

A phase integral approach to irrationality

*This article is dedicated to Professor T. Aoki
in celebration of his 60th birthday*

By

Domenico NAPOLETANI* and Daniele C. STRUPPA**

Abstract

Let $F(x)$ be an analytical, real valued function defined on a compact domain $\mathcal{B} \subset \mathbb{R}$. We show that the potential irrationality of $F(x)$ evaluated at $x_0 \in \mathcal{B}$ can be stated with respect to the lack of convergence of the phase of a suitable integral $I(h)$, defined on an open, bounded domain, for h that goes to infinity. In particular, this allows us to introduce an analytical characterization of rational numbers.

§ 1. Introduction

Establishing the irrationality of special numbers has been a slow process dominated by powerful theories that, however, apply to the pointwise evaluation of limited classes of special, relatively simple functions (for an overview of the field we refer for example to the classic [2]), while establishing directly the irrationality of series, and of pointwise evaluation of complex functions, has been, mostly, the domain of ad-hoc methods (see for example [3, 6]).

In this paper we suggest that, for real irrational numbers, another viewpoint is available, that transforms the problem of the irrationality of $F(x_0)$ into the geometric problem of finding zeros of a systems of equations on a four dimensional open, bounded domain. This problem is then phrased in terms of a phase integral method we develop specifically to solve general real geometry problems for analytic functions. This

Received February 26, 2014. Revised May 23, 2014. Accepted May 26, 2014.

2010 Mathematics Subject Classification(s): 65H10, 65H20, 11D99, 11J72.

Key Words: phase integrals, rationality test, Diophantine equations.

*Institute for Quantum Studies, Chapman University, Orange, CA 92866, USA.

e-mail: napoleta@chapman.edu

**Schmid College of Science and Technology, Chapman University, Orange, CA 92866, USA.

e-mail: struppa@chapman.edu

characterization of the irrationality of function evaluations may offer a new perspective on some old problems and we suggest at the end of this contribution some potential ways in which our main results may allow analytical techniques for the asymptotic and non-perturbative study of complex phase integrals to become relevant for irrationality problems.

§ 2. Characterization of the irrationality of analytic functions at a point

The main concern of our paper is to characterize when $F(x_0)$ is irrational, assuming that $F(x)$ is analytical in a neighborhood of x_0 . To this purpose, we consider the system of equations

$$\begin{aligned}
 (2.1) \quad & F(x) - \alpha = 0, \quad x \in [x_0 - \delta, x_0 + \delta] \\
 & x - x_0 = 0, \quad \alpha \in F([x_0 - \delta, x_0 + \delta]) \\
 & \sin \frac{\pi}{m} = \sin \frac{\pi}{n} = 0, \quad m, n \in (0, 1] \\
 & \alpha m - n = 0
 \end{aligned}$$

and we note that the system in (2.1) has a solution if and only if $F(x_0)$ is a rational number. The problem is however not simpler in this geometric context, as it involves the determination of the zero set of a system of real analytic equations, and we lack a theoretical frame to approach these problems in their generality.

Despite this difficulty, it is our belief that the real geometry of solutions of analytic equations is rich enough to allow for a coherent and structured approach to a variety of seemingly unrelated problems, following an initial suggestion we made in [7], where we showed how a problem on the global robustness properties of dynamical systems could be translated into a problem about the existence (and stability under perturbations) of real solutions of an appropriate analytic equation. Moreover, in [7] we showed how there is a canonical way to move from real analytical geometric problems to questions about the asymptotical properties of the complex phase of appropriate “geometric” path integrals. Indeed, while these geometric path (and higher dimensional field) integrals are suitable in principle not only to study the existence of solutions of real analytical equations, but also of solutions of real analytical-differential equations, simpler phase integrals are sufficient to approach the problem of the existence of zeros of the system in (2.1).

