

Hamiltonian formulation of two-dimensional motion of an ideal fluid and a finite-mode hydrodynamic system

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1 Introduction

The fact that the total kinetic energy is conserved in the motion of an ideal fluid is a manifestation of the fundamental property of mechanics. However, restricting to two-dimensional motions, it is well-known that there exist an infinite number of invariants for the ideal fluid (see §2). Computer simulations of the fluid motions are carried out inevitably by means of *finite*-mode approximation to the exact infinite system. In those studies of two-dimensional motion performed so far, the above property of multiple invariants has not been considered seriously.

Recently, Zeilin [1] proposed a modified dynamical system, based on the $SU(N)$ algebras studied in the paper by Fairlie & Zachos [2]. This work has established connection between algebras of diffeomorphisms of the domain occupied by the flow and $SU(N)$ -algebras in the limit $N \rightarrow \infty$. The Zeilin's hydrodynamic system of the $O(N^2)$ -mode truncation in Fourier space can be shown to have $O(N)$ invariants. Accordingly, as the number of modes increases, the number of invariants increases arbitrarily.

2 Formulation from the hydrodynamics

2.1 Vorticity equation

Two-dimensional motion of an incompressible fluid in (x, y) plane is described by a streamfunction $\psi(x, y, t)$, giving the velocity $\mathbf{v} = (u, v)$ as

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \quad (1)$$

which satisfy the solenoidal relation:

$$\partial_x u + \partial_y v = 0 . \quad (2)$$

The vorticity

$$\omega = \partial_x v - \partial_y u = -(\partial_x^2 + \partial_y^2)\psi \quad (3)$$

is governed by the following evolution equation derived from the Euler's equation of motion for the velocity field:

$$\frac{D}{Dt} \omega = \partial_t \omega + u \partial_x \omega + v \partial_y \omega = 0 , \quad (4)$$

which may be called again Euler equation. The above definition of u and v yields

$$\partial_t \omega = \frac{\partial(\psi, \omega)}{\partial(x, y)} = \{\psi, \omega\} , \quad (5)$$

where the right hand side is the Poisson bracket and the middle is the Jacobian. Since D/Dt stands for the Lagrange derivative, *i.e.* material derivative, the equation (4) represents that the vorticity ω is invariant with respect to each fluid particle in motion. The property (4) leads immediately to

$$\frac{D}{Dt} \omega^n = 0 \quad (6)$$

for arbitrary integer n .

2.2 Motion on the torus T^2

Consider a fluid motion on the torus $T^2 = \{x, y; \text{mod } 2\pi\}$ with periodic boundary condition. It is not difficult to show that the equations (6) and (2) yield

$$\Omega_n = \int_D \omega^n(x, y, t) dx dy = \text{const}, \quad (7)$$

where $D: 0 \leq x, y \leq 2\pi$. This means that there exist an infinite number of invariants for a system of infinite number of degree-of-freedom. The total kinetic energy is given by

$$K = \frac{1}{2} \int_D (u^2 + v^2) dx dy = \frac{1}{2} \int_D \psi \omega dx dy, \quad (8)$$

which is an additional invariant.

2.3 Fourier representation

It is convenient to use the Fourier representation for the analysis on the torus T^2 with the Fourier bases,

$$e_{\mathbf{k}} = \exp(i\mathbf{k} \cdot \mathbf{x}), \quad \text{where } \mathbf{x} = (x, y), \mathbf{k} = (k_x, k_y),$$

where k_x and k_y are integers. The streamfunction ψ and vorticity ω are expanded as

$$\psi = \sum_{\mathbf{k}} \psi_{\mathbf{k}}(t) e_{\mathbf{k}}, \quad \omega = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(t) e_{\mathbf{k}}.$$

Then the equations (3) and (5) lead to

$$\omega_{\mathbf{k}} = k^2 \psi_{\mathbf{k}}, \quad (9)$$

$$\dot{\omega}_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \frac{1}{q^2} \mathbf{p} \times \mathbf{q} \omega_{\mathbf{p}} \omega_{\mathbf{q}} = \frac{1}{q^2} \mathbf{p} \times \mathbf{q} \omega_{\mathbf{p}} \omega_{\mathbf{q}} \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}). \quad (10)$$

where the two expressions on the right hand side are understood to be identical. This is the evolution equation of the vorticity $\omega_{\mathbf{k}}$ in Fourier space, here called again Euler equation. This interesting form of the equation will be reconsidered below.

