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Studies of the Magnetic Dipoles of the Earth and the Planets

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Abstract This paper investigates the dipole moments of the Earth and the planets. For the terrestrial magnetic field, our interest is focused on its eccentricity, namely its asymmetry with respect to the Earth's center. One of the conventional ways of representing this asymmetry of the geomagnetic field is the introduction of an eccentric dipole, that is, to adopt a dipole whose vector moment is the same as that of the centered dipole and to adjust its location so that the quadrupole terms it introduces would best fit to the (observed) quadrupole of the geomagnetic field. One problem here is that the higher harmonics above the quadrupole in the geomagnetic potential are neglected in the introduction of the eccentric dipole. Another problem is the moment of the eccentric dipole is fixed and not varied during the fitting, while its location is adjusted to give the best fit. This is probably because it has long been thought that an eccentric dipole, whose moment is different from that of the geomagnetic centered dipole, cannot be the best fit due to the invariance of the dipole moment. However, it is proven in this paper (chapter II) that an eccentric dipole of different moment is the best fit in some cases in spite that the invariance of the dipole moment certainly holds. For this purpose an idealized simple example of an eccentric distribution of magnetic field is considered, and an eccentric dipole is fitted to this field by performing all the calculations analytically to avoid possible errors arising from approximations or truncations. The result implies that a dipole of moment smaller than that in the original potential is the best fit in this case. A rough estimation suggests a reduction of several tens of nT and a correction of about 80km respectively to the moment and location of the conventional eccentric dipole for the geomagnetic field.

The above prediction is confirmed by defining and calculating a dipole whose moment as well as whose location are adjusted so that it would best fit to all the harmonics of the geomagnetic field (chapter III). The dipoles thus defined are named 'the LSM-dipoles' in this paper. The optimum moment and location of the dipole depend on the minimizing criteria during the least squares fitting. Derivation based on four different minimum conditions are presented; for example, the total magnetic energy integrated over the whole space outside the Earth's core is minimized. The four LSM-dipoles determined are located 4 to 23 degrees west of, 4 to 12 degrees south of, and within ± 20 km in the radial direction from the ordinary eccentric dipole position, and migrate with speeds a little larger than the latter. Qualitatively speaking, however, the manner in which the LSM-dipoles drift is basically the same as the way of the conventional eccentric dipole's drift. This suggests that the secular variation of an LSM-dipole is mainly controlled by the quadrupole, while its position is dependent both on the (centered) quadrupole and higher harmonics.

The author also had his eyes on the recent success of some spacecraft missions in exploring the planets. In chapter IV, comparison is made between the magnetic dipole moments of the planets, and a new scaling law of the planetary magnetism is derived from the basic equations. Since the planetary magnetic fields are not measured in such detail as in the case of geomagnetic field, we restricted ourselves to the centered dipole of the planetary magnetism (as opposed to the case of geomagnetic field in which we analyzed its eccentricity in detail). All the basic equations are treated in the quasi-vectorial form by decomposing them into the toroidal and poloidal components. From the scaling law, not only we can predict the dipole moments of the planets, but also can we estimate the toroidal magnetic field intensity in the planetary core; the result implies a typical toroidal magnetic field intensity of 100 [G], and the toroidal and the poloidal velocity fields respectively of the orders of 1×10^{-5} [ms⁻¹] and 4×10^{-7} [ms⁻¹] in the Earth's core. Since the present study is based on an $\alpha\omega$ -dynamo model, the resultant scaling law depends on the efficiency of the α -effect. If we adopt the dependence of the form $\alpha \propto \Omega$ (with Ω the angular velocity of the planet's self-rotation), the magnetic dipole moment M of a planet scales as $M \propto (\text{characteristic length})^{7/2} (\text{mean density})^{1/2} (\text{angular velocity})$. The predictions agree well with the observations except Mars.

The mystery of the Earth's magnetization is one of the still unanswered problems in the classical physics. Really basic seems the problem (which is usually termed as the dynamo problem), as it is governed by the most fundamental equations of classical electromagnetism and fluid-mechanics. Nevertheless, this problem has still been rejecting our challenges to it, and just because of this, it has attracted a number of scientists. It is even more exciting to know that some of the recent spacecraft missions succeeded in revealing the magnetic fields of other planets. It has become almost evident that the magnetization of a cosmic body is not a special phenomenon to some 'elites', but is a quite universal property of the cosmic bodies.

The theory of magnetic field generation at cosmic bodies begins at the idea of the model of a disk dynamo proposed by LARMOR (1919) to account for the solar magnetic field. This is an idea that the solar magnetism originates from the electric currents flowing in the electrically conducting region inside the Sun. The currents are maintained against Ohmic loss by converting the kinetic energy of the fluid through electromagnetic induction process. We have already accepted that the magnetic field (and other physical quantities concerning to the dynamo problem) are governed by the full set of magnetohydrodynamic equations. In 1946, a shocking and discouraging theorem was proven by COWLING (1946). No axisymmetric velocity field can work as a dynamo. The dynamo problem became essentially three-dimensional, and investigations for analytic solutions turned out to become difficult. Investigations were made numerically, by considering the non-zonal modes of harmonics of lower degrees in the spherical harmonic expansions of the velocity and magnetic fields. The model of BULLARD and GELLMAN (1954) is the most famous one among those models. In their formulation an eigenvalue problem is derived, giving the critical Reynolds number as the eigenvalue beyond which the dynamo becomes possible. In the beginning positive results were reported, and everyone believed in the solution of dynamo problem. However, the existence/convergence of the eigenvalues was denied later by the more intensive re-calculations performed on largely developed computers. After the failure of laminar dynamo models of Bullard-and-Gellman type, we moved into turbulent dynamo theory in which STEENBECK *et al.* (1966) took small helical motions inferred to exist in the Earth's core as the turbulence. This is an extension of PARKER'S (1955) cyclonic event. On the other hand, Braginskii had his eyes on the asymmetry of the field and took the small departure of the field from axial symmetry as the alternative for the turbulence giving rise to electromotive force in the Earth's core. In those turbulent dynamo models, Cowling's theorem does not apply even if the average part of the fields can be taken as being axisymmetric, as the perturbations are usually non-axisymmetric.

Thus we find that the asymmetry of the field is indispensable for a dynamo to have self-excitation process. This reminds us of the slight asymmetry of geomagnetic field with respect to the Earth's center. The magnetization of the Earth had been already known to the Chinese people from the early stage of their history of civilization, and they had already noticed that the north pointed by a piece of magnet slightly differs from the north determined by geological or astronomical methods. Mathematical formulation of this slight asymmetry of geomagnetic field was done by SCHMIDT (1934), by applying the concept of THOMSON'S (1872) "magnetic center" to the geomagnetic field. The determined magnetic center of geomagnetic field was about 500 km apart from the Earth's center. Schmidt represented the geomagnetic field approximately by a dipole placed at the magnetic center of the Earth. This dipole is so called the eccentric dipole for the geomagnetic field. The slight asymmetry of the geomagnetic field represented by the eccentric dipole, in the author's opinion, is the manifestation of the departure from axisymmetric field, which is essential for a dynamo to work, as a consequence of Cowling's theorem which prohibits the existence of any dynamos of axisymmetric kind. The asymmetry, represented by an eccentric dipole, is thus found to be worth studying in the investigations of the dynamo problem. It is also reported that the time variations in the westward drift velocity of the eccentric dipole correlated with the variations in the Earth's rotation (KAHLE *et al.*, 1969).

Usually, the geomagnetic potential is expanded into a series of spherical harmonic functions. The spherical harmonic coefficients give the perfect description of the field (up to the truncation level), but at the same time we easily lose insight to the whole configuration of the field by this method of representation. Although an eccentric-dipole-representation of the geomagnetic field reproduces the field only

approximately, it gives us a very intuitive description of the asymmetry of the field. However, only the dipole and the quadrupole terms are taken into account in the definition of the conventional eccentric dipole of Schmidt, namely, it is derived by neglecting the higher harmonics above the quadrupole. Hence a question can be raised whether or not the conventional eccentric dipole is really a good representation of the whole geomagnetic field. No work, however, has been done attempting inclusion of higher harmonics into the definition of the geomagnetic eccentric dipole, except BOCHEV'S (1969) work in which he fitted an eccentric dipole directly to the surface field itself (which involves the higher harmonics). This paper presents another approach to defining an eccentric dipole than Bochev's method. Chapters I and II in this paper are devoted to this new definition and its interpretation of an eccentric dipole with the higher harmonics above the quadrupole taken into account. The newly defined eccentric dipole differs from the conventional eccentric dipole also in that the vector moment of the dipole is adjusted during the minimization process. It is also different from Bochev's eccentric dipole in that the newly defined dipole is fitted to the whole (namely, to the continuous) distribution of the geomagnetic field by minimizing some kind of surface/volume integrals. This is just in contrast with Bochev's dipole which was fitted to a finite number of discrete data sets obtained on the Earth's surface.

Magnetic dipoles of the planets are also studied in this paper. The recent Voyager-1 and Voyager-2 spacecraft missions provided us with the data of the magnetic fields of the giant planets. Unfortunately, since the magnetic fields at the planets could not be measured in such detail as the geomagnetic field by only two fly-bys, application of the new definition of the eccentric dipole discussed in chapters I and II seems still difficult due to inaccuracy of the higher harmonics above the quadrupole at those planets. Hence only the axial, centered dipoles are treated and compared between the planets in chapter III. A simple relation between the dipole moments of the planets, referred to as a 'scaling law', is derived from the basic equations of dynamo problem. Beginning with BLACKETT'S (1947) proportionality between the angular momentum and the magnetic dipole moment, scaling law tests of several kinds have been already done by several workers until now. On the other hand, there are also some workers who have denied the existence of such kind of uniting scaling laws. They have argued that the type of dynamo should differ from planet to planet, for the mechanism of dynamo is thought to depend on the internal structure of each planet. In the author's opinion, however, if the dynamos in the solar system are described by a set of equations originating from the common basic equations, there can exist a common relation that roughly predicts the magnetic fields of the planets beyond differences in the type of the planetary dynamos.

Most of the scaling laws proposed so far concentrated on the scalar(!) torque balance in the equation of motion. Other equations such as the induction equation of the magnetic field are disregarded. Vectorial treatments were not tried in the derivation of a scaling law. Starting with the basic equations of turbulent dynamo problem, a new scaling law for the planetary magnetism is proposed in this paper. The incompressible vector fields were decomposed into two components (toroidal and poloidal components) to treat them in the vectorial form. Note that an incompressible vector field can be specified by two components along with the divergence-free condition. Comparison between observational data and the result of applying the scaling law to the planetary magnetism reveals that most of the planets lie on a line, implying that the scaling law test was successful. This can be interpreted as giving a back-up for the turbulent dynamo theory.

In addition, the present scaling law also gives an estimate for the toroidal magnetic field intensity as a consequence of decomposition into toroidal and poloidal components. What we can actually observe on the ground (or on the planetary surfaces) is the poloidal component of the geomagnetic field, and the toroidal magnetic field is not directly observable from outside the planetary core; it is completely hidden in the source region. The estimation of the toroidal magnetic field intensity is an important problem for the dynamo problem, since the amount of total energy required to drive the dynamo crucially depends on the magnetic field intensity in the planetary core. Also the estimation of the driving energy of dynamo is an important problem for geophysics, since it is directly associated with the estimation of other core properties such as velocity distribution, heat flux, kinetic viscosity etc. The largest estimation of the toroidal magnetic field intensity, at the present stage, is that of BRAGINSKII (1975) (strong field model), who derived a value of as much as ~ 800 [G] in the Earth's core. This is indeed a magnetic field of by two orders of magnitude larger than that of poloidal magnetic field in the Earth's core. (Note that the typical surface field intensity

on the Earth is ~ 0.4 [G].) In his formulation a torque balance between the Lorentz and the Coriolis forces is taken (magnetostrophic balance), and a large magnetic field is required to sustain the strong Coriolis force in the planetary core. The smallest estimation is one presented by BUSSE (1975) (Weak field model). The toroidal magnetic field intensity is almost of the same order of magnitude ($\lesssim 10$ [G]) as the inferred poloidal magnetic field intensity in this model. He conjectured that the strong Coriolis force might be basically balanced by the pressure gradient (geostrophic balance) just as in the terrestrial atmosphere, and that the torque balance between the remaining parts would hold. Dominant in the remaining parts would be the Lorentz and the inertia forces. Hence the term 'geostrophic balance' is often used to refer to the torque balance between those two forces in the context of magnetohydrodynamics. Certainly, the type of torque balance in the planetary core is one of the dominant factors to determine the appropriate toroidal magnetic field intensity in the core. However, it is not the only one. Another, equally important, is the efficiency of the feed-back mechanism between the poloidal and the toroidal magnetic fields. The magnetic field intensities should be determined so as to satisfy both the requirements from the torque balance (equation of motion) and from the efficiency of feed-back (induction equation).

Although tiny and still far from completeness may be this work, it is a great pleasure for the author to have a chance of participating in the mankind's brave and incessant challenges to further understanding of the unsolved problem — the dynamo problem. It is also a great pleasure for him to publish this thesis which, he hopes, can contribute to make a step forward to our understanding of the dynamo problem.

2-1 Introduction

SCHMIDT (1934) defined an eccentric dipole which best fits the first eight coefficients of the spherical harmonic expansion to the geomagnetic field. In his definition, transformation of the coordinates was performed, and the position of the eccentric dipole was chosen such that the power of the quadrupole term of the transformed potential has the least value.

