

AUTOMORPHISMS ON K3 SURFACES

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This is an expository note on our recent works with K. Oguiso. In the present note, we shall often use the following notations and assumptions :

(*) X is a projective K3 surface, σ an automorphism on X of order m ($m \geq 2$), $\zeta_m := \exp(2\pi\sqrt{-1}/m)$, ω a non-zero holomorphic 2-form on X and $T_X = (\text{Pic}X)^\perp \subseteq H^2(X, \mathbf{Z})$ the transcendental lattice of X [BPV].

Theorem 1 [Ni 1]. *With the notations and assumptions in (*), suppose further that σ^* acts trivially on the 1-dimensional space $H^0(X, \mathcal{O}_X(K_X))$ of holomorphic 2-forms, i.e., $\sigma^*\omega = \omega$. Then $m \leq 8$.*

In view of Theorem 1, we consider the following hypothesis:

(**) With the notations and assumptions in (*), assume further that $\sigma^*\omega = \zeta_m\omega$.

Under the hypothesis (**), the Euler number $\varphi(m)|rkT_X$, and hence one has [Ni 1]:

$m \leq 66$ and $p \leq 19$ for every prime factor p of m .

The following result can be obtained by the Hodge index theorem and by considering the diagonalization of σ^* at its fixed-points.

Lemma 2. *Assume that the pair (X, σ) satisfies the hypothesis (**).*

(1) *The set $X^{<\sigma>} = \{x \in X | \sigma^i(x) = x \text{ for some } \sigma^i \neq \text{id}\}$ of points with non-trivial stabilizer, is a disjoint union of smooth curves and isolated points.*

(2) *If $X^{<\sigma>}$ contains a curve C of genus ≥ 2 , then X^σ is a disjoint union of C , smooth rational curves and isolated points.*

Theorem 3 (see [Z3, Theorems 3 and 3']). *Let X be a projective K3 surface with an involution σ such that $\sigma^*\omega = -\omega$.*

(1) *The fixed locus X^σ is a disjoint union of r smooth curves for some $r \leq 10$.*

If $r = 10$ then X^σ is a union of 9 smooth rational curves and a smooth curve C of genus 0, 1 or 2.

(2) *All pairs (X, σ) , modulo isomorphisms, in (1) with $r = 10$ and $g(C) = 1$ (resp. $g(C) = 2$) are parametrized by a subset of $\mathbf{P}^1$ (resp. of $\mathbf{P}^3$).*

(3) *There is a unique (modulo isomorphisms) pair (X, σ) such that X^σ is a union of 10 rational curves. Such X has Picard number 20 and discriminant 4 and hence has infinite automorphism group $\text{Aut}X$.*

Remark 4. (1) For X in Theorem 3(3) " $\#\text{Aut}X = \infty$ " was first proved by T. Shioda–H. Inose [SI]; such X is called one of the two most algebraic K3 surfaces by E. B. Vinberg [V] who also determined $\text{Aut}X$ (see Remark 15, and Example 6, also for the construction of X).

(2) Nikulin claimed that some results in Theorem 3 has been proved in [Ni 2], though

the author has not found any clear statements similar to Theorem 3 above and will try to read [Ni 2] again later.

Theorem 5 [OZ1, Theorems 3 and 4]. *Let X_m be a projective K3 surface with an automorphism σ of order m where $m = 3$ (resp. 2) such that*

- (i) $\sigma^*\omega = \zeta_m\omega$,
- (ii) *there is no any σ -fixed curve (point-wise) of genus ≥ 2 , and*
- (iii) *there are at least 6 (resp. 10) σ -fixed rational curves.*

Then such a pair is unique upto isomorphisms, and isomorphic to Shioda-Inose's pair $(S_m, < g_m >)$ to be defined below.

Example 6 (see [OZ1, Examples 1 and 2] for details). Let $\zeta := \exp(2\pi\sqrt{-1}/3)$ and let $E_\zeta := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\zeta)$ be the elliptic curve of period ζ . Let $S_3 \rightarrow \bar{S}_3 := E_\zeta^2 / < \text{diag}(\zeta, \zeta^2) >$ be the minimal resolution of the quotient surface $\bar{S}_3$ [SI, Lemma 5.1].

Then S_3 is the unique projective K3 surface of Picard number 20 and discriminant 3. Let g_3 be the automorphism of S_3 induced by the action $\text{diag}(\zeta, 1)$ on E_ζ^2 . Then this Shioda-Inose pair (S_3, g_3) satisfies all conditions in Theorem 5 with $m = 3$.

Let $E_{\sqrt{-1}} := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\sqrt{-1})$ be the elliptic curve of period $\sqrt{-1}$. Let $S_2 \rightarrow \bar{S}_2 := E_{\sqrt{-1}}^2 / < \text{diag}(-\sqrt{-1}, \sqrt{-1}) >$ be the minimal resolution of the quotient surface $\bar{S}_2$ [SI, Lemma 5.2].