More specifically, consider p real analytic functions $F_1, \dots, F_p$ defined on a compact set $\mathcal{B} \subset \mathbb{R}^p$, and the vector function $\mathbf{F}(x) = [F_1(x), \dots, F_p(x)]$. Then we can construct the associated function $L(x) = \sum_{i=1}^p F_i(x)^2$; clearly $L(x) = 0$ if and only if $F_i(x) = 0$ for all i . Moreover, it is immediate to see that every point such that $L(x) = 0$ is also a critical point for $L(x)$. The relations between critical points of $L(x)$ and solutions of

the system of equations $\mathbf{F}(x) = 0$ can be made more compelling by building a suitable phase integral, whose asymptotic behavior depends on the existence of solutions to the system itself. Indeed the following theorem holds, [8]:

Theorem 2.1. *Let $\mathbf{F}(x) = [F_1(x), \dots, F_p(x)]$ be an analytical, vectorial function defined on a compact domain $\mathcal{B} \subset \mathbb{R}^p$, let $L(x) = \sum_{i=1}^p F_i(x)^2$ have only isolated critical points in $\mathcal{B}$, and let $\mathcal{A}$ be a closed interval in $\mathbb{R}$ that does not contain the origin. Consider the complex valued integral*

$$(2.2) \quad I(h) = \int_{\mathcal{A}} \int_{\mathcal{B}} e^{ihL(x)y^2} dx dy, \quad y \in \mathcal{A} \subset \mathbb{R}, \quad x \in \mathcal{B} \subset \mathbb{R}^p,$$

and denote by $\phi(I(h))$ the complex phase (argument) of $I(h)$. Then the system $\mathbf{F}(x) = 0$ has a solution in $\mathcal{B}$ if and only if $\phi(I(h))$ has a limit for h going to infinity.

We call $L(x)$ the *geometric Lagrangian* associated to $\mathbf{F}(x) = 0$. We use this terminology in analogy to the Lagrangian functions used in defining path and field integrals [1], trusting that it will be suggestive of further crossfertilization of ideas and methods.

We will sketch the proof of this result at the end of this paper, and we only note here that the result depends on the evaluation of the phase contribution of critical points of $L(x)$ in $\mathcal{B}$, using standard stationary phase approximation methods [5, 9], to which we refer for the necessary background material.

We now use Theorem 2.1 to study the system in (2.1). We build to this purpose the geometric Lagrangian function:

$$(2.3) \quad L(x, \alpha, m, n) = (F(x) - \alpha)^2 + (x - x_0)^2 + \sin^2 \frac{\pi}{m} + \sin^2 \frac{\pi}{n} + (\alpha m - n)^2$$

Again, $L(x, \alpha, m, n) = 0$ if and only if the previous system has a solution, and we will show that the limit for $h \rightarrow \infty$ of the phase of the following integral characterizes the rationality of $F(x_0)$:

$$(2.4) \quad I_L(h) = \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_\delta} e^{ihL(\omega)y^2} d\omega dy, \quad 0 \notin \mathcal{A}$$

where $\omega = (x, \alpha, m, n)$ and we denote by Ω_δ the tensor product of the domains allowed for each of the components of ω in (2.1).

The main complication, with respect to the setting of Theorem 2.1, is the existence of infinitely many critical points, every time there is at least one point such that $L(\omega) =$

0. Indeed a critical point of $L(\omega)$ has to satisfy:

$$\begin{aligned}
 (2.5) \quad & \frac{\partial L}{\partial x} = 2(F(x) - \alpha) \frac{dF(x)}{dx} + 2(x - x_0) = 0 \\
 & \frac{\partial L}{\partial \alpha} = -2(F(x) - \alpha) + 2(\alpha m - n) = 0 \\
 & \frac{\partial L}{\partial m} = 2 \sin \frac{\pi}{m} \cos \frac{\pi}{m} \left(-\frac{\pi}{m^2}\right) + 2(\alpha m - n)\alpha = 0 \\
 & \frac{\partial L}{\partial n} = 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \left(-\frac{\pi}{n^2}\right) - 2(\alpha m - n) = 0,
 \end{aligned}$$