It was shown in Schmidt's paper that the definition he adopted is equivalent to the original definition of THOMSON (1872) that two of the dipole terms and three of the quadrupole terms vanish. He also proved that the dipole terms of the transformed potential remain unchanged under the transformation, that is, that the dipole moment of the given potential is invariant with respect to the selection of the origin of the coordinates. JAMES and WINCH (1967) presented a different approach to derive the conventional eccentric dipole more readily. In their analysis, the eccentric dipole was expressed as a superposition of an infinite number of centered multipoles, a similar approach to that of HURWITZ (1960).

The present analysis is a first step to the examination of the possibility of deducing a best-fit eccentric dipole with a minimizing condition different from that chosen by Schmidt. Instead of performing a coordinate transformation, we adopt a test dipole which would reproduce the distribution of a given magnetic field as closely as possible in the least squares sense. The points examined are:

- (1) Whether or not one can determine the moment and position of such an eccentric dipole under a minimizing condition different from the conventional condition.
- (2) Can the higher harmonics above the quadrupole that are neglected in the conventional definition be included in the definition?

In relation to (1), one can raise a question of whether or not the position and moment of the eccentric dipole depend upon the minimizing condition employed. It may appear almost evident that different definitions give different dipole positions, while for the moment of the dipole the answer to this question is not self-evident. It is, of course, possible that the optimum dipole moment is subject to the minimizing condition employed in the definition. However, attention should be paid to the invariance of the dipole moment. In Schmidt's definition the dipole moment is an already known parameter and need not be adjusted—this is a direct consequence from the invariance of the dipole moment. Then it may be thought that the optimum dipole moment is independent of the minimizing condition employed because of the invariance of the dipole moment. In our analysis, however, we treat the moment as well as the position of a test dipole as unknown parameters to be determined by least squares fitting. As a result, if the dipole moment need not be varied, one simply obtains the same moment as that in the given, original potential.

Before an application of the least squares method to the real geomagnetic field, we must answer the question whether or not the moment of the fitted test dipole needs to be varied. This paper gives an answer to this question by taking an idealized simple example but by carrying out all the calculation exactly with no approximations. An axisymmetric two-dipole system is employed, and a test dipole is fitted to the field of this system. This gives an example in which the optimum moment of the test dipole has a different value from the sum of the moments of the two dipoles in the original system. Thus we find that, even in such a very simplified axisymmetric model, the moment of the test dipole should be varied in the process of least squares fitting. Hence the aim of this paper—to find out at least one example in which the moment of the test dipole should not be fixed in the process of least squares fitting—is attained. Thus we conclude that the moment of an eccentric dipole should generally be treated as a variable in the process of least squares fitting applied to some given distribution of magnetic field, and that the geomagnetic field is also no exception.

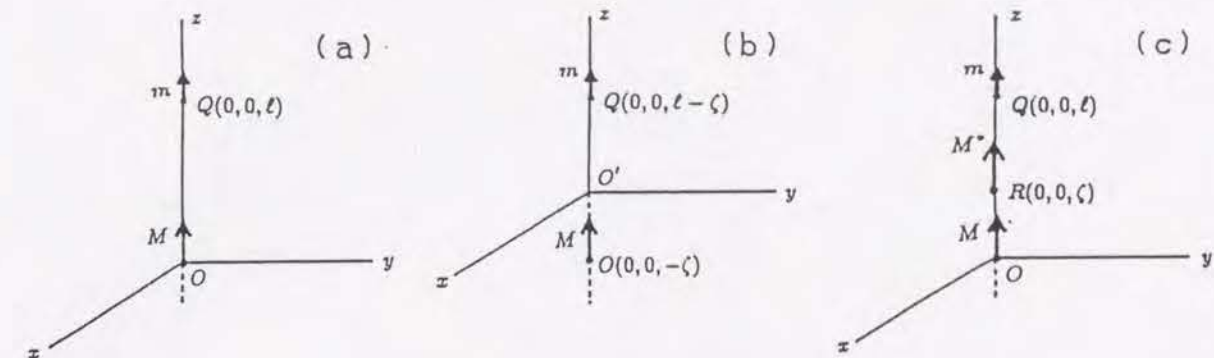


Fig. 1. (a) Two-dipole system assumed. One dipole M is located at the origin (i.e. at the Earth's center) and the other dipole m at a point $Q(0,0,\ell)$. The coordinate system is referred to as the K -frame. (b) The same system as is described in Fig.1(a), but the origin is shifted to point O' from the Earth's center O . The coordinate system is referred to as the K' -frame. (c) The two-dipole system and a test dipole of moment M^* placed at a point $R(0,0,\zeta)$ in the K -frame.

2-2 Schmidt's formula

We consider a system consisting of two dipoles. First consider a coordinate system K whose origin O coincides with the center of the Earth and whose z -axis coincides with the Earth's rotation axis. One of the dipoles, of moment M , is placed at the center of the Earth (i.e. at the origin O), and another dipole of moment m is placed at a point Q on the z -axis. The distance OQ is denoted by ℓ . These two dipoles are assumed to be directed in the $+z$ direction (See Fig.1(a)).

HURWITZ (1960) derived a general formula which expresses the magnetic potential of an arbitrary eccentric dipole in terms of an infinite series of (centered) multipoles. By making use of this formula we find that the potential of the eccentric dipole of moment m is expressed as

$$V_m = a \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n+1} m \cdot n \left(\frac{\ell}{a}\right)^{n-1} P_n(\cos \theta) \quad (2.1)$$

Here we assume that the Earth is a sphere of radius a and that $\ell < a$; (r, θ, φ) are polar coordinates; and $P_n(\cos \theta)$ denotes the Legendre function of degree n . Then the magnetic potential of this two-dipole system (hereafter referred to as the *original* magnetic field) is given by

$$V = a \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n+1} g_n^0 P_n(\cos \theta) \quad (2.2)$$

$$\text{with } g_n^0 = n \left(\frac{\ell}{a}\right)^{n-1} \cdot m + \delta_{n1} \cdot M, \quad (2.3)$$

$$\delta_{n1} = \begin{cases} 1 & (n=1) \\ 0 & (n \neq 1) \end{cases}$$

Next we take another coordinate system K' whose origin O' lies on the z -axis and $OO' = \zeta < a$, as shown in Fig.1(b). As we have two eccentric dipoles in this frame of reference, the potential originally given by (2.2) is transformed into

$$V' = a \sum_{n=1}^{\infty} \left(\frac{a}{r'}\right)^{n+1} g_n'^0 P_n(\cos \theta') \quad (2.4)$$

where the transformed coefficients are given by

$$g_n^0 = n \left(\frac{\ell - \zeta}{a} \right)^{n-1} \cdot m + n \left(-\frac{\zeta}{a} \right)^{n-1} \cdot M \quad (2.5)$$

Especially, the dipole term in the K' -frame satisfies the relation

$$g_1^0 = m + M \quad (2.6)$$

irrespective of the value of ζ . This confirms that the dipole term is invariant with respect to the selection of the origin of the coordinate system. We shall term this dipole moment the 'invariant' dipole moment in this paper.

By Schmidt's definition, we will choose a new origin O' with respect to which the spectral power of the quadrupole of the transformed potential defined by

$$(n+1) \{g_n^0\}^2 \Big|_{n=2} = \frac{1}{4\pi a^2} \int_{S'} |\mathbf{B}_{n=2}|^2 dS \quad (2.7)$$

(MAUERSBERGER, 1956; LOWES, 1966) becomes a minimum. The integral is carried out over the sphere S' of radius a and with center at O' . In our case the quadrupole term g_2^0 can be made zero by an optimum choice of the origin. Since it follows from (2.5) that

$$g_2^0 = \frac{2}{a} \{ \ell m - (m+M)\zeta \} \quad (2.8)$$

Schmidt's eccentric dipole position ζ_0 is determined as

$$\zeta_0 = OO' = \frac{m}{m+M} \ell \quad (2.9)$$

Thus, the conventional definition of Schmidt leads to an eccentric dipole of moment $m+M$ placed at the above optimum location $O'(0,0,\zeta_0)$ in the original K frame of reference.

2-3 Variable dipole moment applied to a field with dipole and quadrupole constituents

We seek for an eccentric dipole which reproduces as closely as possible the given distribution of magnetic field on the Earth's surface, i.e. on a sphere of radius a and with center at O , rather than at O' . For this purpose we place a test dipole of moment M^* (hereafter referred to as the *fitted* dipole) at a point R on the z -axis (with $OR = \zeta$) as is shown in Fig.1(c). We expand its magnetic potential in the K -frame as

$$V^* = a \sum_{n=1}^{\infty} \left(\frac{a}{r} \right)^{n+1} A_n^0 P_n(\cos \theta) \quad (2.10)$$

and require that the power of the residual field defined by

$$P = \sum_{n=1}^N w_n (A_n^0 - g_n^0)^2 \quad (2.11)$$

be a minimum. In (2.11), N and w_n denote respectively the truncation level and the weights used in the definition of the power spectrum. Here the type of the weights w_n is still arbitrary. For the weights, $w_n = n+1$ are often used, since this type of weights corresponds to the surface integral of the total force of the magnetic field derived from the magnetic potential (See eq.(2.7)). Here we will not restrict ourselves to this type of weights but treat them in general form, since various other types of weights are associated with

the surface or volume integrals of the square of other geomagnetic quantities. From the Hurwitz formula we have

$$A_n^0 = n \left(\frac{\zeta}{a} \right)^{n-1} \cdot M^* \quad (2.12)$$

The power P is given by

$$P = \sum_{n=1}^N w_n \left[n \left(\frac{\zeta}{a} \right)^{n-1} \cdot M^* - g_n^0 \right]^2 \quad (2.13)$$

In this approach the optimum set of parameters that specifies the moment and the location of the best fit eccentric dipole is determined from

$$\frac{\partial P}{\partial M^*} = 0 \quad \text{and} \quad \frac{dP}{d\zeta} = 0 \quad (2.14)$$

The total differentiation with respect to ζ means that the dipole moment M^* should be treated as a function of ζ in addition to the explicit ζ in the expression (2.13). When $N = 2$ we have

$$P = w_1 [M^* - (M+m)]^2 + 4w_2 \left[\frac{\zeta}{a} M^* - \frac{\ell}{a} m \right]^2 \quad (2.15)$$

from which we find that P is a minimum when

$$M^* = \frac{w_1(M+m) + 4w_2 \frac{\ell\zeta}{a^2} m}{w_1 + 4w_2 \frac{\zeta^2}{a^2}} \quad (2.16)$$

This is the optimum value of the dipole moment for a given fixed value of ζ . This optimum dipole moment M^* varies with ζ . The corresponding minimum value in the power P is given by

$$P = w_1(M+m)^2 + 4w_2 \frac{\ell^2}{a^2} m^2 - \frac{\left\{ w_1(M+m) + 4w_2 \frac{\ell\zeta}{a^2} m \right\}^2}{w_1 + 4w_2 \frac{\zeta^2}{a^2}} \quad (2.17)$$

We now seek for the optimum ζ that minimizes (2.17). Since

$$\frac{dP}{d\zeta} = - \frac{8w_1w_2 \left\{ w_1(M+m) + 4w_2 \frac{\ell\zeta}{a^2} m \right\}}{(w_1 + 4w_2 \frac{\zeta^2}{a^2})^2} [\ell m - (M+m)\zeta] \quad (2.18)$$

it is found that the optimum location based on our definition is still given by

$$\zeta = \frac{m}{M+m} \ell \quad (= \zeta_0 \text{ in (2.9)}) \quad (2.19)$$

regardless of the choice of weights in (2.11). This optimum location is exactly equal to the optimum location ζ_0 based on the optimization criteria of Schmidt. From (2.16) and (2.19) it can readily be shown that the optimum dipole moment at the optimum dipole location is

$$M^* = M + m \quad (2.20)$$

Thus we find that the two definitions are equivalent as long as the terms higher than the quadrupole are neglected and as long as the optimum state is considered. It should be noted, however, that our definition does not give the same dipole moment as the invariant dipole moment except for the optimum state, while by the conventional definition by Schmidt the same dipole moment is always obtained.

2-4 Variable dipole moment applied to a field with higher harmonics

It is of interest to find out whether or not the relation (2.20) remains valid even when the infinite terms are included in the definition. When $N \rightarrow \infty$, we have

$$M^* = \frac{\sum_{n=1}^{\infty} w_n \cdot n \left(\frac{\zeta}{a}\right)^{n-1} g_n^0}{\sum_{n=1}^{\infty} w_n \cdot n^2 \left(\frac{\zeta^2}{a^2}\right)^{n-1}} \quad (2.21)$$

and

$$P = \sum_{n=1}^{\infty} w_n (g_n^0)^2 - \frac{\left\{ \sum_{n=1}^{\infty} w_n \cdot n \left(\frac{\zeta}{a}\right)^{n-1} g_n^0 \right\}^2}{\sum_{n=1}^{\infty} w_n \cdot n^2 \left(\frac{\zeta^2}{a^2}\right)^{n-1}} \quad (2.22)$$

Although we have treated the weights in general form in the previous section, we now choose a specific set of weights in order to carry out and simplify the calculations. The weights adopted are $w_n = 1/n^2$, which are selected purely for a mathematical reason of obtaining analytic solutions. We believe that the qualitative aspect of the following discussion will not change even if a different weight is chosen.