The integral (7) gives

$$I_n = \frac{\Omega_n}{(2\pi)^2} = \sum_{\mathbf{k}_1} \cdots \sum_{\mathbf{k}_n} \omega_{\mathbf{k}_1} \omega_{\mathbf{k}_2} \cdots \omega_{\mathbf{k}_n}, \quad (\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_n = 0). \quad (11)$$

In particular for $n=2$, we have the enstrophy integral,

$$\frac{\Omega_2}{(2\pi)^2} = \sum_{\mathbf{k}} |\omega_{\mathbf{k}}|^2 = \text{const}. \quad (12)$$

The kinetic energy (8) is reduced to

$$H = \frac{K}{(2\pi)^2} = \frac{1}{2} \sum_{\mathbf{p}+\mathbf{q}=0} a^{\mathbf{p}\mathbf{q}} \omega_{\mathbf{p}} \omega_{\mathbf{q}}, \quad (13)$$

where

$$a^{\mathbf{p}\mathbf{q}} = \frac{1}{p^2} \delta(\mathbf{p} + \mathbf{q}). \quad (14)$$

3 Hamiltonian formulation

3.1 Algebraic structure

In order to derive the Euler equation (10) in Fourier space from a Hamiltonian formalism, let us first define a commutator (Kirillov bracket) by

$$\{f, g\}_K \equiv c_{pq}^k \omega_k \frac{\partial f}{\partial \omega_p} \frac{\partial g}{\partial \omega_q} \quad (15)$$

(the summation convention is understood for repeated indices) for two arbitrary functions of ω_k , where the structure constant c_{pq}^k has the two properties:

$$1) \quad c_{pq}^k = -c_{qp}^k, \quad (16)$$

$$2) \quad c_{pk}^s c_{sr}^q + c_{kr}^s c_{sp}^q + c_{rp}^s c_{sk}^q = 0. \quad (17)$$

The Kirillov bracket provided with these properties is characterized by (i) bilinearity with respect to f and g , (ii) antisymmetric relation: $\{f, g\} = -\{g, f\}$, and (iii) Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad (18)$$

for any three functions f, g and h of ω_k . Hence this forms a Lie algebra. For the elements like $f = \omega_k$, the bracket (15) takes the form

$$\{\omega_p, \omega_q\}_K = c_{pq}^k \omega_k. \quad (19)$$

By this relation and the expression (13) for H , the Euler equation may be written in the following Hamiltonian form,

$$\dot{\omega}_k = \{H, \omega_k\}_K = a^{pr} c_{rk}^q \omega_p \omega_q. \quad (20)$$

Let us introduce the structure constant defined by

$$c_{\mathbf{p}\mathbf{q}}^{\mathbf{k}} = (\mathbf{p} \times \mathbf{q}) \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}), \quad (21)$$

where the boldface indices $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{k}$ stand for 2-vectors with two integer components, *e.g.* $\mathbf{p} = (p_1, p_2)$. Using the definition (14), we recover the Euler equation (10):

$$\dot{\omega}_{\mathbf{k}} = \frac{1}{p^2} \delta(\mathbf{p} + \mathbf{r}) \mathbf{r} \times \mathbf{k} \delta(\mathbf{q} - \mathbf{r} - \mathbf{k}) \omega_{\mathbf{p}} \omega_{\mathbf{q}} = \frac{1}{q^2} \mathbf{p} \times \mathbf{q} \omega_{\mathbf{p}} \omega_{\mathbf{q}} \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}). \quad (22)$$

3.2 Matrix formulation

The dynamical system has a matrix representation with some set of basis matrices L_i , satisfying the following commutation relation,

$$[L_p, L_q] = (\mathbf{p} \times \mathbf{q}) L_{\mathbf{p}+\mathbf{q}} . \quad (23)$$

