

ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS

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ABSTRACT. Let $\mathcal{S}$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The subclasses of $\mathcal{S}$ whose coefficients a_n belong to a quadratic field have been studied by Friedman [3] and Bernardi [1]. Linis [7] gave a short proof of Friedman's theorem which states that if all the a_n are "rational integers" then f is rational and has nine forms. In this paper, we consider what will happen if all the a_n are "half-integers"; that is, $2a_n \in \mathbb{Z}$.

1. PRELIMINARIES

1.1. Notation and Definitions. A *domain* is an open connected set in the complex plane $\mathbb{C}$. The *unit disk* $\mathbb{D}$ consists of all points $z \in \mathbb{C}$ of modulus $|z| < 1$. A single-valued function f is said to be *univalent* in a domain $D \subset \mathbb{C}$ if it is injective; that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$. The function f is said to be *locally univalent* at a point $z_0 \in D$ if it is univalent in some neighborhood of z_0 . For analytic functions f , the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0 .

We shall be concerned primarily with the class $\mathcal{S}$ of functions f analytic and univalent in $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in \mathcal{S}$ has a Taylor series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1.$$

The important example of a function in the class $\mathcal{S}$ is the *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots.$$

1.2. Bieberbach's Conjecture. In 1916, Bieberbach estimated the second coefficient a_2 of a function in the class $\mathcal{S}$. (See [2, p. 30].)

Theorem 1. *If $f \in \mathcal{S}$ then $|a_2| \leq 2$. Equality occurs if and only if f is the Koebe function or one of its rotations.*

This suggests the general problem to find

$$A_n := \sup_{f \in \mathcal{S}} |a_n|, \quad n = 2, 3, \dots$$

In a footnote, he wrote "Vielleicht ist überhaupt $A_n = n$ (Perhaps it is generally $A_n = n$). Since the Koebe function plays the extremal role in so many problems for the class $\mathcal{S}$, it is natural to suspect that it maximizes $|a_n|$ for all n . This is the famous conjecture of Bieberbach, first proposed in 1916.

Many partial results were obtained in the intervening years, including results for special subclasses of $\mathcal{S}$ and for particular coefficients, as well as asymptotic estimates and estimates for general n . Finally, de Branges [4] gave a remarkable proof in 1985. (See [6].)

Theorem 2. If $f \in \mathcal{S}$ then

$$|a_n| \leq n, \quad n = 2, 3, \dots \quad (1)$$

Equality occurs if and only if f is the Koebe function or one of its rotations.

1.3. Prawitz' Inequality. Let $f \in \mathcal{S}$. Set $F(z) = z/f(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$F(z) = 1 - a_2 z + (a_2^2 - a_3) z^2 + \dots$$

Hence, we have $b_0 = 1$, $b_1 = -a_2$, $b_2 = a_2^2 - a_3, \dots$. The coefficient b_n ($n \geq 1$) can be computed by the relation

$$b_n = (-1)^n \begin{vmatrix} a_2 & 1 & \dots & 0 \\ a_3 & a_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n+1} & a_n & \dots & a_2 \end{vmatrix}.$$

Prawitz [8] discovered an estimate for the coefficient b_n . It is a generalization of the Gronwall area theorem (see [2, p. 29]) and may be formulated as follows:

Theorem 3. Let $f \in \mathcal{S}$ and $[z/f(z)]^{\alpha/2} = \sum_{n=0}^{\infty} \beta_n z^n$. Then

$$\sum_{n=0}^{\infty} \frac{(2n - \alpha)}{\alpha} |\beta_n|^2 \leq 1$$

for all real α .

In particular, for $\alpha = 2$ we have the following

Corollary 1. Let $f \in \mathcal{S}$ and $z/f(z) = \sum_{n=0}^{\infty} b_n z^n$. Then

$$\sum_{n=1}^{\infty} (n-1) |b_n|^2 \leq 1. \quad (2)$$

This corollary is essentially equivalent to the Gronwall area theorem.

2. MOTIVATION

2.1. Friedman's Theorem. Friedman [3] proved the following theorem which is a part of Salem's theorem on univalent functions [10]:

Theorem 4. Let $f \in \mathcal{S}$. If all the coefficients a_n are rational integers then $f(z)$ is one of the following nine functions:

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}.$$

Proof. Set $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, then the coefficients b_n are rational integers. Since $b_1 = -a_2$ and $|a_2| \leq 2$, it follows that $|b_1| \leq 2$. Applying the inequality (2), we have $|b_2| \leq 1$ and $b_n = 0$ for $n \geq 3$. Therefore, the possible values for b_n are:

$$b_1 = 0, \pm 1, \pm 2; \quad b_2 = 0, \pm 1; \quad b_n = 0 \text{ for } n \geq 3.$$

From the combination of these values we obtain 15 functions. However, the following six functions must be rejected as having zeros in $\mathbb{D}$:

$$1 \pm 2z, \quad 1 \pm 2z - z^2, \quad 1 \pm z - z^2.$$

The remaining nine functions prove the theorem. □

2.2. Extensions of Friedman's Theorem. The method of the proof of Friedman's theorem in the previous section was given by Linis [7]. He also proved the following

Theorem 5. *Let $f \in \mathcal{S}$. If all the coefficients a_n are Gaussian integers then f has 15 forms. Here, a Gaussian integer is a complex number whose real and imaginary part are both rational integers.*

Royster [9] extended the method of the proof given by Linis to quadratic fields with negative discriminant as follows:

Theorem 6. *Let $f \in \mathcal{S}$. If all the coefficients a_n are algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{d})$ for some square-free rational negative integer d , then f has 36 forms.*

As mentioned above, they have obtained new results by replacing the condition "rational integers" with other conditions.

3. MAIN RESULT

3.1. Subclass of $\mathcal{S}$ Having Half-integer Coefficients. Now, we shall consider what will happen if all the coefficients a_n are half-integers. Here, a_n is said to be a *half-integer* if $2a_n$ is a rational integer.

In a similar way used in the proof of Friedman's theorem in the second chapter, we set $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, then the coefficients b_n are rational numbers. In the case when the a_n are rational integers, we could obtain all the possible values for the b_n . But, in this case we cannot obtain them immediately. However, using the inequalities (1) and (2), we can examine the possibilities of coefficients one by one, and obtain the following

Theorem 7. *Let $f \in \mathcal{S}$. If all the coefficients a_n are half-integers then $f(z)$ is one of the following 13 functions:*

$$z, \quad z \pm \frac{1}{2}z^2, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}, \quad \frac{z(2 \pm z)}{2(1 \pm z)^2}.$$

The detailed proof of this theorem is given in [5].

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