With the choice of weights $w_n = 1/n^2$, we obtain

$$M^* = \frac{1 - \frac{\zeta^2}{a^2}}{1 - \frac{\zeta^2}{a^2}} \left[m + \left(1 - \frac{\ell}{a^2} \zeta\right) M \right] \quad (2.23)$$

and

$$P = \left(M^2 + 2Mm + \frac{m^2}{1 - \frac{\ell^2}{a^2}} \right) - \frac{1 - \frac{\zeta^2}{a^2}}{\left(1 - \frac{\ell}{a^2} \zeta\right)^2} \left[m + \left(1 - \frac{\ell}{a^2} \zeta\right) M \right]^2 \quad (2.24)$$

We first seek for ζ such that it makes the optimum dipole moment M^* equal to the invariant dipole moment $m + M$. From (2.23) we obtain

$$\left(1 - \frac{\zeta^2}{a^2}\right) \left\{ (m + M) - \frac{\ell}{a^2} M \zeta \right\} - (m + M) \left(1 - \frac{\ell}{a^2} \zeta\right) = 0 \quad (2.25)$$

Apart from $\zeta = 0$, (2.25) admits a solution

$$\zeta = \alpha \equiv \frac{a}{2\ell M} \left\{ a(m + M) - \sqrt{a^2(m + M)^2 - 4\ell^2 M m} \right\} \quad (2.26)$$

(which is always real and is between 0 and a). This gives the distance $\zeta = \alpha$ at which the fitted dipole possesses the same moment as the dipole in the original potential. The next step to take is to investigate whether the power P is a minimum at this $\zeta = \alpha$.

For the power P in Equation (2.24), $dP/d\zeta$ yields

$$\frac{dP}{d\zeta} = \frac{2 \left[m + \left(1 - \frac{\ell}{a^2} \zeta\right) M \right]}{a^2 \left(1 - \frac{\ell}{a^2} \zeta\right)^3} \cdot g(\zeta) \quad (2.27)$$

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with

$$g(\zeta) \equiv -\ell m + (m + M)\zeta - \frac{2\ell M}{a^2} \zeta^2 + \frac{\ell^2 M}{a^4} \zeta^3 \quad (2.28)$$

The minimum in the power of the residual field is given by $g(\zeta) = 0$. Since this equation can be rewritten as

$$\frac{M}{m} = \frac{a^4(\ell - \zeta)}{\zeta(\ell\zeta - a^2)^2} \quad (2.29)$$

we find that the solutions of $g(\zeta) = 0$ are given as the crossing points of $y = M/m$ and $y = h(\zeta)$, where $h(\zeta)$ stands for the right hand side of (2.29). Since $h(\zeta)$ is a monotonically decreasing function of ζ in $0 \leq \zeta \leq a$ if $9\ell^2 - 8a^2 \leq 0$, we obtain only one real solution $\zeta = \beta$, at which the power of the residual field P becomes a minimum. If β coincides with α , the optimum dipole moment M^* also coincides with the invariant dipole moment $m + M$. If not, M^* takes a different value. Since $\zeta = \beta$ is the only real solution of $g(\zeta) = 0$ in $0 \leq \zeta \leq a$, and since $g(0) < 0$ and $g(a) > 0$, we find that

$$g(\zeta) < 0 \quad \text{for } 0 < \zeta < \beta$$

and

$$g(\zeta) > 0 \quad \text{for } \beta < \zeta < a$$

From the direct substitutions

$$\begin{aligned} g(\alpha) &= -\frac{\ell M}{a^2} \alpha^2 \left(1 - \frac{\ell \alpha}{a^2}\right) < 0 \\ g(\beta) &= 0 \\ g(\ell) &= \left(1 - \frac{\ell^2}{a^2}\right) \ell M > 0 \end{aligned} \quad (2.30)$$

we obtain an inequality

$$0 < \alpha < \beta < \ell < a \quad (2.31)$$

This is the most important consequence of this paper since it indicates that $M^*(\beta) \neq m + M$. Since $M^*(\alpha) = m + M$ and $M^*(a) = 0$, we conclude that

$$0 < M^*(\beta) < m + M \quad (2.32)$$

The relations (2.31) and (2.32) are clearly seen in Fig.2, which shows a case when $m/M = 1.0$ and $\ell/a = 0.8$. The upper panel (Fig.2(a)) shows the line $y = M/m$ and the curve $y = h(\zeta)$. The crossing point is at $\zeta/a = \beta/a = 0.650$, while (2.26) gives $\alpha/a = 0.500$. The middle panel (Fig.2(b)) shows the change in the power of the residual field with ζ . It is a minimum at $\zeta = \beta$. The inequality of the optimum dipole moment M^* to $m + M$ is seen in the bottom panel (Fig.2(c)). The dipole moment M^* becomes identical with $m + M$ only when $\zeta = 0$ or $\zeta = \alpha$. Since $\zeta = \alpha$ does not give a minimum in the power of the residual field, the optimum dipole moment $M^*(\beta)$ differs from the invariant dipole moment $m + M$. In the present case, the ratio $M^*/(m + M)$ is found to be 0.89. Note that the conventional eccentric dipole position defined by $\zeta/a = m\ell/a(m + M)$ is 0.400, which is not only different from $\zeta = \beta$ but is also different from $\zeta = \alpha$.

2-5 Generalization of Schmidt's definition

It is also possible to generalize Schmidt's definition of an eccentric dipole by including all harmonics up to the infinite degree. In the K' -frame the total power is given by

$$P' = \sum_{n=1}^{\infty} w_n (g_n^0)^2 \quad (2.33)$$

-10-

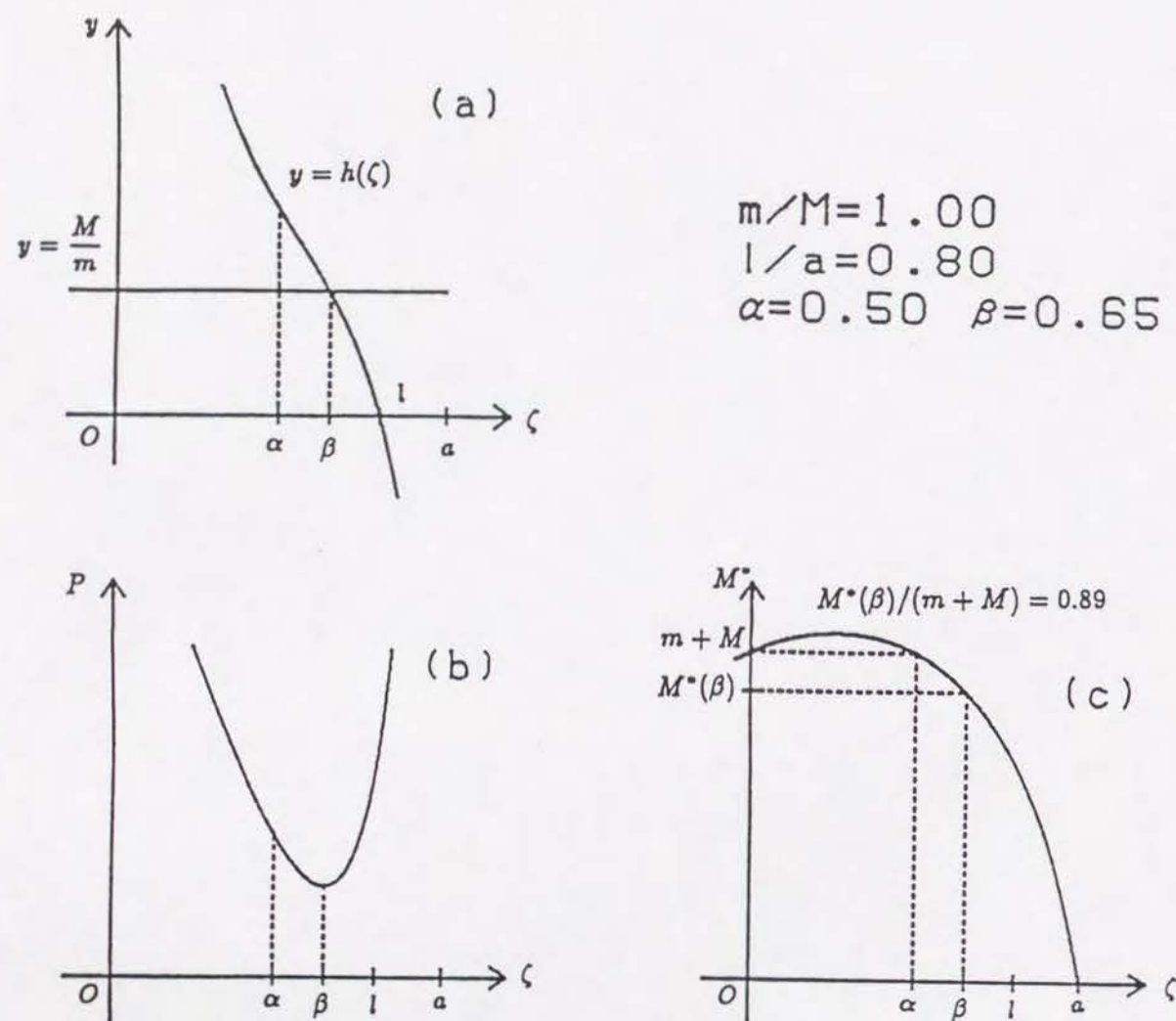


Fig. 2. (a) The line $y = M/m$ and the curve $y = h(\zeta)$ in the case $m = M$ and $l/a = 0.8$. The crossing point is given by $\beta/a = 0.650$. The solution for $M^* = m + M$ is found to be $\alpha/a = 0.500$ (See Fig. 2(c)). (b) The power of the residual field against ζ . It is seen that the power is a minimum when $\zeta = \beta$, but is not when $\zeta = \alpha$. (c) The optimum dipole moment M^* against ζ . M^* is identical with the invariant dipole moment $m + M$ only when $\zeta = 0$ or $\zeta = \alpha$. The optimum dipole moment $M^*(\beta)$ corresponding to the minimum in the power of the residual field is smaller than $m + M$ (with the ratio $M^*(\beta)/(m + M) = 0.89$).

With the transformed coefficients (2.5) and $w_n = 1/n^2$, we find

$$P' = \frac{m^2}{1 - \frac{(\ell - \zeta)^2}{a^2}} + \frac{2Mm}{1 + \frac{\zeta(\ell - \zeta)}{a^2}} + \frac{M^2}{1 - \frac{\zeta^2}{a^2}} \quad (2.34)$$

In this definition, as in the original definition by Schmidt, the dipole term is always given by

$$g_1^0 = m + M \quad (2.35)$$

which is constant, independent of ζ . Hence the minimum in the power of the residual field $P' - w_1(g_1^0)^2$ is equivalent to the minimum in P' itself. Then the generalized Schmidt's eccentric dipole position should be defined from $dP'/d\zeta = 0$. But this yields a ninth order equation in ζ and cannot be solved analytically. It is noted that for a special case $m = M$, (2.34) implies the position $\zeta = 0.5\ell$. For those parameters adopted in Fig. 2, the position of this generalized eccentric dipole is $\zeta/a = 0.400$, which is again different from $\zeta = \beta$. Our definition does not agree with Schmidt's definition even when the original definition of Schmidt is expanded including all harmonics up to infinite degree.

2-6 Discussion

In the limit $m \ll M$ or $m \gg M$, we can evaluate β in an approximate form. When $m \ll M$, we find from $g(\beta) = 0$ that

$$\beta \simeq \ell \cdot \frac{m}{M} \left\{ 1 - \left(1 - \frac{\ell^2}{a^2} \right) \frac{m}{M} \right\} \quad (2.36)$$

To the same level of approximation we obtain

$$\alpha \simeq \ell \cdot \frac{m}{M} \left\{ 1 - \left(1 - \frac{\ell^2}{a^2} \right) \frac{m}{M} \right\} \quad (2.37)$$

Hence the difference in the dipole positions $\beta - \alpha$ is estimated to be

$$\beta - \alpha \simeq \frac{\ell^2}{a^2} \left(\frac{m}{M} \right)^2 \cdot \ell \quad (2.38)$$

which is of second order in m/M . In the opposite limit $m \gg M$ we obtain

$$\alpha \simeq \ell \cdot \left\{ 1 - \left(1 - \frac{\ell^2}{a^2} \right) \frac{M}{m} \right\} \quad (2.39)$$

and

$$\beta \simeq \ell \cdot \left\{ 1 - \left(1 - \frac{\ell^2}{a^2} \right)^2 \frac{M}{m} \right\} \quad (2.40)$$

The difference $\beta - \alpha$ is given by

$$\beta - \alpha \simeq \frac{\ell^2}{a^2} \left(1 - \frac{\ell^2}{a^2} \right) \frac{M}{m} \cdot \ell \quad (2.41)$$

which differs from (2.38) in that it is linear in the ratio M/m . Since it follows from the position (2.19) of Schmidt's eccentric dipole that

$$\zeta_0 = \frac{m}{m+M} \ell \simeq \begin{cases} \frac{m}{M} \ell \left(1 - \frac{m}{M} \right) & (m \ll M) \\ \ell \left(1 - \frac{M}{m} \right) & (m \gg M) \end{cases} \quad (2.42)$$

we obtain

$$\begin{cases} \alpha \simeq \zeta_0 + \frac{\ell^2}{a^2} \left(\frac{m}{M}\right)^2 \cdot \ell \\ \beta \simeq \zeta_0 + \frac{2\ell^2}{a^2} \left(\frac{m}{M}\right)^2 \cdot \ell \end{cases} \quad (m \ll M) \quad (2.43)$$

and

$$\begin{cases} \alpha \simeq \zeta_0 + \frac{\ell^2}{a^2} \left(\frac{M}{m}\right) \cdot \ell \\ \beta \simeq \zeta_0 + \left(\frac{2\ell^2}{a^2} - \frac{\ell^4}{a^4}\right) \left(\frac{M}{m}\right) \cdot \ell \end{cases} \quad (m \gg M) \quad (2.44)$$

For the Earth, the upper bound for ℓ is the core radius 3440km, while $a=6370$ km. Although the actual geomagnetic field cannot be regarded as being such a two-dipole system, we introduce an effective dipole moment ratio m/M based on (2.42) as

$$\frac{m}{M} \left(1 - \frac{m}{M}\right) = \frac{\zeta_0}{\ell}$$

It is well-known that Schmidt's formula gives the value $\zeta_0 \sim 500$ km. Hence by taking $m/M \sim 0.2$ we can estimate the upper limit of the amount of shift from the conventional location as

$$\beta - \zeta_0 \simeq \frac{2\ell^2}{a^2} \left(\frac{m}{M}\right)^2 \cdot \ell \sim 80\text{km} \quad (2.45)$$

which is not small enough to be neglected. In the same manner we can express the difference ΔM in the dipole moment as

$$\Delta M \simeq \frac{dM^*}{d\zeta}(\alpha) \cdot \Delta\zeta \quad (2.46)$$

where $\Delta\zeta = \beta - \alpha$. After some algebraic manipulation we find

$$\begin{aligned} \Delta M &\sim -\frac{M\alpha}{a^2} \frac{a^2 - 2\ell\alpha + \alpha^2}{a^2 - \left(\frac{a}{\ell} + \ell\right)\alpha + \alpha^2} \cdot \Delta\zeta \\ &\sim -\left(\frac{\ell}{a}\right)^4 \left(\frac{m}{M}\right)^3 \cdot M \sim -23\text{nT} \end{aligned} \quad (2.47)$$

Although this is a small correction to the total dipole moment, being third order in m/M , the correction has the same order of magnitude as the reduction of the geomagnetic dipole moment due to secular variation in one year.