Then the Euler equation may be rewritten in the matrix form:

$$\dot{W} = [W, \Psi] \quad (24)$$

where

$$W = \omega_i L_i , \quad \Psi = a^{lm} \omega_l L_{-m} . \quad (25)$$

In fact, substituting (25) into (24), one obtains

$$\dot{\omega}_i L_i = a^{lm} \omega_k \omega_l [L_k, L_{-m}] = \frac{1}{l^2} \mathbf{k} \times \mathbf{l} \omega_k \omega_l \delta(\mathbf{i} - \mathbf{k} - \mathbf{l}) L_i . \quad (26)$$

This is equivalent to (10). From the matrix equation (24), it is readily shown that $\text{Trace}(W^n)$ is conserved for any integer n (Casimir functions) :

$$I_n = \text{Tr}(W^n) = \sum_{\mathbf{k}_1} \cdots \sum_{\mathbf{k}_n} \omega_{\mathbf{k}_1} \omega_{\mathbf{k}_2} \cdots \omega_{\mathbf{k}_n} , \quad (\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_n = 0) . \quad (27)$$

3.3 Finit-mode analogue

An attempt to construct a finite-mode system closely connected with (10) has been made by Zeitlin [1]. This is based on the fact that there exists a special basis for $\text{SU}(N)$ -algebras [2] in which the commutator takes the form,

$$[L_p, L_q] = -2i \sin \frac{2\pi}{N} (\mathbf{p} \times \mathbf{q}) L_{\mathbf{p}+\mathbf{q} \bmod N} . \quad (28)$$

Here L_p is a set of special $N \times N$ matrices defined by

$$L_p = \alpha^{p_1 p_2 / 2} G^{p_1} H^{p_2} ; \quad L_{-p} = L_p^* , \quad (29)$$

where the superscript $*$ denotes taking the complex conjugate. For odd N , α is given as $e^{i4\pi/N}$ which is a primitive N th root of unity. The 2-vector $\mathbf{p}$ is (p_1, p_2) with p_1 and p_2 being integers. A basis for the $\text{SU}(N)$ algebras is built from the following two unitary unimodular matrices:

$$G = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & \alpha^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha^{N-1} \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad (30)$$

$$G^N = H^N = 1, \quad H G = \alpha G H.$$

The formula of matrix multiplication defined by

$$L_p L_q = \alpha^{\frac{1}{2} p \times q} L_{p+q \bmod N}$$

leads to the commutation relation (28). Renormalizing the generator L_p and taking the limit $N \rightarrow \infty$, the commutator (28) reduces to the relation (23).

The matrix $W = \omega_i L_i$ is a hermitean traceless matrix, hence there are $N - 1$ functionally independent invariants $\text{Tr } W^n$ (Casimir invariants) for $n = 2, \dots, N$:

$$I_n^{(N)} = \text{Tr } (W^n) = \sum_{\mathbf{k}_1 + \dots + \mathbf{k}_n = 0 \bmod N} \omega_{\mathbf{k}_1} \cdots \omega_{\mathbf{k}_n} \text{Tr } (L_{\mathbf{k}_1} \cdots L_{\mathbf{k}_n}) \quad (31)$$

3.4 Examples

Let us illustrate the above results by two lowest-mode systems.

(A) $N = 3$ system

Minimal system is the $\text{su}(3)$ -system in which $\alpha = e^{i4\pi/3}$: (i) take eight points on the plane with coordinates k_1, k_2 taking the values $(-1, 0, +1)$; (ii) assign to each point except the origin $(0, 0)$ the complex quantity $\omega_{\mathbf{k}}$; (iii) identify $\omega_{-\mathbf{k}} = \omega_{\mathbf{k}}^*$. As a result, we have three integrals of motion:

$$H = \frac{1}{2} \sum_{\mathbf{k} \neq 0} \frac{1}{k^2} |\omega_{\mathbf{k}}|^2 \quad (\text{kinetic energy}),$$

$$I_2^{(3)} = \frac{1}{2} \sum_{\mathbf{k} \neq 0} |\omega_{\mathbf{k}}|^2,$$

$$I_3^{(3)} = \sum_{\mathbf{p}, \mathbf{q} \neq 0} \cos \frac{2\pi}{3} (\mathbf{p} \times \mathbf{q}) \omega_{\mathbf{p}} \omega_{\mathbf{q}} \omega_{-\mathbf{p}-\mathbf{q} \bmod 3}.$$

