho \geq 0, \text{Tr}[\rho] = 1$) と観測量 H (自己共役作用素: $H^* = H$) との間のある種の非可換性の度合いを表す情報量として次の Wigner-Yanase skew information が知られている:

$$I_\rho(H) \equiv \frac{1}{2} \text{Tr} [(i[\rho^{1/2}, H])^2]. \quad (1)$$

ここで $[X, Y] \equiv XY - YX$ である. また Dyson による一般化

$$I_{\rho, \alpha}(H) \equiv \frac{1}{2} \text{Tr} [(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])], \quad \alpha \in [0, 1] \quad (2)$$

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が Wigner-Yanase-Dyson skew information として知られている. 近年この種の skew information と不確定性関係に関する研究が盛んになされている [2, 3, 4]. 量子状態 ρ と観測量 X, Y に対する Heisenberg の不確定性関係は

$$V_\rho(X)V_\rho(Y) \geq \frac{1}{4} |\text{Tr} [\rho[X, Y]]|^2 \quad (3)$$

である. ここで分散は $V_\rho(H) \equiv \text{Tr} [\rho(H - \text{Tr}[\rho H]I)^2]$ で定義される. これよりも強い結果として Schrodinger の不確定性関係

$$V_\rho(X)V_\rho(Y) - |\text{Cov}_\rho(X, Y)|^2 \geq \frac{1}{4} |\text{Tr} [\rho[X, Y]]|^2$$

が知られている. ただし $\text{Cov}_\rho(X, Y) \equiv \text{Tr} [\rho(X - \text{Tr}[\rho X]I)(Y - \text{Tr}[\rho Y]I)]$ である. 下記の不等式 (6) の意味で不等式 (3) より強い結果として

$$I_\rho(X)I_\rho(Y) \geq \frac{1}{4} |\text{Tr} [\rho[X, Y]]|^2$$

が考えられたがこれは不成立であった. その後 Luo [5] は古典的な混合を取り除いた量子的な不確定性を表す量

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2} \quad (4)$$

を導入し次の不等式を導いた:

$$U_\rho(X)U_\rho(Y) \geq \frac{1}{4} |\text{Tr} [\rho[X, Y]]|^2. \quad (5)$$

ここで以下の関係に注意する.

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (6)$$

そこで (4) 式の一般化

$$U_{\rho, \alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho, \alpha}(H))^2} \quad (7)$$

に対して不等式 (5) に相当するものを考えるのは自然であるが (7) 式を直接用いた一般化はわかっていない. ここでは次の定義を与える.

Definition 1 量子状態 ρ と観測量 H およびパラメータ $0 \leq \alpha \leq 1$ に対して

$$I_{\rho, \alpha}(H) \equiv \frac{1}{2} \text{Tr} [(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])]$$

と

$$J_{\rho, \alpha}(H) \equiv \frac{1}{2} \text{Tr} [\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}]$$

を定義する. ただし $H_0 \equiv H - \text{Tr}[\rho H]I$ であり $\{X, Y\} \equiv XY + YX$ である. また煩雑さを避けるために後で

$$A_\alpha(H) \equiv i[\rho^\alpha, H_0], \quad B_\alpha(H) \equiv \{\rho^\alpha, H_0\}$$

という記号を用いる. このとき

$$I_\rho(H) \geq I_{\rho,\alpha}(H), \quad J_\rho(H) \leq J_{\rho,\alpha}(H), \quad (8)$$

$$U_{\rho,\alpha}(H) \equiv \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)} \quad (9)$$

であり

$$0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H) \quad (10)$$

が成り立つ.

このとき次の定理がなりたつ.

Theorem 1 量子状態 ρ と観測量 X, Y に対して

$$\tilde{U}_{\rho,\alpha}(X)\tilde{U}_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2.$$

ただし

$$\tilde{U}_{\rho,\alpha}(X) \equiv \frac{1}{2} \sqrt{\left(\text{Tr} \left[\frac{A_\alpha(X)^2 + A_{1-\alpha}(X)^2}{4} \right] + I_{\rho,\alpha}(X) \right) \left(\text{Tr} \left[\frac{B_\alpha(X)^2 + B_{1-\alpha}(X)^2}{4} \right] + J_{\rho,\alpha}(X) \right)}.$$

Remark 1 次の2点から上の定理は *trivial* とは言い切れない.