The choice of the weights $w_n = 1/n^2$ is crucial for the estimations (2.45) and (2.47). These estimations can vary if other types of weights are adopted in the definition of the power by (2.11). The shift $\beta - \zeta_0$ and the difference ΔM are expected to become larger if the physically meaningful type of weights $w_n = n+1$ are used instead of $1/n^2$, because the relative contribution from the higher harmonics to the power, (2.11), is more emphasized by this type of weights than by $w_n = 1/n^2$. In this sense (2.45) and (2.47) may be underestimate if $w_n = n+1$ are used in the definition. We conjecture that the larger the increasing rate of the weights with increasing degree n , the larger the amount of shift from the conventional eccentric dipole position and the difference in the dipole moment.

Finally we remark briefly on the possible cause of the difference ΔM in the dipole moment. As the definition of the power (2.7) and its physical interpretation imply, there are physical differences in the optimum criteria between Schmidt's and the present definitions. In Schmidt's definition, one takes a virtual sphere with its center at the eccentric dipole position and requires that the eccentric dipole give the best-fit on the surface of this virtual sphere. This virtual sphere moves with the translation of the origin (i.e. of the eccentric dipole position) during the minimization. It is on the surface of this virtual sphere and not on the Earth's surface that the conventional eccentric dipole is the best-fit. Note that the equation (2.25) always

admits a solution $\zeta = 0$. In Schmidt's case the dipole is always located at the center of the virtual sphere, and hence no difference arises in the moments between the dipole in the original potential and the test dipole. In contrast, the dipole presented in this paper gives the best-fit on the Earth's surface, although the dipole itself is not located at the center of the Earth. The difference in the dipole moment stems from this circumstance.

2-7 Conclusion

We have adopted an optimum condition different from the conventional condition, and have derived an eccentric dipole by also taking into account the higher harmonics above the quadrupole. Those higher harmonics are neglected in the conventional definition. If the problem were merely the neglect of higher harmonics, we could generalize the conventional definition and introduce a dipole which would best-fit the field of a given magnetic potential up to some finite or infinite degree (§2-5). However, the dipole used in this analysis is determined under an optimum condition different from that adopted by Schmidt. We note that there are physical differences in the optimum criteria between Schmidt's and the present definitions. In Schmidt's definition, one takes a virtual sphere with its center at the eccentric dipole position and requires that the eccentric dipole give the best-fit on the surface of this virtual sphere. The dipole presented in this paper gives the best fit on the Earth's surface.

With the above distinctions in mind, we examined four eccentric dipoles: the conventional eccentric dipole, a generalization of the conventional eccentric dipole, a dipole determined from the least squares but neglecting higher harmonics above the quadrupole, and the same dipole but including all harmonics. A dipole of moment equal to the moment in the original given potential is obtained from a generalization of the conventional definition. The same dipole is also derived from our definition when the harmonics above the quadrupole term are truncated. In contrast, when the harmonics above the quadrupole term are included, both the location and moment of the optimum dipole on the basis of our definition are generally different from those derived from the conventional definition. This is not inconsistent with the invariance of the dipole moment; the dipole moment in the original potential is certainly invariant. Nevertheless, when we require the best-fit on the Earth's surface, an eccentric dipole of different moment can be the best-fit.

Chapter III. New definitions of the eccentric dipole for the geomagnetic field

3-1 Introduction

The concept of an eccentric dipole first appeared in THOMSON'S (1872) work and applied by SCHMIDT (1934) to the geomagnetic field (see also BARTELS 1936; CHAPMAN and BARTELS 1940). The geomagnetic potential, V , which satisfies Laplace's equation, is expressed in spherical coordinates (r, θ, φ) as

$$V = V_1 + V_2 + V_3 + \dots$$

$$V_n = a \sum_{m=0}^n \left(\frac{a}{r}\right)^{n+1} [g_n^m \cos m\varphi + h_n^m \sin m\varphi] P_n^m(\cos \theta) \quad (3.1)$$

where a is the radius of the Earth, P_n^m are the Schmidt normalized associated Legendre polynomials, and g_n^m and h_n^m , the spherical harmonic coefficients (SHC). Usually, we take the origin, O , of the coordinate system to coincide with the center of the Earth. In Schmidt's definition the same potential is expressed in a coordinate system whose origin, O' , is different from that of the above system and whose orientation is arbitrary, as

$$V = V'_1 + V'_2 + V'_3 + \dots$$

$$V'_n = a \sum_{m=0}^n \left(\frac{a}{r'}\right)^{n+1} [g'_n{}^m \cos m\varphi' + h'_n{}^m \sin m\varphi'] P_n^m(\cos \theta') \quad (3.2)$$

Here (r', θ', φ') are polar coordinates in this frame of reference and $g'_n{}^m$ and $h'_n{}^m$, transformed SHC. The conventional eccentric dipole is defined by minimizing the surface integral, over a sphere of given radius, of V'_2 , which represents the potential of the quadrupole. In the process of the minimization, the dipole moment vector remains constant, namely, the leading term, V'_1 , is exactly equal to V_1 given by (3.1). In other words, the dipole terms do not depend on the location of the origin, i.e. that the dipole moment vector of the geomagnetic potential is invariant under a parallel translation or a rotation of the coordinate system. Because of this invariance, the dipole moment vector of the eccentric dipole has been defined to be exactly equal to that of the centered dipole. With the above minimum condition SCHMIDT (1934) first derived the formulas for the location of the eccentric dipole. His well-known formulas express the conventional eccentric dipole location in terms of the first eight coefficients in the spherical harmonic expansion, i.e. in terms of the dipole and the quadrupole components. JAMES and WINCH (1967) presented a different derivation of the conventional eccentric dipole, by which one can arrive at the same definition but more readily than was done by Schmidt. In their analysis, they adopted a model dipole, expanded its magnetic potential into a Taylor series, and compared this potential with the dipole and quadrupole terms of the geomagnetic potential.

Since the location of the eccentric dipole is thus determined only from the first eight SHC, a question can be raised if the eccentric dipole so derived adequately represents the actual geomagnetic field when the neglected higher harmonics are also taken into account. BOCHEV (1965, 1969) fitted an eccentric dipole directly to the geomagnetic field observed on the ground by the least squares method. He determined the position and the three components of the moment of the optimum dipole. In this sense he fitted a dipole to a field including higher harmonics. The position and moment of the dipole he determined were close to, but were slightly different from, those of the conventional eccentric dipole.

This paper examines the definition of the eccentric dipole. The definition of an eccentric dipole is presented, involving all the terms in spherical harmonic coefficients. The manner in which we defined it is

different from that presented in Bochev's work. The criteria for the optimum dipole are also re-examined; the conventional definition is based on a direct coordinate transformation applied to the geomagnetic potential itself, while the present analysis adopts a model eccentric dipole which gives the best fit to the geomagnetic field by the least squares method. As was done by Bochev, the vector moment of the model dipole is determined as well as the dipole location in the minimization process. It is shown that the best-fit dipole is not necessarily the one whose dipole moment has the same magnitude and direction as those of the centered dipole when the higher degree harmonics are included in the definition. The best-fit dipole is obtained by the least squares method in a way that is not inconsistent with the independence of the dipole moment on the selection of the location of the origin of the coordinate system, as is discussed in the previous chapter (and also in SANO (1991)). Finally, the secular variations of the variously defined dipoles are studied.

3-2 Expression for the optimum dipole moment

We adopt a spherical polar coordinate system in which the origin is at the center of the Earth and in which the polar axis coincides with the Earth's rotation axis. Let us consider a dipole of moment $\vec{M} = (M_r, M_\theta, M_\varphi)$ placed at a point $O'(d, \beta, \alpha)$, where M_r, M_θ, M_φ are the r, θ , and φ components of the dipole moment vector, d is the distance OO' , and β and α are colatitude and longitude of the point O' , respectively. The magnetic potential, V_{model} , of this offset dipole can be expanded, with N expressing the truncation level, as

$$V_{model} = a \sum_{n=1}^N \sum_{m=0}^n \left(\frac{a}{r}\right)^{n+1} [A_n^m \cos m\varphi + B_n^m \sin m\varphi] P_n^m(\cos \theta) \quad (3.3)$$

Note that the origin remains at the center of the Earth. The coefficients, A_n^m and B_n^m , are linear functions of M_r, M_θ , and M_φ , and nonlinearly depend on d, β , and α . They have been given in a closed form by HURWITZ (1960) as

$$A_n^m = M_r \tilde{A}_{nr}^m + M_\theta \tilde{A}_{n\theta}^m + M_\varphi \tilde{A}_{n\varphi}^m \quad (3.4)$$

$$B_n^m = M_r \tilde{B}_{nr}^m + M_\theta \tilde{B}_{n\theta}^m + M_\varphi \tilde{B}_{n\varphi}^m \quad (3.5)$$

$$\text{where } \tilde{A}_{nr}^m = \left(\frac{d}{a}\right)^{n-1} \cdot n P_n^m(\cos \beta) \cos m\alpha \quad (3.6)$$

$$\tilde{B}_{nr}^m = \left(\frac{d}{a}\right)^{n-1} \cdot n P_n^m(\cos \beta) \sin m\alpha \quad (3.7)$$

$$\tilde{A}_{n\theta}^m = \left(\frac{d}{a}\right)^{n-1} \cdot \frac{d}{d\beta} P_n^m(\cos \beta) \cos m\alpha \quad (3.8)$$

$$\tilde{B}_{n\theta}^m = \left(\frac{d}{a}\right)^{n-1} \cdot \frac{d}{d\beta} P_n^m(\cos \beta) \sin m\alpha \quad (3.9)$$

$$\tilde{A}_{n\varphi}^m = -\left(\frac{d}{a}\right)^{n-1} \cdot \frac{m}{\sin \beta} P_n^m(\cos \beta) \sin m\alpha \quad (3.10)$$

$$\tilde{B}_{n\varphi}^m = \left(\frac{d}{a}\right)^{n-1} \cdot \frac{m}{\sin \beta} P_n^m(\cos \beta) \cos m\alpha \quad (3.11)$$

Thus one finds that an eccentric dipole can be expressed equivalently by the superposition of an infinite number of centered multipoles. As the potential, V_{obs} , of the observed geomagnetic field is usually expressed in the form

$$V_{obs} = a \sum_{n=1}^N \sum_{m=0}^n \left(\frac{a}{r}\right)^{n+1} [g_n^m \cos m\varphi + h_n^m \sin m\varphi] P_n^m(\cos \theta) \quad (3.12)$$

we regard V_{model} as giving the best approximation to V_{obs} when the following quantity is made a minimum:

$$f = \sum_{n=1}^N w_n \sum_{m=0}^n [(g_n^m - A_n^m)^2 + (h_n^m - B_n^m)^2] \quad (3.13)$$

Here one has the choice of the weights w_n , leading to different definitions of the eccentric dipole. For the conventional eccentric dipole, A_1^0 , A_1^1 , and B_1^1 are made equal to g_1^0 , g_1^1 , and h_1^1 , and the sum is made only over $n = 2$, namely, we require that the quadrupole terms which the eccentric dipole (whose dipole moment is fixed) introduces would best fit the quadrupole terms in the original potential (3.12). In this case, the location of the eccentric dipole does not depend on the choice of w_n , because the minimum condition reduces to

$$w_2 \sum_{m=0}^{m=2} [(g_2^m - A_2^m)^2 + (h_2^m - B_2^m)^2] = \text{a minimum} \quad (3.14)$$

where w_2 is merely a constant independent of d , α , and β . In our method, we have a general truncation level N (≥ 2); we require that the eccentric dipole (whose moment is variable) best fit all the harmonics including the dipole and the quadrupole up to some finite truncation level under various types of weight.