and we can see that if $\omega_0 = (x_0, \alpha_0, m_0, n_0)$ is a solution of $L(\omega_0) = 0$, then it is also a critical point of L . However, also $\omega_i = (x_0, \alpha_0, m_i, n_i)$ will be a zero and a critical point of L , where $m_i = \frac{m_0}{i}$ and $n_i = \frac{n_0}{i}$, i any integer (this can be seen by simple substitution in $\alpha m - n = 0$, assuming $\alpha_0 m_0 - n_0 = 0$). Note that all critical points with $L(\omega) = 0$ need to have $x = x_0$ and $\alpha = \alpha_0 = F(x_0)$.

To gain some degree of control on the proliferation of critical points we cut the domain of m and n as $m \in [M, 1]$ and $n \in [N, 1]$ with $0 < M, N < 1$ and define the domain

$$(2.6) \quad \Omega_\delta(M, N) = [x_0 - \delta, x_0 + \delta] \times F([x_0 - \delta, x_0 + \delta]) \times [M, 1] \times [N, 1].$$

We then have the following theorem:

Theorem 2.2. *Let $F(x)$ be an analytical function in the interval $[x_0 - \delta, x_0 + \delta]$, with δ sufficiently small, and assume $F'(x_0) \neq 0$. Consider the following phase integral, the restriction of $I_L(h)$ to the domain $\Omega_\delta(M, N)$:*

$$(2.7) \quad I_L(h, M, N) = \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_\delta(M, N)} e^{ihL(\omega)y^2} d\omega dy, \quad 0 \notin \mathcal{A}$$

where L is defined in (2.3). Let $\phi(I_L(h, M, N))$ be the complex phase of $I_L(h, M, N)$. $F(x_0)$ is a rational number if and only if the following limit converges:

$$(2.8) \quad \lim_{M, N \rightarrow 0} \lim_{h \rightarrow \infty} \phi(I_L(h, M, N)).$$

Proof. We will present a full proof of this theorem elsewhere, and for the time being we refer the reader to [8]. We only note here that the core of the argument consists in proving that, for δ small enough, all critical points in Ω_δ are isolated, and in controlling the size of the Hessian determinant at those points. $\square$

Note that the phase integral in (2.4) depends *functionally* on $F(x)$, so that the local behavior of $F(x)$ for $x \sim x_0$ becomes relevant for the irrationality of $F(x_0)$. This

shift of perspective may prove itself to be the key in approaching some old problems with our method. We formally write only one such potential application for a number, the gamma constant γ , whose irrationality is not known. The Digamma function Ψ can be used to define the Euler-Mascheroni γ constant as $\Psi(1) = -\gamma$, and since $\Psi(x)$ is analytical at $x = 1$, with $\Psi'(1) = \frac{\pi^2}{6} \neq 0$ we can state the following Corollary to Theorem 2.2, where we assume δ has already been chosen sufficiently small:

Corollary 2.3. *Consider the geometric Lagrangian associated to the Digamma function Ψ :*

$$(2.9) \quad L(x, \alpha, m, n) = (\Psi(x) - \alpha)^2 + (x - 1)^2 + \sin^2 \frac{\pi}{m} + \sin^2 \frac{\pi}{n} + (\alpha m - n)^2.$$

The Euler-Mascheroni constant γ is rational if and only if the following limit converges

$$(2.10) \quad \lim_{M, N \rightarrow 0} \lim_{h \rightarrow \infty} \phi(I_L(h, M, N)).$$

We can also use Theorem 2.2 to characterize explicitly the set of rational numbers with phase integrals. More particularly we have the following result:

Corollary 2.4. *Let $a_0.a_1a_2a_3\dots$ be the decimal expansion of a real number A , and consider the function $F(x) = \sum_{i=0}^{\infty} \frac{a_i}{10^i} x^i$. Consider the following geometric Lagrangian:*

$$(2.11) \quad L(x, \alpha, m, n) = (F(x) - \alpha)^2 + (x - 1)^2 + \sin^2 \frac{\pi}{m} + \sin^2 \frac{\pi}{n} + (\alpha m - n)^2.$$

Then A is rational if and only if the following limit converges

$$(2.12) \quad \lim_{M, N \rightarrow 0} \lim_{h \rightarrow \infty} \phi(I_L(h, M, N)).$$

.