Then S_2 is the unique projective K3 surface of Picard number 20 and discriminant 4. Let g_2 be the automorphism of S_2 induced by the action $\text{diag}(-1, 1)$ on $E_{\sqrt{-1}}^2$. Then this Shioda-Inose pair (S_2, g_2) satisfies all conditions in Theorem 5 with $m = 2$.

Corollary 7 [OZ1, Theorems 1 and 2]. *There is only one isomorphism class of rational*

log Enriques surface of Type D_{19} , and only one of Type A_{19} (see Definitions below).

Let Z be a rational normal projective surface with at worst isolated quotient singular points. Z is a (rational) *log Enriques surface* if a positive multiple mK_Z of the canonical Weil divisor K_Z is linearly equivalent to zero.

$m := \min\{n \in \mathbf{Z}_{>0} | nK_Z \sim 0\}$ is called the *index* of Z . Let

$$\pi : Y := \text{Spec}_{\mathcal{O}_Z} \bigoplus_{i=0}^{m-1} \mathcal{O}_Z(-iK_Z) \rightarrow Z$$

be the *canonical* Galois $\mathbf{Z}/m\mathbf{Z}$ -covering. By the definition, we have:

Lemma 8. (1) π is unramified over the smooth part $Z - \text{Sing}Z$.

(2) Y is a projective K3 surface with at worst Du Val (= rational double) singular points. Let $g : X \rightarrow Y$ be a minimal resolution.

(3) Let σ be an order- m automorphism on X (or on Y) coming from the map π so that $\text{Gal}(Y/Z) = \langle \sigma \rangle$. Then $\sigma^*\omega = \zeta_m\omega$, after replacing σ by a new generator of $\text{Gal}(Y/Z)$.

By Lemma 8(3), we can apply Lemma 2 (here $m \geq 2$ because $Z = Y/\sigma$ is rational).

Remark 9. The following two things are essentially equivalent:

(A) A pair (X, σ) , where X is a projective K3 surface and σ an order m ($m \geq 2$) automorphism on X such that $\sigma^*\omega = \zeta_m\omega$ and that $X^{\langle \sigma \rangle}$ is non-empty but consists of only rational curves and isolated points. By Lemma 2, $X^{\langle \sigma \rangle}$ is now a disjoint union of smooth rational curves and isolated points.

(B) A (rational) log Enriques surface Z of index m .

In fact, for (B) $\Rightarrow$ (A), we define X, σ as in Lemma 8.

For (A) $\Rightarrow$ (B), we let $X \rightarrow Y$ be a contraction of a σ -stable divisor D containing all curves in $X^{<\sigma>}$, into Du Val singular points. Now Define $Z := Y/\sigma$.

Question 10. Let Z be a rational log Enriques surface of index m with $\pi : Y \rightarrow Z$ as its canonical covering.

We know that Y is a projective K3 surface with at worst singular points of Dynkin types A_r ($r \geq 1$), D_s ($s \geq 4$) and E_t ($t = 6, 7, 8$).

What is the possible combination of Dynkin types of singular points on Y ?

Definition 11. A log Enriques surface Z is of *Type* $A_r + D_s + E_t + \dots$ if the canonical covering Y of Z satisfies $\text{Sing } Y = A_r + D_s + E_t + \dots$.

Remark 12. The sum of "weights" $r + s + t + \dots$ in Definition 11 has an upper bound 19, because the Picard number of a K3 surface has an upper bounded 20.

Z is an *extremal log Enriques surface* if this sum $r + s + t + \dots$ equals 19.

Theorem 13 [OZ3, Main Theorem]. *There are exactly 7 isomorphism classes of extremal (rational) log Enriques surfaces. Their Types are as follows:*

$$D_{19}, D_{16} + A_3, D_{13} + A_6,$$

$$D_7 + A_{12}, D_7 + D_{12}, D_4 + A_{15}, A_{19}.$$

Example 14. Let (S_m, g_m) ($m = 2, 3$) be Shioda-Inose's pairs in Example 6. On S_m where $m = 3$ (resp. 2), there are 24 normal crossing (g_m -stable) smooth rational curves shown in [OZ1, Figures 1 and 2] or [SI, Figures 2 and 3]; among these 24, there are divisors Δ_i of the first six Dynkin types (resp. divisor Δ_7 of Dynkin type A_{19}) in Theorem 13.

Let $S_m \rightarrow \bar{S}_m$ be the contraction of Δ_i and let $Z(i) := \bar{S}_m/g_m$. Then $Z(i)$'s are nothing but 7 extremal rational log Enriques surfaces in Theorem 13.