In the following analysis we use four types of w_n (the derivation is given later), namely:

$$\begin{aligned} (1) \quad w_n &= n+1; & (2) \quad w_n &= (n+1) \cdot \left(\frac{a}{r_c}\right)^{2n+4}; \\ (3) \quad w_n &= \frac{n+1}{2n+1}; & (4) \quad w_n &= \frac{n+1}{2n+1} \cdot \left(\frac{a}{r_c}\right)^{2n+1} \end{aligned} \quad (3.15)$$

where r_c is the radius of the Earth's core. Defining the residual parts of the magnetic potential and of the magnetic field respectively by

$$V_{obs} = V_{model} + V_{res}, \quad \vec{B}_{obs} = \vec{B}_{model} + \vec{B}_{res} \quad (3.16)$$

the four types of w_n correspond respectively to minimizing:

- (1) the average intensity of the (squared) residual field on the Earth's surface S

$$\frac{1}{4\pi a^2} \int_S \vec{B}_{res}^2 dS \quad (= (n+1) [(g_n^m - A_n^m)^2 + (h_n^m - B_n^m)^2]) \quad (3.17)$$

For the evaluation of the integral, see MAUERSBERGER (1956) and LOWES (1966).

- (2) the average intensity of the (squared) residual field on the surface of the Earth's core S_c

$$\frac{1}{4\pi r_c^2} \int_{S_c} \vec{B}_{res}^2 dS \quad (= (n+1) \left(\frac{a}{r_c}\right)^{2n+4} [(g_n^m - A_n^m)^2 + (h_n^m - B_n^m)^2]) \quad (3.18)$$

- (3) the total magnetic energy of the residual field integrated over the whole volume $\hat{V}$ outside the Earth's surface (BENTON and ALLDREDGE, 1987)

$$\frac{1}{2\mu_0} \int_{\hat{V}} \vec{B}_{res}^2 dV \quad (\propto \frac{n+1}{2n+1} [(g_n^m - A_n^m)^2 + (h_n^m - B_n^m)^2]) \quad (3.19)$$

- (4) the total magnetic energy of the residual field integrated over the whole volume $\hat{V}_c$ outside the Earth's core

$$\frac{1}{2\mu_0} \int_{\hat{V}_c} \vec{B}_{res}^2 dV \quad (\propto \frac{n+1}{2n+1} \cdot \left(\frac{a}{r_c}\right)^{2n+1} [(g_n^m - A_n^m)^2 + (h_n^m - B_n^m)^2]) \quad (3.20)$$

In the ordinary eccentric dipole case the weights w_n are taken as

$$w_1 = 0, \quad w_2 \neq 0, \quad w_n = 0 \quad (n \geq 3) \quad (3.21)$$

Table 1

n	(1)	(2)	(3)	(4)
1	1.00	1.00	1.00	1.00
2	1.50	5.19	0.90	3.11
3	2.00	23.91	0.86	10.25
4	2.50	103.36	0.83	34.45
5	3.00	428.91	0.82	116.98
6	3.50	1730.31	0.81	399.30
7	4.00	6837.95	0.80	1367.59
8	4.50	26600.46	0.79	4694.20
9	5.00	102201.56	0.79	16137.08
10	5.50	388741.81	0.79	55534.54

Table 1. Relative magnitudes of the weights w_n normalized to respective w_1 for the cases (1), (2), (3), and (4).

with A_1^0 , A_1^1 , and B_1^1 fixed respectively to g_1^0 , g_1^1 , and h_1^1 . Therefore we will refer to the ordinary eccentric dipole case as case (0).

The weights are increasing functions of the degree n except for the case (3) in which the weight hardly varies with n . The relative magnitudes of the weights normalized to respective w_1 are listed in Table 1. The ratio a/r_c is approximately 1.8 for the Earth, and the power of this ratio increases rapidly with increasing n . The other factor such as $n+1$ and $(n+1)/(2n+1)$ modulates the magnitude of the weights. For a fixed n (≥ 2), the magnitude of the weights increases in the order of cases (3), (1), (4), and (2).

The moment of the ordinary eccentric dipole is defined to be the same as that of the centered dipole, so the minimum condition on the quantity f (for $n = 2$ only) gives the values of d , β , and α only. It is possible, however, to treat not only the location of an eccentric dipole but also the three components of the dipole moment vector as unknown parameters to be determined by a minimum condition on the quantity f . Differentiating f in (3.13) with respect to M_r , one obtains the following equation:

$$2 \sum_{n=1}^N w_n \sum_{m=0}^n [\tilde{A}_{nr}^m (M_r \tilde{A}_{nr}^m + M_\theta \tilde{A}_{n\theta}^m + M_\varphi \tilde{A}_{n\varphi}^m - g_n^m) + \tilde{B}_{nr}^m (M_r \tilde{B}_{nr}^m + M_\theta \tilde{B}_{n\theta}^m + M_\varphi \tilde{B}_{n\varphi}^m - h_n^m)] = 0 \quad (3.22)$$

Similar equations are derived by differentiating f with respect to M_θ and M_φ . It then follows that

$$\sum_{n=1}^N w_n \sum_{m=0}^n U_n^m \begin{pmatrix} M_r \\ M_\theta \\ M_\varphi \end{pmatrix} = \sum_{n=1}^N w_n \sum_{m=0}^n \vec{K}_n^m \quad (3.23)$$

$$\text{with} \quad U_n^m = \left(\frac{d}{a}\right)^{2n-2} \cdot \begin{pmatrix} n^2 (P_n^m(\cos \beta))^2 & \frac{n}{2} \frac{d}{d\beta} (P_n^m(\cos \beta))^2 & 0 \\ \frac{n}{2} \frac{d}{d\beta} (P_n^m(\cos \beta))^2 & \left(\frac{d}{d\beta} P_n^m(\cos \beta)\right)^2 & 0 \\ 0 & 0 & \left(\frac{m P_n^m(\cos \beta)}{\sin \beta}\right)^2 \end{pmatrix} \quad (3.24)$$

$$\text{and} \quad \vec{K}_n^m = \left(\frac{d}{a}\right)^{n-1} \cdot \begin{pmatrix} n P_n^m(\cos \beta) [g_n^m \cos m\alpha + h_n^m \sin m\alpha] \\ \frac{d}{d\beta} P_n^m(\cos \beta) [g_n^m \cos m\alpha + h_n^m \sin m\alpha] \\ \frac{m}{\sin \beta} P_n^m(\cos \beta) [h_n^m \cos m\alpha - g_n^m \sin m\alpha] \end{pmatrix} \quad (3.25)$$

It is easy to show from the addition theorem

$$P_n(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \varphi) = \sum_{m=0}^n P_n^m(\cos \theta) P_n^m(\cos \theta') \cos m\varphi \quad (3.26)$$

that

$$\sum_{m=0}^n (P_n^m(\cos \beta))^2 = P_n(1) = 1 \quad (3.27)$$

$$\sum_{m=0}^n \left(\frac{d}{d\beta} P_n^m(\cos \beta) \right)^2 = \sum_{m=0}^n \left(\frac{m P_n^m(\cos \beta)}{\sin \beta} \right)^2 = P_n'(1) = \frac{n(n+1)}{2} \quad (3.28)$$

Using these relations, the expressions for M_r , M_θ , and M_φ are immediately obtained as

$$M_r = \sum_{n=1}^N \sum_{m=0}^n w_n \cdot n \left(\frac{d}{a} \right)^{n-1} P_n^m(\cos \beta) [g_n^m \cos m\alpha + h_n^m \sin m\alpha] / S_N \quad (3.29)$$

$$M_\theta = \sum_{n=1}^N \sum_{m=0}^n w_n \cdot \left(\frac{d}{a} \right)^{n-1} \frac{d}{d\beta} P_n^m(\cos \beta) [g_n^m \cos m\alpha + h_n^m \sin m\alpha] / T_N \quad (3.30)$$

$$M_\varphi = \sum_{n=1}^N \sum_{m=0}^n \frac{m w_n}{\sin \beta} \cdot \left(\frac{d}{a} \right)^{n-1} P_n^m(\cos \beta) [h_n^m \cos m\alpha - g_n^m \sin m\alpha] / T_N \quad (3.31)$$

$$\text{where } S_N = \sum_{n=1}^N n^2 w_n \left(\frac{d}{a} \right)^{2n-2} \text{ and } T_N = \sum_{n=1}^N \frac{n(n+1)}{2} w_n \left(\frac{d}{a} \right)^{2n-2} \quad (3.32)$$

The optimum location (d, β, α) can be determined by a nonlinear least squares method using the above expressions, and the $\vec{M}$ can be found by substitution. We will call this newly defined dipole the 'LSM-dipole' in this paper.

3-3 Application of the new definition of the dipole to the geomagnetic data between 1945–1990

The position (d, β, α) and the three components $(M_r, M_\theta, M_\varphi)$ of the LSM-dipole moment are computed using nine sets of time sequential SHC models for the epoch 1945–1985 under the minimum conditions (0) to (4). The nine sets consist of DGRF/IGRF (the Definitive Geomagnetic Reference Field/International Geomagnetic Reference Field) 1945–1985 (see e.g. IAGA DIVISION I WORKING GROUP 1, 1985). The set of these coefficients is for epochs five years apart and of degrees 8 to 13. In our calculation we made use only of the time-independent terms of the models because the secular change terms may inaccurately represent the time variation of the geomagnetic field over long periods of time. Starting with the conventional dipole location, iterative calculations were performed to determine the location of the dipole, i.e. to determine the values of d , β , and α . At each stage of iteration, the corresponding dipole moment was evaluated from (3.29) to (3.31). We regard the parameters as having converged when the locations $\vec{x}^{(k)}$ and $\vec{x}^{(k+1)}$ of the dipole after k -th and $(k+1)$ -th iterations satisfy a condition $|\vec{x}^{(k+1)} - \vec{x}^{(k)}| < \varepsilon |\vec{x}^{(k+1)}|$ for the first time. In our calculation, ε is taken to be 5×10^{-5} . Convergence was achieved in all cases of calculation.

Apart from the convergence with respect to the number of iteration in the least squares fitting, we should also examine the convergence of the parameters with respect to the increase in the truncation level of the field models used. The slowest convergence is expected for the case (2) where the higher harmonics are more heavily weighted than in other cases. Tables 2a and 2b show results of changing the truncation level, for both cases (1) and (2). The truncation levels were varied from 2 to 10. The model used was IGRF 1985. Relatively large changes are seen in all parameters from $n = 2$ to $n = 4$, and above $n = 4$ rapid convergences are observed. In particular, the optimum values of the parameters show a perfect convergence

Table 2a
 $w_n = n + 1$ case(1)

YEAR=1985.00	IGRF1985 MODEL	Distance	Latitude	Longitude	A_1^0	A_1^1	B_1^1	M
2		505.1	20.74	146.81	-29906.9	-2163.7	5217.4	30435.6
4		478.4	15.28	140.53	-29924.2	-2133.7	5240.5	30454.4
6		478.3	15.27	140.54	-29924.3	-2133.6	5240.6	30454.6
8		478.3	15.27	140.54	-29924.3	-2133.6	5240.6	30454.6
10		478.3	15.27	140.54	-29924.3	-2133.6	5240.6	30454.6

Table 2b
 $w_n = (n+1) \cdot \left(\frac{a}{r_c} \right)^{2n+4}$ case(2)

YEAR=1985.00	IGRF1985 MODEL	Distance	Latitude	Longitude	A_1^0	A_1^1	B_1^1	M
2		511.7	20.45	147.13	-29937.0	-2743.6	4606.1	30413.3
4		544.7	8.08	123.59	-29704.4	-2368.9	4549.6	30143.6
6		535.4	8.15	123.74	-29748.7	-2368.3	4565.7	30190.0
8		536.0	8.14	123.71	-29745.7	-2367.9	4564.3	30186.8
10		536.0	8.14	123.71	-29745.8	-2367.9	4564.3	30186.9

Table 2. (a) Convergence of the parameters with varying truncation levels N . The model used is IGRF1985, and the weights $w_n = n + 1$. (b) The same as Table 2a, but $w_n = (n+1) \left(\frac{a}{r_c} \right)^{2n+4}$.

Table 3
 $w_n = n + 1$

YEAR=1985.00	IGRF1985 MODEL	Distance	Latitude	Longitude	g_1^0	g_1^1	h_1^1	M
2		502.6	20.88	146.67	-29877.0	-1903.0	5497.0	30438.0
2		505.1	20.74	146.81	-29906.9	-2163.7	5217.4	30435.6
10		476.3	15.34	140.46	-29877.0	-1903.0	5497.0	30438.0
10		478.3	15.27	140.54	-29924.3	-2133.6	5240.6	30454.6

Table 3. The determined parameters when $N = 2$ (first and second lines) and $N = 10$ (third and forth lines), and when the dipole moment is fixed (first and third lines) and varied (second and forth lines).

for $N \geq 6$ in the case (1). Even for the case (2), the convergence is attained when $N \geq 8$. Thus we confirm sufficient convergence for the parameters for all types of weights used in the present analysis when the truncation level $N \geq 8$. Thus we may regard that the inhomogeneity of the truncation levels of the models used would have no significant effects on the results. However, all the results presented below are calculated, unless explicitly stated, by truncating the SHC up to degree 8 in order to compare the results under the same condition.

WEIGHT	Distance	Latitude	Longitude	A_1^0	A_1^1	B_1^1	M
(0)	502.6	20.88	146.67	-29877.0	-1903.0	5497.0	30438.0
(X)	—	25.57	144.26	—	—	—	—
(1)	478.3	15.27	140.54	-29924.3	-2133.6	5240.6	30454.6
(2)	536.0	8.14	123.71	-29745.7	-2367.9	4564.3	30186.8
(3)	481.3	16.78	142.30	-29905.9	-2047.0	5339.4	30447.7
(4)	494.1	9.88	130.94	-29910.4	-2275.0	4977.5	30406.9

Table 4. Parameters determined based on IGRF1985 model. The truncation level $N = 8$. Radial distances from the center of the Earth, longitudes and latitudes of the dipole positions, absolute magnitudes and the first three dipole terms A_1^0 , A_1^1 and B_1^1 . The cases (0),(X),(1),(2),(3), and (4) respectively denote the conventional eccentric dipole, Yukutake's approximation, and the four newly defined eccentric dipole cases.