Proof. We only need to prove that $F(x)$ is analytical in a neighborhood of $x = 1$ and that $F'(1) \neq 0$. Now $F(x) = \sum_{i=0}^{\infty} \frac{a_i}{10^i} x^i$ has a radius of convergence ϱ defined by $1/\varrho = \limsup_{i \rightarrow \infty} (a_i/10^i)^{1/i}$, and this supremum limit is either 0, if A has finite nonzero decimal representation, or $1/10$ if the nonzero decimal representation is unending. It follows in both cases that $x = 1$ is well within the interval of convergence, and $F(x)$ is analytical in a neighborhood of 1. Suppose now that $F'(1) = 0$, this implies that $\sum_{i=1}^{\infty} \frac{i \cdot a_i}{10^i} = 0$, but this is a sum of positive numbers, and it cannot equal zero, so we are also assured that $F'(1) \neq 0$ and the corollary follows from Theorem 2.2. $\square$

§ 3. Further Developments

In this section we would like to sketch some further potential developments of our approach. Suppose for example that we are interested in the problem of finding whether $F(x) = 0$, $x \in \mathcal{B}$ has rational solutions, with $\mathcal{B}$ a compact domain in $\mathbb{R}^p$. The method described in Section 2 can be formally applied, using the Geometric Lagrangian:

$$(3.1) \quad L(x, m, n) = F(x)^2 + \sum_{i=1}^p \sin^2 \frac{\pi}{m_i} + \sin^2 \frac{\pi}{n_i} + (x_i m_i - n_i)^2$$

A full adaptation of Theorem 2.2 to this case requires a careful evaluation of convergence of multiple series associated to critical points with $F(x) = 0$, and this is problematic if there are infinitely many discrete rational solutions of $F(x) = 0$ in $\mathcal{B}$. However, the method directly applies when $F(x) = 0$ has finitely many rational solutions on $\mathcal{B}$, as in this case the evaluation of the corresponding phase integral reduces to the evaluation of a finite number of phase integrals each on a domain that has at most a single critical point, reducing the problem to the setting of Theorem 2.2. This consideration implies that a slightly modified version of Theorem 2.2 holds for the study of rational curves of genus bigger than 1 since such curves always have at most finitely many rational points [4].

Diophantine equations could be similarly approached. If we are interested in the existence of finitely many integer solutions of $F(x) = 0$ on $\mathbb{R}^p$, the following geometric Lagrangian would be suitable:

$$(3.2) \quad L(x, m) = F(x)^2 + \sum_{i=1}^p \sin^2 \frac{\pi}{m_i} + (x_i m_i - 1)^2$$

Note that here we assume integration on unbounded domains since we did not restrict ourselves to subsets of $\mathbb{R}^p$. The implication is that the method would distinguish the case of finitely many integer solutions from *both* the case of infinitely many solutions, and the case of no solutions.

Also the problem of determining whether $F(x_0)$ is algebraic of degree K can also be stated in terms of phase integrals on geometric Lagrangians of the type:

$$(3.3) \quad L(x, \alpha, a, m, n) = (F(x) - \alpha)^2 + (x - x_0)^2 + g(\alpha)^2 + \sum_{j=1}^K \sin^2 \frac{\pi}{m_j} + (a_j m_j - 1)^2$$

where $g(\alpha)$ is a polynomial of degree at most K , a_j are its (integer) coefficients and we denote by a the vector of all a_j . We note however that the phase integral defined by

this Lagrangian would require integration over an unbounded domain in the a_j 's, since we cannot restrict a priori the size of the coefficients. It is also not clear whether the phase integral method can be adapted to discriminate algebraic numbers, of any degree K , from transcendental numbers.