(B) $N = 5$ system

Difference from the $N = 3$ system is to take 24 points on the plane with coordinates k_1, k_2 taking the values $(-2, -1, 0, +1, +2)$, and α is $e^{i4\pi/5}$ instead of $e^{i4\pi/3}$. There exist five invariants: energy integral H and $I_n^{(5)}$ ($n = 2, \dots, 5$), where

$$I_n^{(5)} = \sum_{\mathbf{k}_1 + \dots + \mathbf{k}_n = 0 \pmod{5}} \omega_{\mathbf{k}_1} \cdots \omega_{\mathbf{k}_n} \text{Tr}(L_{\mathbf{k}_1} \cdots L_{\mathbf{k}_n}).$$

For example, $I_3^{(5)}$ has the same form as (32) except for 3 being replaced by 5.

A numerical test has been performed, in which only three modes of $\mathbf{k} = (0, 1)$, $(1, 2)$, $(2, 2)$ and their complex conjugate counterparts (hence 6 modes out of 24 modes) are given nonzero initial values. A double-precision calculation has shown that the relative errors of the values of the five invariant functions with respect to the initial values are

$$1.3 \times 10^{-15} \ (H), \quad 1.2 \times 10^{-15} \ (I_2^{(5)}), \quad 17.3 \times 10^{-15} \ (I_3^{(5)}), \\ 0.4 \times 10^{-15} \ (I_4^{(5)}), \quad 4.4 \times 10^{-15} \ (I_5^{(5)}).$$

Figures 1 and 2 illustrate how the energy H and the fifth invariant $I_5^{(5)}$ stay at constant levels. Figure 3 shows the streamlines at the initial ($t = 0$) and final ($t = 10$) time.

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References

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- [2] D.B. Fairlie and C.K. Zachos (1989) *Infinite-dimensional algebras, sine brackets, and $SU(\infty)$* , Phys. Lett. B, **224**, 101-107.