The definition of the dipoles presented in this paper differs from the conventional definition in that the vector dipole moment is allowed to vary, and in that the higher harmonics above the quadrupole are taken into account. Logically, it is necessary to discuss the effects of these two different points separately. For this purpose we calculate dipoles of fixed and variable moments when the level of truncation is 2 and 10. The results are given in Table 3. The weights used are $w_n = n + 1$, and the model used is IGRF 1985. The first line lists the parameters for conventional eccentric dipole. The second, third, and last lines respectively give the parameters for a variable dipole moment fitted to the dipole plus quadrupole, a fixed dipole moment fitted to all harmonics up to degree 10, a variable dipole moment fitted to all harmonics up to degree 10. The table shows an interesting result. Relaxation of the fixation of the dipole moment has only small effects on the eccentric dipole location, as is evidenced by the fact that the dipole locations listed in each of the two pairs of the first and second lines, and the third and fourth lines, are close to each other. Thus we find that the shift of the dipole location is mainly due to inclusion of higher harmonics. In contrast, when the dipole moment is allowed to vary, the dipole moment converges to similar values regardless of the difference in the truncation levels; this is seen by observing that the dipole moments listed in the second and last lines are close to each other.

Table 4 shows the optimum values of the parameters calculated for 1985. We see from this table that the location and the moment of the LSM-dipoles differ from those of the ordinary eccentric dipole (case (0)) to some extent, but not drastically. Longitudes and latitudes of these LSM-dipoles and of the ordinary eccentric dipole are plotted against time in Figs.3a and 3b for the interval 1945 to 1985. In the figure the symbols O,X,*,+,=, and # respectively stand for the cases (0),(X),(1),(2),(3), and (4). The numbers in parentheses show the case numbers for the weight used, and the case (X) is commented on later. The longitude of the ordinary eccentric dipole position varied from 153° (E) to 147° (E) during the forty years. All the other cases are found to be westward of the case (0), with the deviation from the case (0) becoming larger in the order of cases (X),(3),(1),(4), and (2). We find similarly that the eccentric dipole latitude changed from 13° (N) to 21° (N), and that the case (X) is situated at the highest latitude, with latitude decreasing in the order of cases (0),(3),(1),(4), and (2). We then obtain trajectories on the longitude-latitude plane as are shown in Fig.4. Note that what the trajectories show is only the movements of the projection of the point (d, β, α) in three-dimensional space onto a two-dimensional space (β, α) where $\hat{\beta} \equiv (\pi/2) - \beta$. In all cases the migration in time is from southeast to northwest as the arrow indicates. These "inverted S" shaped traces suggest that the westward and the northward components of the drift velocities of these dipoles varied with time rather than remaining constant during the interval of this analysis. The most striking feature is the shift of the traces towards the southwest direction in the order of cases (0),(3),(1),(4), and (2). This order is the same as expressing that the greater the weight given to higher harmonics the larger the amount of shift from case (0).

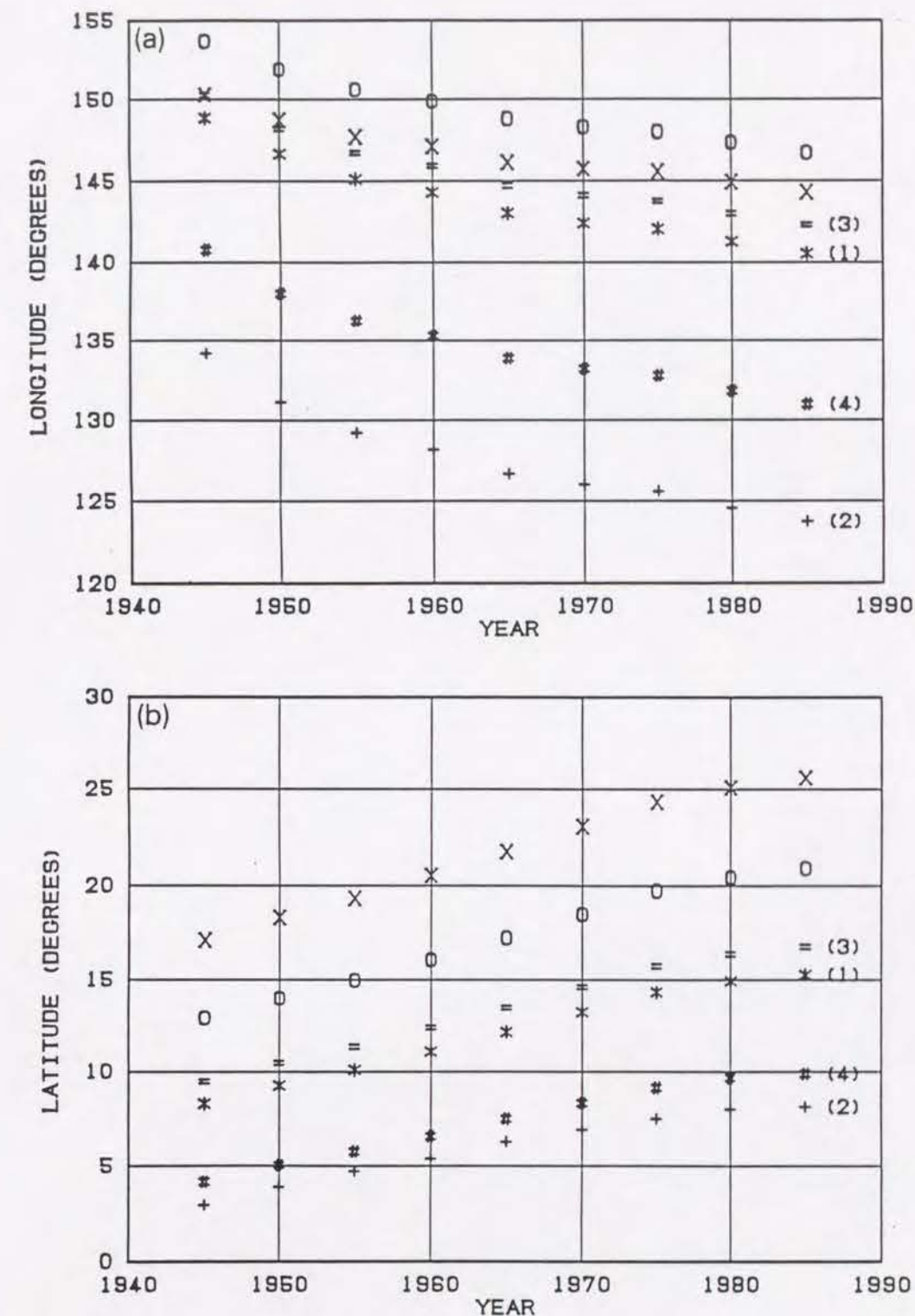


Fig. 3. (a) Longitudes of the conventional eccentric dipole position and the LSM-dipole positions. The cases (0),(X),(1),(2), (3), and (4) are respectively denoted in the figure by (O),(X), (*),(+),(=), and (#). (b) Latitudes of the conventional eccentric dipole position and the LSM-dipole positions. The symbols denote the same type of weights as in Fig.3a.

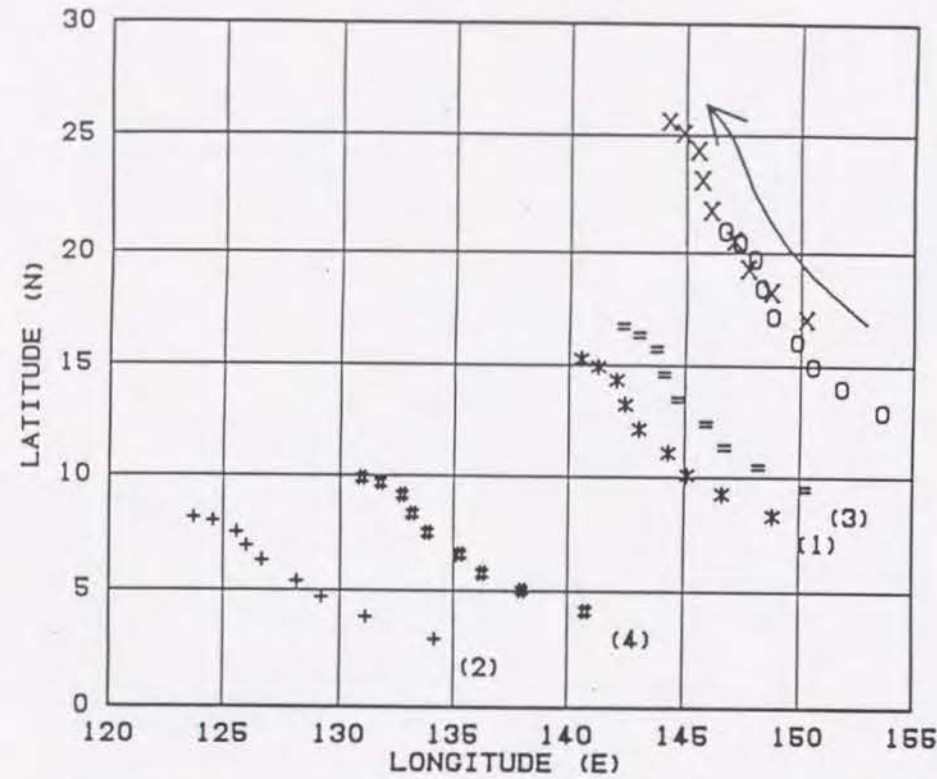


Fig. 4. Longitudes and latitudes of the position of the conventional eccentric dipole, Yukutake's approximating location, and the LSM-dipoles during 1945 to 1985. The symbols denote the same type of weights as in Fig. 3a.

The problem on the goodness of the fit of the eccentric dipole has been once discussed by YUKUTAKE (1973), but from a different viewpoint; he argued that the conventional eccentric dipole is an inadequate representation of the quadrupole field, and questioned the goodness of the conventional eccentric dipole for the representation of the movement of the geomagnetic field as a whole. He pointed out that the longitude and colatitude of the ordinary eccentric dipole are, because of the particular accident of the present numerical values, approximately given by

$$\alpha_X = \pi + \tan^{-1} \frac{h_2^1}{g_2^1} \quad (3.33)$$

$$\beta_X = -\tan^{-1} \frac{\sqrt{3} g_2^0}{2\sqrt{(g_2^1)^2 + (h_2^1)^2}} \quad (3.34)$$

The conventional eccentric dipole location is then expressed as

$$\alpha = \alpha_X + \varepsilon_\varphi \quad (3.35)$$

$$\beta = \beta_X + \varepsilon_\theta \quad (3.36)$$

with small quantities ε_φ and ε_θ . Variations in α_X and $\beta_X \equiv (\pi/2) - \beta$, are shown in Fig. 4 as case (X). The correction terms ε_φ and ε_θ are seen in the figure as the separation of two points (X) and (O) for each epoch. In agreement with Yukutake's result, the parallelism of the trajectories of these points is again evident, with $\varepsilon_\varphi \sim 3$ degrees and $\varepsilon_\theta \sim 4$ degrees. The small quantities ε_φ and ε_θ in the expression (3.35) and (3.36) are interpreted as expressing the effect of taking five terms $g_1^0, g_1^1, h_1^1, g_2^2$, and h_2^2 into account in the definition of the longitude and latitude of the eccentric dipole. In the similar way, we introduce correction terms $\varepsilon_\varphi^{(\ell)}$ and $\varepsilon_\theta^{(\ell)}$ by

$$\alpha^{(\ell)} = \alpha_X + \varepsilon_\varphi^{(\ell)}, \quad \beta^{(\ell)} = \beta_X + \varepsilon_\theta^{(\ell)}, \quad (\ell = 1, 2, 3, 4) \quad (3.37)$$

where $\alpha^{(\ell)}$ and $\beta^{(\ell)}$ respectively denote the longitude and the latitude of the LSM-dipole location of the type of weight (ℓ) ($\ell = 1, 2, 3, 4$). The effect of including the higher harmonics above the quadrupole under the type of weight (ℓ) is expressed by $\varepsilon_\varphi^{(\ell)}$ and $\varepsilon_\theta^{(\ell)}$. With the notations $\varepsilon_\varphi^{(0)} = \varepsilon_\varphi$ and $\varepsilon_\theta^{(0)} = \varepsilon_\theta$, a quantity $\sqrt{\varepsilon_\varphi^{(\ell)^2} + \varepsilon_\theta^{(\ell)^2}}$ ($\ell = 0, 1, 2, 3, 4$) denoting the separation of case (X) and other cases on Fig. 4 increases in the order of cases (0), (3), (1), (4), and (2), as we noted in the previous paragraph. The correction terms, $\varepsilon_\varphi^{(\ell)}$ and $\varepsilon_\theta^{(\ell)}$, however, do not depend on time to a first approximation, since they follow qualitatively similar traces, and hence the relations

$$\dot{\alpha}^{(\ell)} = \frac{d}{dt} \tan^{-1} \frac{h_2^1}{g_2^1} \quad (3.38)$$

$$\dot{\beta}^{(\ell)} = -\frac{d}{dt} \tan^{-1} \frac{\sqrt{3} g_2^0}{2\sqrt{(g_2^1)^2 + (h_2^1)^2}} \quad (3.39)$$

approximately hold, similarly to Yukutake's result. This means that the drift in longitude and latitude of the LSM-dipoles are nearly the same as that in the conventional eccentric dipole (and in Yukutake's approximation). Our interest is the effect of including the higher harmonics above the quadrupole in the definition of the eccentric dipole. The result indicates that the neglected higher harmonics have relatively small effects on the longitudinal and latitudinal movement of the eccentric dipoles. Inclusion or neglect of the higher harmonics leads to nearly the same results, so we should conclude that the dipole and the quadrupole terms of the geomagnetic potential dominate in determining the secular change in the location of the eccentric dipole, and that the conventional representation of the movement of the geomagnetic field by an eccentric dipole may not be as inadequate as was pointed out by Yukutake as far as the longitudinal and latitudinal drifts are concerned.