(1) $\left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2$ と $|\text{Tr}[\rho[X, Y]]|^2$ には大小関係はない.

(2) $U_{\rho,\alpha}(H) \leq U_\rho(H)$ かつ $U_{\rho,\alpha}(H) \leq \tilde{U}_{\rho,\alpha}(H)$ であるが $U_\rho(H)$ と $\tilde{U}_{\rho,\alpha}(H)$ には大小関係はない.

Proof of Theorem 1. $K = \frac{1}{2}(A_\alpha(X) + A_{1-\alpha}(X))x + \frac{1}{2}(B_\alpha(Y) + B_{1-\alpha}(Y))$ とおく. このとき $K^* = K$ である. したがって

$$\begin{aligned}
0 &\leq Tr[KK^*] \\
&= \frac{1}{4}Tr[(A_\alpha(X) + A_{1-\alpha}(X))^2]x^2 + \frac{1}{2}Tr[(A_\alpha(X) + A_{1-\alpha}(X))(B_\alpha(Y) + B_{1-\alpha}(Y))]x \\
&\quad + \frac{1}{4}Tr[(B_\alpha(Y) + B_{1-\alpha}(Y))^2] \\
&= \left(\frac{1}{4}Tr[A_\alpha(X)^2 + A_{1-\alpha}(X)^2] + I_{\rho,\alpha}(X) \right) x^2 \\
&\quad + \frac{1}{2}Tr[(A_\alpha(X) + A_{1-\alpha}(X))(B_\alpha(Y) + B_{1-\alpha}(Y))]x \\
&\quad + \left(\frac{1}{4}Tr[B_\alpha(Y)^2 + B_{1-\alpha}(Y)^2] + J_{\rho,\alpha}(Y) \right).
\end{aligned}$$

したがって

$$\begin{aligned}
&\frac{1}{4} (Tr[(A_\alpha(X) + A_{1-\alpha}(X))(B_\alpha(Y) + B_{1-\alpha}(Y))])^2 \\
&\leq 4 \left(\frac{1}{4}Tr[A_\alpha(X)^2 + A_{1-\alpha}(X)^2] + I_{\rho,\alpha}(X) \right) \left(\frac{1}{4}Tr[B_\alpha(Y)^2 + B_{1-\alpha}(Y)^2] + J_{\rho,\alpha}(Y) \right).
\end{aligned} \tag{11}$$

ここで

$$\begin{aligned}
&Tr[(A_\alpha(X) + A_{1-\alpha}(X))(B_\alpha(Y) + B_{1-\alpha}(Y))] \\
&= Tr[(i[\rho^\alpha, X_0] + i[\rho^{1-\alpha}, X_0])(\{\rho^\alpha, Y_0\} + \{\rho^{1-\alpha}, Y_0\})] \\
&= Tr[i(\rho^\alpha X_0 - X_0 \rho^\alpha + \rho^{1-\alpha} X_0 - X_0 \rho^{1-\alpha})(\rho^\alpha Y_0 + Y_0 \rho^\alpha + \rho^{1-\alpha} Y_0 + Y_0 \rho^{1-\alpha})] \\
&= Tr[i\{(\rho^\alpha + \rho^{1-\alpha})X_0 - X_0(\rho^\alpha + \rho^{1-\alpha})\}\{(\rho^\alpha + \rho^{1-\alpha})Y_0 + Y_0(\rho^\alpha + \rho^{1-\alpha})\}] \\
&= iTr[(\rho^\alpha + \rho^{1-\alpha})X_0(\rho^\alpha + \rho^{1-\alpha})Y_0 + (\rho^\alpha + \rho^{1-\alpha})^2 X_0 Y_0 \\
&\quad - Y_0 X_0(\rho^\alpha + \rho^{1-\alpha})^2 - X_0(\rho^\alpha + \rho^{1-\alpha})Y_0(\rho^\alpha + \rho^{1-\alpha})] \\
&= iTr[(\rho^\alpha + \rho^{1-\alpha})^2 X_0 Y_0 - Y_0 X_0(\rho^\alpha + \rho^{1-\alpha})^2] \\
&= Tr[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X_0, Y_0])] \\
&= Tr[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X, Y])].
\end{aligned}$$