We have already stressed in the previous section the importance of keeping a functional dependence of the phase integral from $F(x)$. This dependence can be used to say a little more about the structure of the geometric Lagrangians defined so far, as they can all be split into three components. Indeed some of the Lagrangians in this section could have been written in more compact ways, but we specifically wanted to keep this tripartite structure. Let us focus for simplicity on the geometric Lagrangian for the irrationality test in Section 2 (the same arguments extend easily to the Lagrangians sketched in this section).

The geometric Lagrangian $L(x, \alpha, m, n) = (F(x) - \alpha)^2 + (x - x_0)^2 + \sin^2 \frac{\pi}{m} + \sin^2 \frac{\pi}{n} + (\alpha m - n)^2$ can be written as $L(x, \alpha, m, n) = L_1(x, \alpha) + L_2(m, n) + L_3(\alpha, m, n)$, now the functions $L_1(x, \alpha) = (F(x) - \alpha)^2 + (x - x_0)^2$ and $L_2(m, n) = \sin^2 \frac{\pi}{m} + \sin^2 \frac{\pi}{n}$ can be seen as two distinct Lagrangians, each leading to geometric phase integrals whose phase is always convergent, while $L_3(\alpha, m, n) = (\alpha m - n)^2$ can be seen as a “coupling Lagrangian” that provides the interaction between the first two Lagrangians.

This discussion is inspired by the language of quantum field theory, where interaction among free fields is often mediated by only some of the terms in the associated Lagrangian ([1], chapter 5). If we push this analogy even further, we can say that, for any small coupling parameter $\beta > 0$ the Lagrangian $L_\beta(x, \alpha, m, n) = (F(x) - \alpha)^2 + (x - x_0)^2 + \sin^2 \frac{\pi}{m} + \sin^2 \frac{\pi}{n} + \beta(\alpha m - n)^2 = L_1(x, \alpha) + L_2(m, n) + L_{3,\beta}(\alpha, m, n)$ is just as suitable to study the irrationality of $F(x_0)$. For β very small, this modification allows to expand the phase integral associated to L_β in terms of powers of $L_{3,\beta}$, since, for β sufficiently small, $L_{3,\beta} = \beta L_3 = \beta(\alpha m - n)^2$ will also be small on Ω_δ . More particularly, for β very small, and under the conditions of Theorem 2.2, we have the following equalities:

$$\begin{aligned}
 (3.4) \quad & \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_\delta(M, N)} e^{ihL_\beta(\omega)y^2} d\omega dy = \\
 & \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_\delta(M, N)} e^{ih(L_1(\omega) + L_2(\omega))y^2} e^{ihL_{3,\beta}(\omega)y^2} d\omega dy = \\
 & \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_\delta(M, N)} e^{ih(L_1(\omega) + L_2(\omega))y^2} \sum_{j=0}^{\infty} \frac{(ihL_{3,\beta}(\omega)y^2)^j}{j!} d\omega dy.
 \end{aligned}$$

And, being mindful of the contrasting tension between the requirement $h \rightarrow \infty$ and $\beta \rightarrow 0$, the study of the convergence of the phase of the first integral in (3.4) could be replaced by the study of the convergence of the phase of the following series of integrals,

potentially allowing perturbative and renormalization methods to be relevant here:

$$(3.5) \quad \sum_{j=0}^{\infty} \frac{(ih\beta)^j}{j!} \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_{\delta}(M, N)} L_3(\omega)^j y^{2j} e^{ih(L_1(\omega) + L_2(\omega))y^2} d\omega dy.$$