To a first approximation the two-dimensional drifts of the eccentric dipole are similar (Fig. 4). However, a closer inspection of Fig. 4 reveals that the inverted S shaped curve representing the drift becomes more inclined as one advances from case (0) to case (2), meaning an increasing westward drift and a decreasing northward drift in the same order.

Figures 5a and 5b respectively show the time variations in the radial distance d and in the magnitude of the moments of the LSM-dipoles and the conventional eccentric dipole. From Fig. 5a we see that the distances between the Earth's center and the locations of the LSM-dipoles and the conventional eccentric dipole increase with time with an average speed of $2 \sim 3$ km/yr.; the dipoles migrated in the radial direction by $80 \sim 110$ km during the forty years between 1945 and 1985. The distance d of the dipole from the Earth's center becomes greater in the order of cases (1), (3), (4), (0), and (2), except in 1950 and in 1955 when the order of two cases (4) and (0) is reversed. The order is different from the previous order (0), (3), (1), (4), and (2) which expresses the order of greater emphasis on the higher degree terms. The interpretation of the order (1), (3), ..., (2) is difficult; the distance once decreased from (0) to (3) to (1) and then increased to (4) and (2). It thus appears that the lower degree part of the non-dipole field has the effect of reducing the radial distance of the dipole while the higher degree part of it has the effect of increasing it. The magnitudes of the total vector dipole moments are shown in Fig. 5b. Like the moment of the centered dipole, the moments of the LSM-dipoles decrease with time. The decreases in the total dipole moments are found to be roughly 800 nT in cases (0), (1), (3), and (4), and 1000 nT in case (2) during the interval of this analysis, with the average rate of decrease of $20 \sim 25$ nT/yr. The symbols in the figure are almost overlapping with each other except for the symbols standing for the case (2). This indicates that the magnitudes of the determined dipole moments did not differ greatly from that of the centered dipole except in case (2). This does not mean, however, that our treatment was trivial. Let us turn our attention to the terms A_1^0, A_1^1, B_1^1 and M listed in Table 4. These terms were calculated from the equations (3.4) to (3.11) using the determined three components of the dipole moment M_r, M_θ , and M_φ for the LSM-dipole cases (case numbers (1) through (4)), and were taken to be equal to g_1^0, g_1^1 , and h_1^1 for the conventional eccentric dipole case (case number (0)). The determined values A_1^0, A_1^1 , and B_1^1 are different from g_1^0, g_1^1 , and h_1^1 , but $M^2 \equiv (A_1^0)^2 + (A_1^1)^2 + (B_1^1)^2$ being almost equal to $(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2$ implies that the

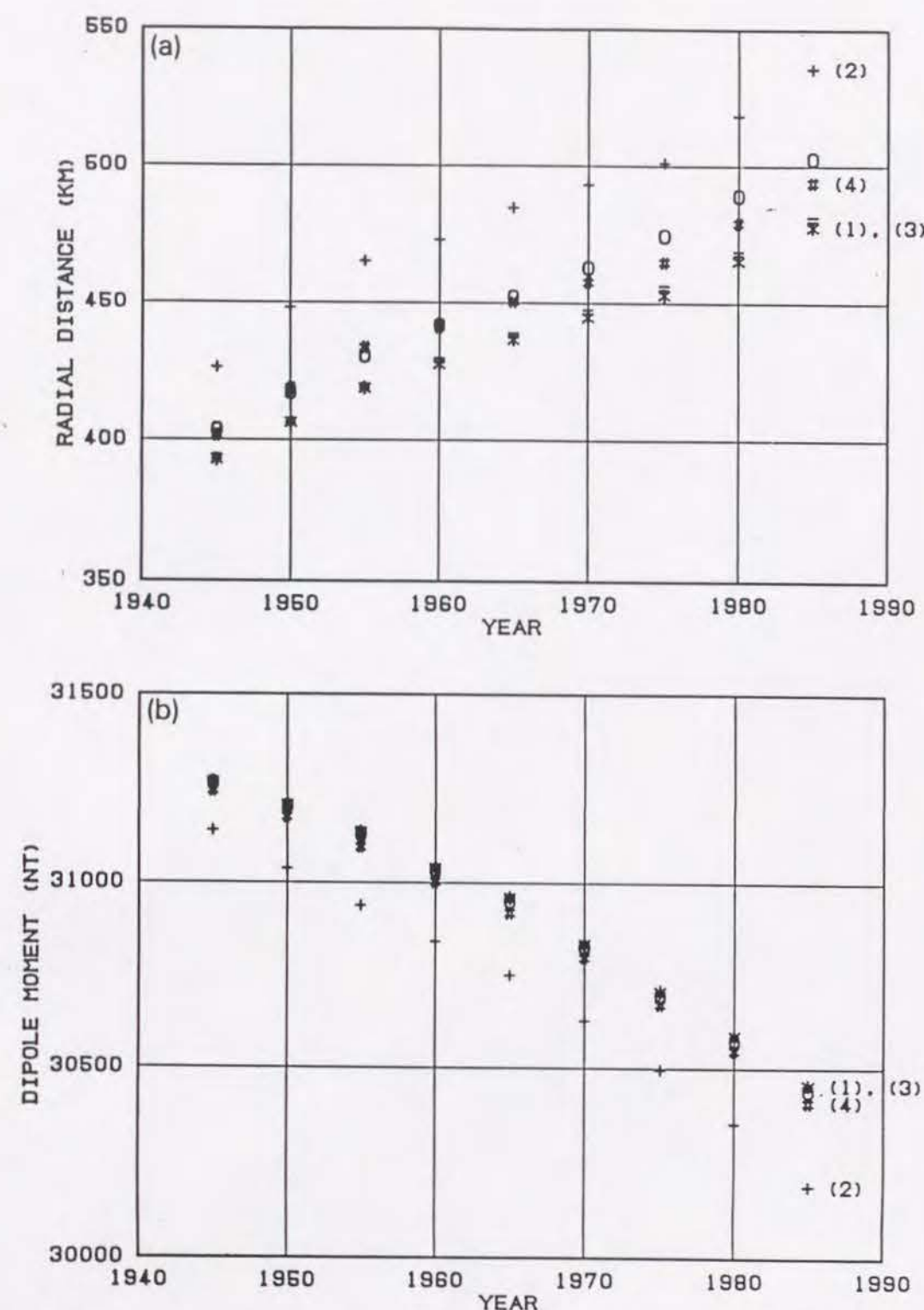


Fig. 5. (a) Temporal changes in the radial distance between the Earth's center and the locations of the conventional and the LSM-dipoles. The symbols denote the same type of weights as in Fig. 3a. (b) Magnitudes of the dipole moments of the conventional and the LSM-dipoles versus time. The symbols denote the same type of weights as in Fig. 3a.

quantity f can be made a minimum by adopting a dipole whose vector moment is rotated from the vector moment of the centered dipole (and consequently, of the conventional eccentric dipole).

3-4 Discussion

We have chosen four cases for the minimum condition to be imposed on the quantity f . The higher degree terms that would rapidly increase with increasing depth are not accurately determined from observation made near the Earth's surface. A dipole moment determined by minimizing a physical quantity such as the average intensity of the residual field on the Earth's surface may not be a good representation of the whole field including all components from the lowest to the highest degrees. Therefore we impose a new condition, i.e. minimizing a physical quantity on the core surface; it is on the surface of the core that the higher degree terms have amplitudes of the same order of magnitude as those of the lower degree terms (MEYER *et al.*, 1983; MEYER, 1985). Strictly speaking, this procedure, however, has two potential problems. One is related to the accuracy with which the terms are determined. The higher degree terms are regrettably less well determined compared with the lower degree terms. Thus it is not expected that the geomagnetic field can be accurately extrapolated to the core surface. Since the heavy weights for the higher degree terms amplify errors, the determined values of d , β , and α and the three components of the dipole moment M_r , M_θ , and M_ϕ may be subject to errors. In this sense there is an advantage in dealing with the parameters at the surface of the Earth, where the poorly determined higher terms have the least influence. All the cases except the cases (0) and (X) are affected by the accuracy of the higher degree terms.

The other problem stems from the effect of the non-zero conductivity of the mantle. This can affect the results of integration on the core surface or over the total volume outside the core-mantle boundary, namely, for the cases (2) and (4). The lower mantle conductivity is not as yet accurately known, but is supposed to be less than $100 \sim 200$ S/m (e.g. DUCRUIX *et al.*, 1980), which is not small enough to be totally neglected. For periods shorter than several decades the mantle cannot be regarded as being a perfect insulator. Therefore the geomagnetic field within the mantle does not strictly satisfy Laplace's equation for these periods, and we cannot simply extrapolate the field to the core surface on the basis of the potential determined from observations on and above the Earth's surface. Nevertheless, we assume a perfectly insulating mantle for practical reasons; in any case, no observations exist providing the distribution of the geomagnetic field within the Earth.

Referring to cases (1) and (2), we have imposed a minimum condition that a physical quantity take the smallest value when integrated over the surface of a sphere concentric with the Earth. This comes from the idea that the best fit dipole moment would be the one that would reproduce as closely as possible the geomagnetic field on that surface. The scalar potential representation of the geomagnetic field, on the other hand, is assumed to be valid over the whole space outside the Earth's core, and therefore it should be possible to consider an eccentric dipole that would give the best fit everywhere in specified space. Such a fit would be attained by minimizing the volume integral of the square of a physical quantity over the whole space outside the Earth's core, i.e. over the whole region where the scalar potential representation is valid, instead of minimizing a surface integral over a chosen surface. This approach is taken in cases (3) and (4).

The spherical harmonic coefficients g_n^m and h_n^m are determined by the least squares method based on geomagnetic data from the surface observatories and low altitude satellites. Logically, it may be thought to be a roundabout way to obtain an eccentric dipole by determining a set of coefficients by the least squares method using a set of magnetic field data calculated from a geomagnetic field model, when the spherical harmonic coefficients themselves in the model are already determined by the least squares method using observational data. However, this is a philosophical question of the definition of the 'eccentric dipole.' One can directly determine an eccentric dipole from the observational data, just as BOCHEV (1965, 1969) did. He compiled data from 61 ground observatories, and determined the location and the vector moment of a dipole which would give as good as possible fits to the measured magnetic field at those stations. The optimum moment of the dipole he determined is expected to differ from the dipole moment of the geomagnetic field, since the number of data points he treated is not infinite, the distribution of the data points is not continuous, nor extending all over the Earth's surface. He simply fitted an eccentric dipole

directly to a set of observations, namely, to the observed magnetic fields at the observatories he had chosen. In contrary, we took an indirect way that an eccentric dipole be fitted to the whole geomagnetic field through a minimization process performed onto some kinds of surface or volume integrals that are expressed in terms of the gauss coefficients. This is because our point of view is totally different from Bochev's. We are not thinking of the observations as a starting point, but we are interested in determining a dipole when a magnetic potential of the form (3.1) is given. Here the possible observational errors in determining the SHC in (3.1) are set aside, and we deal with the ideal case when all the SHC are determined exactly. Then the mathematical expression of the magnetic potential (3.1) guarantees a spatially continuous distribution of exact magnetic field (outside the source region), and the evaluation of surface/volume integrals on the basis of the magnetic potential (3.1) is found to be equivalent to the determination of a dipole fitted to the infinite number of complete observations. Probably, we should here refer to the finiteness in the number of SHC in the potential (3.1), since the actual models used in this analysis are of finite degrees. Finiteness of the terms in the series (3.1) has absolutely nothing to do with the spatial continuity of the derived magnetic field (= infiniteness in the number of data points dealt with). One should pay attention to the fact that even a simple centered dipole aligned to the Earth's rotation axis (which has only one term if expressed in the form of (3.1)) gives a spatially continuous distribution of magnetic field. Moreover, although the expressions (3.29) to (3.31) give a vector dipole moment which is dependent on the highest degree N of the series (3.1), and although the actual models used in this analysis are of finite degrees, we can logically introduce a potential of infinite degrees even by adding zeros above the truncation level to obtain an equivalently infinite series, and determine a vector dipole moment from the expression (3.29) to (3.31). Since $d \ll r_c$, the sums S_N and T_N both converge very rapidly as $N \rightarrow \infty$ even when the weights w_n increase by $w_n \propto (a/r_c)^{2(n-1)}$. The rapid convergence in S_N and T_N guarantees rapid convergence of the results of M_r , M_θ , and M_ϕ above the truncation level; in a realistic situation where $N \gtrsim 10$, truncation of the terms of degrees $n > N$ may be regarded as having little effect. Indeed, the results listed in Tables 2a and 2b show that the vector dipole moment of the LSM-dipoles converges as $N \rightarrow \infty$, but to a vector whose three components are, if converted to A_1^0 , A_1^1 , and B_1^1 , definitely different (although by small amounts) from g_1^0 , g_1^1 , and h_1^1 in the original potential (3.1). In this respect, it is clear that the differences in the dipole moment did not arise from truncation errors. Had we fitted an eccentric dipole directly to the observational data, these relationships would not have been clarified.