したがって (11) は次と同値になる.

$$\begin{aligned}
&\frac{1}{4} (Tr[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X, Y])])^2 \\
&\leq 4 \left(\frac{1}{4}Tr[A_\alpha(X)^2 + A_{1-\alpha}(X)^2] + I_{\rho,\alpha}(X) \right) \left(\frac{1}{4}Tr[B_\alpha(Y)^2 + B_{1-\alpha}(Y)^2] + J_{\rho,\alpha}(Y) \right).
\end{aligned} \tag{12}$$

同様にして

$$\begin{aligned}
&\frac{1}{4} |Tr[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X, Y])]|^2 \\
&\leq 4 \left(\frac{1}{4}Tr[A_\alpha(Y)^2 + A_{1-\alpha}(Y)^2] + I_{\rho,\alpha}(Y) \right) \left(\frac{1}{4}Tr[B_\alpha(X)^2 + B_{1-\alpha}(X)^2] + J_{\rho,\alpha}(X) \right).
\end{aligned} \tag{13}$$

よって (12) と (13) をかけて両辺の平方根をとると

$$\begin{aligned} & \left\{ \frac{1}{4} (Tr[(\rho^\alpha + \rho^{1-\alpha})^2(i[X, Y])])^2 \right\}^2 \\ & \leq 4 \left(\frac{1}{4} Tr[A_\alpha(X)^2 + A_{1-\alpha}(X)^2] + I_{\rho, \alpha}(X) \right) \left(\frac{1}{4} Tr[B_\alpha(Y)^2 + B_{1-\alpha}(Y)^2] + J_{\rho, \alpha}(Y) \right) \\ & \quad 4 \left(\frac{1}{4} Tr[A_\alpha(Y)^2 + A_{1-\alpha}(Y)^2] + I_{\rho, \alpha}(Y) \right) \left(\frac{1}{4} Tr[B_\alpha(X)^2 + B_{1-\alpha}(X)^2] + J_{\rho, \alpha}(X) \right). \end{aligned}$$

したがって

$$\begin{aligned} & \frac{1}{4} (Tr[(\rho^\alpha + \rho^{1-\alpha})^2(i[X, Y])])^2 \\ & \leq 2 \sqrt{\left(\frac{1}{4} Tr[A_\alpha(X)^2 + A_{1-\alpha}(X)^2] + I_{\rho, \alpha}(X) \right) \left(\frac{1}{4} Tr[B_\alpha(Y)^2 + B_{1-\alpha}(Y)^2] + J_{\rho, \alpha}(Y) \right)} \\ & \quad 2 \sqrt{\left(\frac{1}{4} Tr[A_\alpha(Y)^2 + A_{1-\alpha}(Y)^2] + I_{\rho, \alpha}(Y) \right) \left(\frac{1}{4} Tr[B_\alpha(X)^2 + B_{1-\alpha}(X)^2] + J_{\rho, \alpha}(X) \right)}. \end{aligned}$$

ゆえに

$$\frac{1}{4} \left(Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 (i[X, Y]) \right] \right)^2 \leq \tilde{U}_\alpha(\rho, X) \tilde{U}_\alpha(\rho, Y).$$

最後に

$$Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] = -Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right]$$

より

$$Re \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] = 0$$

となるので

$$Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] = i \text{Im} Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right].$$

したがって

$$\begin{aligned} & \left(Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 (i[X, Y]) \right] \right)^2 \\ & = - \left(Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right)^2 \end{aligned}$$

$$\begin{aligned}
&= - \left(i \operatorname{Im} \operatorname{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right)^2 \\
&= \left(\operatorname{Im} \operatorname{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right)^2 \\
&= \left| \operatorname{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2.
\end{aligned}$$

以上より結論を得る.

q.e.d.