Not only, it is possible to construct an entire family of Lagrangians $\{L_{\beta}\}$, and study the structure of the “flow” of the associated phase integrals as $\beta \rightarrow 0$. Note that, for any $\beta \neq 0$, if $F(x_0)$ is irrational, there will be no phase convergence of $I(h, M, N)$ as defined in Theorem 2.2, but for $\beta = 0$ there will always be phase convergence since the Lagrangians L_1 and L_2 will be decoupled in that case, and there will always be solutions to the associated geometric problem. So the problem of irrationality of $F(x_0)$ can also be approached as an abrupt qualitative transition, at $\beta = 0$, of the structure of the family of phase integrals associated to the Lagrangians $\{L_{\beta}\}$, again enriching irrationality problems with the methodologies that have been developed to study phase transitions in physics.

While this heuristic discussion is brief and very informal, it is included in the paper to be suggestive of the significant conceptual shift that is possible, by using Theorem 2.2 as a starting point for a renewed study of irrationality.

§ 4. Sketch of the proof of Theorem 2.1

We will use standard stationary phase approximation methods [9, 5] to calculate the integral in (2.2), for $h \rightarrow \infty$, with respect to the critical points of $L(x)$ in $\mathcal{B}$. By denoting with $H(x_*)$ the Hessian matrix of $L(x)$ evaluated at x_* , and σ_* the signature of $H(x_*)$, and by assuming that there is at least one critical point with Hessian different from zero, we can evaluate the limit

$$(4.1) \quad \lim_{h \rightarrow \infty} I(h) = \int_{y \in \mathcal{A}} \sum_{L(x_i)=0} \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{y^p (\det H(x_i))^{1/2}} e^{i\frac{\pi}{4}\sigma_i} dy + \int_{y \in \mathcal{A}} \sum_{L(x_j) \neq 0} \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{y^p (\det H(x_j))^{1/2}} e^{ihL(x_j)y^2 + i\frac{\pi}{4}\sigma_j} dy$$

by considering separately its two summands. To begin with, the first summand can be written as

$$(4.2) \quad \lim_{h \rightarrow \infty} I_1(h) = \lim_{h \rightarrow \infty} \int_{y \in \mathcal{A}} \sum_{L(x_i)=0} \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{y^p (\det H(x_i))^{1/2}} e^{i\frac{\pi}{4}\sigma_i} dy = \lim_{h \rightarrow \infty} \sum_{L(x_i)=0} \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{(\det H(x_i))^{1/2}} e^{i\frac{\pi}{4}\sigma_i} S$$

where $S = \int_{\mathcal{A}} \frac{1}{y^p} dy$, and, being the last term a finite sum, and factoring out $\frac{1}{h^{p/2}}$, the phase of $I_1(h)$ is shown to be independent of h and dependent only on the critical points x_i 's, or more exactly, on the signatures σ_i .

As to the second summand in (4.1), we see that

$$(4.3) \quad I_2(h) = \int_{y \in \mathcal{A}} \sum_{L(x_j) \neq 0} \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{y^p (\det H(x_j))^{1/2}} e^{ihL(x_j)y^2 + i\frac{\pi}{4}\sigma_j} dy =$$

$$(4.4) \quad = \int_{y \in \mathcal{A}} \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{y^p (\det H(x_j))^{1/2}} e^{ihL(x_j)y^2 + i\frac{\pi}{4}\sigma_j} dy =$$

$$(4.5) \quad = \left(\frac{2\pi}{h}\right)^{\frac{p}{2}} \frac{1}{(\det H(x_j))^{1/2}} e^{i\frac{\pi}{4}\sigma_j} \int_{y \in \mathcal{A}} \frac{1}{y^p} e^{ihL(x_j)y^2} dy$$

which is a phase integral in y computed over an interval that does not include a critical point ($y = 0$). Such integral decreases at least like $O(\frac{1}{hL(x_j)})$, the leading contribution from the boundary points of $\mathcal{A}$ ([9], page 488; [5], page 52). And therefore

$$(4.6) \quad \lim_{h \rightarrow \infty} I_2(h) = \lim_{h \rightarrow \infty} (2\pi)^{\frac{p}{2}} \sum_{L(x_j) \neq 0} \frac{1}{(\det H(x_j))^{1/2}} e^{i\frac{\pi}{4}\sigma_j} O\left(\frac{1}{h^{p/2+1}L(x_j)}\right)$$