ENERGY

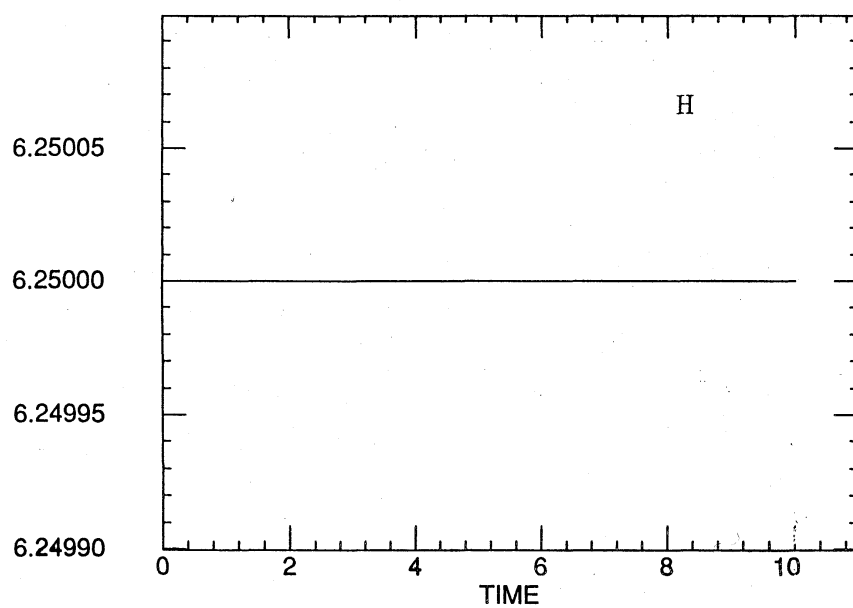


Figure 1

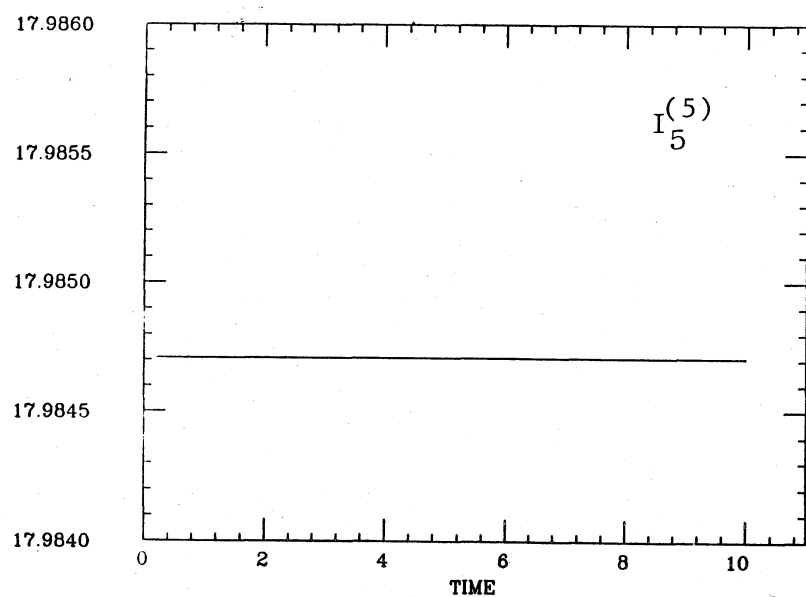


Figure 2

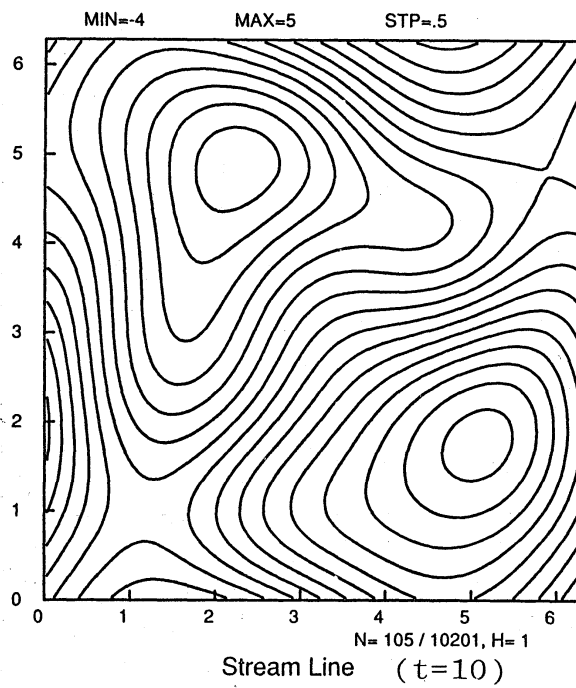
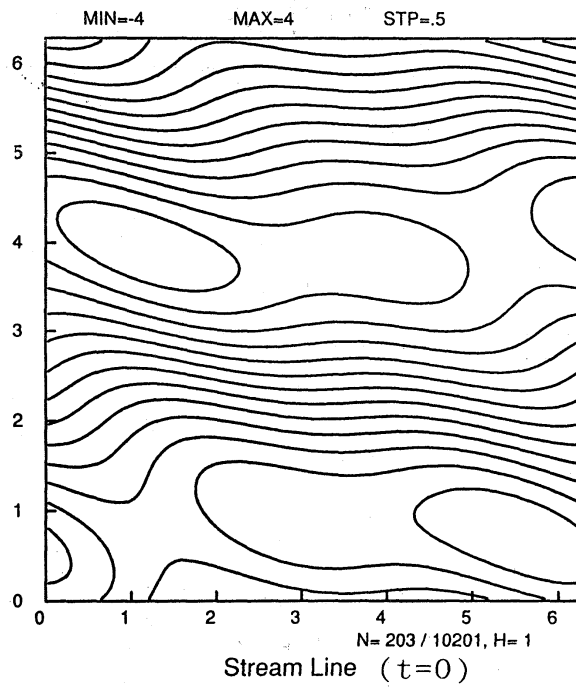


Figure 3