もう1つの定理が成り立つ.

Definition 2 量子状態 ρ と観測量 H およびパラメータ $0 \leq \alpha \leq 1$ に対して

$$W_{\rho, \alpha}(H) \equiv \frac{1}{4} \sqrt{\operatorname{Tr} [(i[\rho^\alpha, H_0])^2] \operatorname{Tr} [(i[\rho^{1-\alpha}, H_0])^2] \operatorname{Tr} [\{\rho^\alpha, H_0\}^2] \operatorname{Tr} [\{\rho^{1-\alpha}, H_0\}^2]}$$

と定義する.

このとき次の定理が成り立つ.

Theorem 2 量子状態 ρ と観測量 X, Y に対して

$$\sqrt{W_{\rho, \alpha}(X) W_{\rho, \alpha}(Y)} \geq \frac{1}{4} |\operatorname{Tr} [\rho^{2\alpha} [X, Y]] \operatorname{Tr} [\rho^{2(1-\alpha)} [X, Y]]|.$$

Remark 2 次の (1), (2) から定理 1 の不等式と定理 2 の不等式との強弱関係はない.

(1) $4W_{\rho, \alpha}(X)$ と

$$\left(\operatorname{Tr} \left[\frac{A_\alpha(X)^2 + A_{1-\alpha}(X)^2}{4} \right] + I_{\rho, \alpha}(X) \right) \left(\operatorname{Tr} \left[\frac{B_\alpha(X)^2 + B_{1-\alpha}(X)^2}{4} \right] + J_{\rho, \alpha}(X) \right)$$

との大小関係はない. 即ち

$$\sqrt{\operatorname{Tr} [(i[\rho^\alpha, X_0])^2] \operatorname{Tr} [(i[\rho^{1-\alpha}, X_0])^2]} \text{ と}$$

$$\operatorname{Tr} \left[\frac{(i[\rho^\alpha, X_0])^2 + (i[\rho^{1-\alpha}, X_0])^2}{4} \right] + \frac{1}{2} \operatorname{Tr} [(i[\rho^\alpha, X_0])(i[\rho^{1-\alpha}, X_0])]$$

および $\sqrt{\operatorname{Tr} [\{\rho^\alpha, X_0\}^2] \operatorname{Tr} [\{\rho^{1-\alpha}, X_0\}^2]}$ と

$$\operatorname{Tr} \left[\frac{\{\rho^\alpha, X_0\}^2 + \{\rho^{1-\alpha}, X_0\}^2}{4} \right] + \frac{1}{2} \operatorname{Tr} [\{\rho^\alpha, X_0\} \{\rho^{1-\alpha}, X_0\}]$$

との大小関係はない.

- (2) $|Tr [\rho^{2\alpha}[X, Y]] Tr [\rho^{2(1-\alpha)}[X, Y]]|$ と $|Tr [(\frac{\rho^\alpha + \rho^{1-\alpha}}{2})^2[X, Y]]|^2$
 との大小関係はない. 即ち
 $|Tr [\rho^{2\alpha}[X, Y]]|$ と $|Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right]|$
 および $|Tr [\rho^{2(1-\alpha)}[X, Y]]|$ と $|Tr \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right]|$
 との大小関係はない.

(3) $\alpha = 1/2$ のとき定理 1 と定理 2 はどちらも *Luo* の結果を得る.