Since there is at least one point x_j with $\det H(x_j) \neq 0$, and we know there are finitely many critical points for which $L(x_j) \neq 0$ we have that all the values $L(x_j)$ can be bounded away from 0, and the entire sum above can be estimated as

$$(4.7) \quad \lim_{h \rightarrow \infty} I_2(h) = O\left(\frac{1}{h^{p/2+1}}\right)$$

This is a negligible quantity with respect to $I_1(h) \sim \frac{1}{h^{p/2}}$. We can conclude that the limit for $h \rightarrow \infty$ of $I(h) = I_1(h) + I_2(h)$ has constant phase if $L(x) = 0$ for at least a specific x_j . If there are no values for which $L(x) = 0$, the phase will not converge, this is easy to see in the case we do have at least a critical point x_j with $L(x_j) \neq 0$ and $\det H(x_j) \neq 0$, since in that case the term $e^{ihL(x_j)y^2}$ in $I_2(h)$ will each continue to change phase as h goes to infinity.

Suppose finally that there are no critical points at all, then the integral in (2.2) is dominated by the evaluation of some derived phase integral on the boundary of $\mathcal{A} \times \mathcal{B}$, more particularly, it is true that (adapted from [9], page 488):

$$(4.8) \quad I(h) \sim -\frac{i}{h} \int_{\partial(A \times B)} G e^{ihL(x)y^2} da$$

where $\partial(\mathcal{A} \times \mathcal{B})$ is the boundary of $\mathcal{A} \times \mathcal{B}$, da is a suitable measure on the boundary, and G is a multiplier function dependent on $L(x)y^2$.

Now, $\mathcal{A} \times \mathcal{B}$ is an hypercube, and a recursive application of the result in (4.8), to lower and lower dimensional boundaries of its hyperfaces, will reduce the asymptotic evaluation of $I(h)$ to a sum of suitable multiples of evaluations of $e^{ihL(x)y^2}$ at the vertices of the hypercube. None of these values is independent of h , since we assumed there are no critical points of L on $\mathcal{A} \times \mathcal{B}$, and therefore $L(x)y^2 \neq 0$ everywhere. This implies that $\lim_{h \rightarrow \infty} \phi(I(h))$ does not exist when there are no critical points on $\mathcal{A} \times \mathcal{B}$.

References

- [1] A. Altland, B. D. Simons, *Condensed Matter Field Theory*. Cambridge University Press; 2 edition, 2010.
- [2] A. Baker, *Transcendental Number Theory*. Cambridge University Press, 1990.
- [3] F. Beukers, A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$. Bull. London Math. Soc. (1979) 11 (3): 268-272.
- [4] E. Bombieri, The Mordell conjecture revisited. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (1990) 17 (4): 615-640.
- [5] A. Erdelyi, *Asymptotic Expansions*. Dover Publications, 2010.
- [6] P. Erdős and E. G. Straus, On the irrationality of certain series. Pacific J. Math. (1974) 55 (1): 85-92.
- [7] D. Napoletani, E. Petricoin, D. C. Struppa, Geometric Path Integrals. A Language for Multiscale Biology and Systems Robustness. The Mathematical Legacy of Leon Ehrenpreis, Springer Proceedings in Mathematics, Volume 16, 2012, pp 247-260.
- [8] D. Napoletani, D.C. Struppa, Geometric phase integrals and irrationality tests, arxiv.org/abs/1312.2016.
- [9] R. Wang, *Asymptotic Approximation of Integrals*. SIAM: Society for Industrial and Applied Mathematics, 2001.