Remark 15. (1) Every K3 surface of (maximum possible) Picard number 20 satisfies $\text{discr.}X \geq 3$ [SI]. This might be the reason why Vinberg call *the* two K3 surfaces X with Picard number 20 and $\text{discr.}X = 3, 4$, the most algebraic K3 surfaces.

(2) The same rational log Enriques surfaces of Type D_{19} and A_{19} were constructed by "bottom up" (rather than "top down" here) in [Z1]. I. Naruki and M. Reid then asked about the uniqueness of these two surfaces. See Reid [R] for his result towards a kind of uniqueness theorem.

We know that there is a unique rational log Enriques surface of Type D_{19} and one of Type A_{19} . One may ask the same uniqueness question for D_n, A_n with smaller n . The following are some of the answers, where $D_{17} + *$, etc. means $D_{17} +$ something.

Theorem 16 [OZ2, Theorems 1 and 2]. *There is exactly one (resp. two) isomorphism class(es) of rational log Enriques surface(s) of Type $D_{18} + *$ ($= D_{18}$ as a matter of fact) (resp. $A_{18} + *$ ($= A_{18}$ as a matter of fact)).*

Theorem 17. [Z5, Theorem 4]. *There is no any rational log Enriques surface of Type $D_{17} + *$.*

Theorem 18 [OZ4, Z4, Z5]).

(1) *Any rational log Enriques surface of Type $A_{17} + *$ has index 2, 3, 4, or 5.*

(2) *There are exactly two isomorphism classes of rational log Enriques surfaces of*

Type $A_{17} + *$ ($= A_{17}$ as a matter of fact) and index 5 (cf. Theorem 19 and Remark 20).

(3) There are exactly three isomorphism classes of rational log Enriques surfaces of Type $A_{17} + *$ ($= A_{17} + A_1$ as a matter of fact) and index 4.

(4) There is at least one and at most three isomorphism classes of rational log Enriques surfaces of Type $A_{17} + *$ ($= A_{17}$ as a matter of fact) and index 3.

(5) There are exactly three isomorphism classes of rational log Enriques surfaces of Type $A_{17} + A_1$ and index 2.

Theorem 19 (cf. [OZ5, Theorem 4] and [OZ5]). *Let Z be a rational log Enriques surface of Type A_{17} and index 5. Let $Y \rightarrow Z$ be the canonical $\mathbf{Z}/5\mathbf{Z}$ -covering, $X \rightarrow Y$ a minimal resolution, and σ an order-5 automorphism on X (or on Y) such that $\text{Gal}(Y/Z) = \langle \sigma \rangle$. Then we have:*

(***) $\text{discr.} X = 5$, $\langle \sigma \rangle = \text{Ker}(\text{Aut} X \rightarrow \text{Aut}(\text{Pic} X))$, and the Euler number $\varphi(5) = \text{rk} T_X$.

Remark 20. According to the result (announced in a 3-page paper by S. P. Vorontsov but without detailed proof), there is only one isomorphism class of X with a σ satisfying (***) above. We have a detailed proof of the same result [OZ5]. Kondo [Ko] has constructed such a pair $(X, \langle \sigma \rangle)$.

For general m , we have the following results:

Theorem 21 [OZ5]. *Let X be a projective K3 surface with an automorphism σ of order m where $m = 2$ (resp. 3, 5, 7, 11, 13, 17, or 19) such that*

(i) $\sigma^*|_{\text{Pic} X} = \text{id}$,

(ii) $\sigma^*\omega = \zeta_m\omega$, and

(iii) there is no any σ -fixed curve of genus ≥ 2 , and there are at least 10 (resp. 6, 3, 2, 1, 1, 0, or 0) σ -fixed rational curves.

Then such a pair (X, σ) is unique upto isomorphisms. Moreover, $\text{discr}.X = m$.

Let X be a projective K3 surface and let $H_X := \text{Ker}(\text{Aut}X \rightarrow \text{Aut}(\text{Pic}X))$. Then H_X is a finite cyclic group of order m_X say [Ni 1]. By [Ni 1], $\varphi(m_X) | \text{rk}T_X$. Kondo [Ko] determined all possible values of m_X ; in particular, if T_X is non-unimodular, then either m_X is prime with $2 \leq m \leq 19$, or $m_X = 2^r$ ($r = 0, 2, 3, 4$), 3^s ($s = 2, 3$), or 25.

Corollary 22 [OZ5]. *There is a unique projective K3 surface such that m_X is prime and $\varphi(m_X) = \text{rk}T_X$. Moreover, such X satisfies $\text{discr}.X = m_X$.*

When $m = 13, 17$ or 19 , we can prove that all conditions (i), (ii) and (iii) in Theorem 21 will be satisfied automatically. That is, we have:

Corollary 23 [OZ5]. *For each of $m = 13, 17$ and 19 , there is exactly one isomorphism class of projective K3 surface with an automorphism of order m .*

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