Proof of Theorem 2. $K = i[\rho^\alpha, X_0]x + \{\rho^\alpha, Y_0\}$ とおく. このとき $K^* = K$ である. したがって

$$\begin{aligned} 0 &\leq Tr [KK^*] \\ &= Tr [(i[\rho^\alpha, X_0]x + \{\rho^\alpha, Y_0\})^2] \\ &= Tr [(i[\rho^\alpha, X_0])^2] x^2 + 2iTr [[\rho^\alpha, X_0]\{\rho^\alpha, Y_0\}] x + Tr [\{\rho^\alpha, Y_0\}^2] \\ &= Tr [(i[\rho^\alpha, X_0])^2] x^2 + 2iiImTr [\rho^{2\alpha}[X, Y]] x + Tr [\{\rho^\alpha, Y_0\}^2]. \end{aligned}$$

したがって

$$|Tr [\rho^{2\alpha}[X, Y]]|^2 = (ImTr [\rho^{2\alpha}[X, Y]])^2 \leq Tr [(i[\rho^\alpha, X_0])^2] Tr [\{\rho^\alpha, Y_0\}^2].$$

X と Y を入れ替えて

$$|Tr [\rho^{2\alpha}[X, Y]]|^2 \leq Tr [(i[\rho^\alpha, Y_0])^2] Tr [\{\rho^\alpha, X_0\}^2].$$

同様にして

$$|Tr [\rho^{2(1-\alpha)}[X, Y]]|^2 \leq Tr [(i[\rho^{1-\alpha}, X_0])^2] Tr [\{\rho^{1-\alpha}, Y_0\}^2].$$

X と Y を入れ替えて

$$|Tr [\rho^{2(1-\alpha)}[X, Y]]|^2 \leq Tr [(i[\rho^{1-\alpha}, Y_0])^2] Tr [\{\rho^{1-\alpha}, X_0\}^2].$$

ここで

$$\begin{aligned} S_{\rho, \alpha}(X) &\equiv \frac{1}{2}Tr [(i[\rho^\alpha, X_0])^2], \quad T_{\rho, \alpha}(X) \equiv \frac{1}{2}Tr [\{\rho^\alpha, X_0\}^2] \\ S_{\rho, 1-\alpha}(X) &\equiv \frac{1}{2}Tr [(i[\rho^{1-\alpha}, X_0])^2], \quad T_{\rho, 1-\alpha}(X) \equiv \frac{1}{2}Tr [\{\rho^{1-\alpha}, X_0\}^2] \\ S_{\rho, \alpha}(Y) &\equiv \frac{1}{2}Tr [(i[\rho^\alpha, Y_0])^2], \quad T_{\rho, \alpha}(Y) \equiv \frac{1}{2}Tr [\{\rho^\alpha, Y_0\}^2] \end{aligned}$$

$$S_{\rho,1-\alpha}(Y) \equiv \frac{1}{2} \text{Tr} [(i[\rho^{1-\alpha}, Y_0])^2], \quad T_{\rho,1-\alpha}(Y) \equiv \frac{1}{2} \text{Tr} [\{\rho^{1-\alpha}, Y_0\}^2]$$

とおくと次を得る.

$$|\text{Tr} [\rho^{2\alpha}[X, Y]]|^2 \leq 4 \sqrt{S_{\rho,\alpha}(X) T_{\rho,\alpha}(X) S_{\rho,\alpha}(Y) T_{\rho,\alpha}(Y)}.$$

$$|\text{Tr} [\rho^{2(1-\alpha)}[X, Y]]|^2 \leq 4 \sqrt{S_{\rho,1-\alpha}(X) T_{\rho,1-\alpha}(X) S_{\rho,1-\alpha}(Y) T_{\rho,1-\alpha}(Y)}.$$

ここで

$$W_{\rho,\alpha}(X) \equiv \sqrt{S_{\rho,\alpha}(X) S_{\rho,1-\alpha}(X) T_{\rho,\alpha}(X) T_{\rho,1-\alpha}(X)},$$

$$W_{\rho,\alpha}(Y) \equiv \sqrt{S_{\rho,\alpha}(Y) S_{\rho,1-\alpha}(Y) T_{\rho,\alpha}(Y) T_{\rho,1-\alpha}(Y)}$$

とおくと目標の不等式

$$\sqrt{W_{\rho,\alpha}(X) W_{\rho,\alpha}(Y)} \geq \frac{1}{4} |\text{Tr} [\rho^{2\alpha}[X, Y]] \text{Tr} [\rho^{2(1-\alpha)}[X, Y]]|.$$

を得る.

q.e.d.

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