Fig.6 illustrates the Earth's spherical surface (or the surface of the Earth's core), S , with center O , and a spherical surface, S' , of the same radius as S and with center at O' . The potential in (3.1) is expanded relative to the origin O and that in (3.2), relative to O' . In either system, $V_1, V_2, V_3, \dots$ (or $V'_1, V'_2, V'_3, \dots$) represent the dipole, quadrupole, octupole, ... terms, respectively. The invariance of the dipole moment implies the equality of V_1 and V'_1 independent of the choice of O' . Rewriting (3.1) and (3.2),

$$V = V_1 + V_2 + V_3 + \dots \quad (3.1')$$

$$V = V'_1 + V'_2 + V'_3 + \dots \quad (3.2')$$

$$\text{with } V'_1 = V_1$$

Let the magnetic potential of the fitted eccentric dipole be $\tilde{V}$. According to the Hurwitz equations, this potential is expanded, relative to O , as

$$\tilde{V} = \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 \dots \quad (3.40)$$

while, relative to O' we simply have

$$\tilde{V} = \tilde{V}'_1 \quad (3.41)$$

since the fitted dipole is placed at the very location O' itself. In the determination of the conventional eccentric dipole, the square of the field of V'_2 integrated over S' (or any spherical surface centered at O') is made a minimum, since the minimum in the residual of the field of $V - \tilde{V}$ is equivalent to the minimum in the field of V'_2 itself when $\tilde{V}'_1$ is taken to be equal to $V'_1 (= V_1)$ and when $V'_3, V'_4, \dots$ are truncated in the potential (3.2'). What will then happen if we generalize the original definition of Schmidt and choose

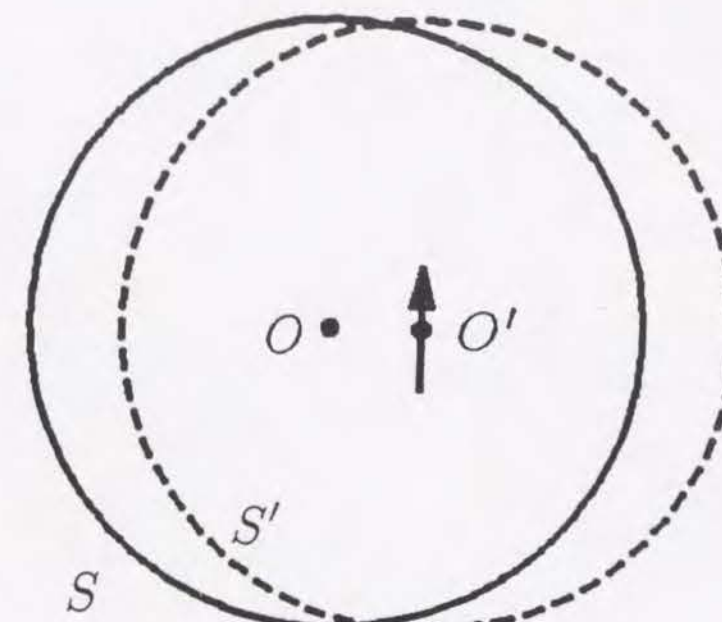


Fig. 6. Illustration of the Earth's surface (or core-mantle boundary), S , and a surface, S' , on which the minimization is performed in the conventional method of determining the eccentric dipole.

an origin, O' , with respect to which the total power, defined by including all the harmonics up to some higher truncation level N , takes the smallest value on the surface S' ? First we can take the origin O' such that the power of the field of $V'_2 + V'_3 + \dots + V'_N$ takes the smallest value, and then we can take $\tilde{V}'_1 = V'_1$. Hence we will obtain a dipole which is placed at a different location from the conventional eccentric dipole location, and whose moment is identical with that in the original potential (3.1). (In a different context HILTON and SCHULTZ (1973) determined $V'_2, V'_3, \dots, V'_N$, but with respect to the conventional eccentric dipole coordinate system.)

However, there does not seem to be any *a priori* reason to choose surface S' as the surface on which the integral of the square of the residual field is made a minimum. This surface may well be chosen to be the Earth's surface (or the core-mantle boundary), S . Besides the pragmatic reasons that the ground-based observations are made on S and not on S' , there can be a more physical basis for the choice of S . For instance, surface S' can conceivably reside partly in a source region, in which the scalar potential representation is not valid. Such a circumstance can happen if the surface S is taken as that of the Earth's core. In the approach presented in this paper, we require that a quantity f in (3.13) representing the field of the residual of the potentials (3.1) and (3.40) be a minimum when integrated on the surface S (in cases (2) and (4), integration over the whole space outside S). Since the higher harmonics $\tilde{V}_2, \tilde{V}_3, \dots, \tilde{V}_N$ are not independent of $\tilde{V}_1$ as can be seen from the Hurwitz equations, the quantity f cannot be made a minimum when $\tilde{V}_1 = V_1$; V_k and $\tilde{V}_k$ ($k = 2, 3, \dots, N$) are not orthogonal when integrated on S , and the optimum value of $\tilde{V}_1$ is dependent on $V_2, V_3, \dots, V_N$ through $\tilde{V}_2, \tilde{V}_3, \dots, \tilde{V}_N$. Thus we expect to have a different dipole location and a moment from those by the conventional definition. This circumstance, $\tilde{V}_1 \neq V_1$, is not inconsistent with the invariance of the dipole moment which never implies that $\tilde{V}_1 = V_1$, but in reality does imply that $V'_1 = V_1$.

The approach taken in this paper is motivated from these considerations. In the conventional method the integral of the square of V'_2 is minimized. However, there is no logical reason to think that the surface integral of the square of V'_2 is the only quantity that is to be minimized. The content of the meaning of

'best fit' could be any of other minimization criteria. This is the essential point of the argument presented in this paper. It is important to appreciate that the question of the invariance of the dipole component as generally conceived is an entirely separate problem and that the discussions presented in this paper by no means contradicts the concept of the invariance of the dipole.

The conventional eccentric dipole is derived by neglecting the higher harmonics above the quadrupole. While by the present method all non-zero terms are used in the determination of an eccentric dipole. We recognize that the definition of the conventional eccentric dipole has the pragmatic advantage in that the first eight terms of the spherical harmonic coefficients specify the eccentric dipole through simple analytic relations. However, the conventional definition of the eccentric dipole offers no answer to the question of whether or not the neglect of the higher harmonics has no serious influence on the results obtained.

3-5 Conclusion

An approach is presented defining an eccentric dipole that would best fit the observed geomagnetic field. In the conventional definition of the eccentric dipole, the dipole and quadrupole terms in a spherical harmonic expression of the geomagnetic field uniquely determine an eccentric dipole (which is parallel to, and of moment equal to that of, the centered dipole). In the present method all the harmonic terms are used to determine the position, orientation and magnitude of an eccentric dipole by the least squares method. In the least squares formulation four different physical quantities are minimized, leading to four different eccentric dipoles. This illustrates that an eccentric dipole can be defined in a number of different ways. The numerical results indicate that the positions of these newly defined eccentric dipoles differ from the location of the conventional eccentric dipole by 4 to 23 degrees in longitude, 4 to 12 degrees in latitude, and by ± 20 km in radial distance, while to a first approximation the drifts of the new eccentric dipoles are nearly the same as the drift of the conventional eccentric dipole.

Chapter IV. Dipole moments of the planets

4-1 Introduction

Observations have revealed that a number of cosmic bodies possess their intrinsic magnetic fields. BLACKETT (1947) was the first who discussed the existence of a scaling law (or a magnetic Bode's law) for the magnetic fields of cosmic bodies. He proclaimed that the magnetic dipole moment of a cosmic body scaled as its own angular momentum. He argued that a rotating body, even if electrically neutral, gave rise to magnetic field through some unknown unification theory of gravity and electromagnetism. Now it is already accepted that the intrinsic magnetic fields of cosmic bodies, including those of the planets, are due to the dynamo action operative in the electrically conducting fluid core of the cosmic bodies (e.g. textbooks of MOFFATT, 1978; JACOBS, 1987). Although some workers have denied the existence of scaling laws in the opinion that each planet should have each dynamo, there are also several workers who relied on its possibility and have attempted to deduce or to test scaling laws for the planetary magnetism based on the dynamo theory (BUSSE, 1976; RUSSELL, 1978, 1979; JACOBS, 1979). Different may be the type of dynamo from planet to planet, we still recognize a possibility of existence of some kind of relation between the magnetic fields of the planets if any type of dynamo at the planets can be finally attributed to the same type of basic equations. Although a simple order estimation can by no means replace the full solutions (which may be dependent on the actual type of dynamo at each planet), it has an advantage that it can provide a rough estimation of the physical quantities without getting into any details of the actual dynamos of the planets. CURTIS and NESS (1986) derived a scaling law starting with torque balance between Coriolis and Lorentz forces (Magnetostrophic balance). Two different scaling laws were presented by MIZUTANI *et al.* (1992) also by assuming the magnetostrophic balance. Most of the scaling laws above were derived by treating the vector equations in the scalar form.

In this paper we decompose incompressible vector fields into the toroidal and the poloidal components in order to improve the scalar treatment, and derive a scaling law of the planetary magnetism treating these two components separately. This approach also allows us to estimate the toroidal magnetic field intensity, which is unobservable from outside the planet's core. Of course the estimation of the toroidal magnetic field intensity is dependent on what one employs as the model. For instance, BRAGINSKII (1975, 1980) constructed a nearly symmetric dynamo model on the basis of magnetostrophic balance, and predicted very strong toroidal magnetic field in the Earth's core (Strong field model). On the other hand, BUSSE (1976) adopted the geostrophic balance, namely, a balance between Coriolis and pressure gradient forces, and made an estimation that the toroidal magnetic field was of the same order of magnitude as that of the poloidal magnetic field (Weak field model). Thus, the estimation of the toroidal magnetic field intensity in the planetary core is closely related to the type of torque balance one assumes in the planetary core. The results presented in this paper favors the magnetostrophic balance rather than the geostrophic balance in the planetary core, giving the toroidal magnetic field intensity of one order larger than that of the poloidal field for the Earth. At the same time, we suggest that this ratio is not invariant among the planets; it depends on several physical parameters and hence can vary from planet to planet.

4-2 Torque balance in the vectorial form

The turbulent dynamo problem is described by the following mean-field MHD equations:

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}(\mathbf{v} \times \mathbf{B} + \alpha \mathbf{B}) + \frac{1}{\mu\sigma} \nabla^2 \mathbf{B} \quad (4.1)$$

$$\rho \left\{ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right\} = -\nabla p + \frac{1}{\mu} \text{rot} \mathbf{B} \times \mathbf{B} + 2\rho \mathbf{v} \times \boldsymbol{\Omega} + (\text{Buoyancy, Gravity, etc.}) \quad (4.2)$$

$$\text{div} \mathbf{v} = 0 \quad (4.3)$$

Here $\mathbf{B}$ is the magnetic field, $\mathbf{v}$ velocity field, μ permeability, σ electric conductivity, ρ density of the planetary core, p pressure, and $\boldsymbol{\Omega}$ angular velocity of the planetary core. The term $\alpha \mathbf{B}$ in eq. (4.1) expresses the mean electromotive force arising from interactions between turbulent velocity and magnetic fields (STEENBECK *et al.*, 1966). The terms $(\mathbf{v}' \cdot \nabla) \mathbf{v}'$ and $\text{rot} \mathbf{B}' / \mu \times \mathbf{B}'$ with $\mathbf{v}'$ and $\mathbf{B}'$ fluctuations of $\mathbf{v}$ and $\mathbf{B}$ might be important in some cases, but these terms are neglected in eq. (4.2) for simplicity. We have also assumed incompressible fluid and no kinetic viscosity. Given a divergence-free vector field $\mathbf{a}$, we can introduce the toroidal-poloidal decomposition of this vector field by

$$\mathbf{a} = \text{rot}(\Psi \mathbf{r}) + \text{rot} \text{rot}(\Phi \mathbf{r}) \quad (4.4)$$

where Ψ and Φ are appropriate scalar functions. The first term denotes a toroidal field and the second term a poloidal field (see appendix. See also MOFFATT, 1978). We then decompose the velocity and the magnetic fields in the core of a planet into their toroidal and poloidal components:

$$\mathbf{v} = \mathbf{v}_T + \mathbf{v}_P, \quad \mathbf{B} = \mathbf{B}_T + \mathbf{B}_P \quad (4.5)$$

Here the subscript T refers to the toroidal component and the subscript P to the poloidal component, respectively. The equation of continuity (4.3) is automatically satisfied by the velocity field of the form (4.4). If convective cells are formed in the planetary core, the non-axisymmetric modes of the poloidal velocity field will also become important, but the average contributions from those modes can be included in the turbulent electromotive force term $\alpha \mathbf{B}_T$ and $\alpha \mathbf{B}_P$ as small deviations from axisymmetry (BRAGINSKII, 1975). Thus, for the axisymmetric part, we write down the toroidal and the poloidal parts of the induction and the Navier-Stokes equations as

$$\frac{\partial \mathbf{B}_T}{\partial t} = \text{rot}(\mathbf{v}_T \times \mathbf{B}_P + \mathbf{v}_P \times \mathbf{B}_T + \alpha \mathbf{B}_P) + \frac{1}{\mu\sigma} \nabla^2 \mathbf{B}_T \quad (4.6)$$

$$\frac{\partial \mathbf{B}_P}{\partial t} = \text{rot}(\mathbf{v}_P \times \mathbf{B}_P + \alpha \mathbf{B}_T) + \frac{1}{\mu\sigma} \nabla^2 \mathbf{B}_P \quad (4.7)$$

$$\rho \left\{ \frac{\partial \mathbf{v}_T}{\partial t} + (\mathbf{v}_T \cdot \nabla) \mathbf{v}_P + (\mathbf{v}_P \cdot \nabla) \mathbf{v}_T \right\} = \frac{1}{\mu} \text{rot} \mathbf{B}_T \times \mathbf{B}_P + 2\rho \mathbf{v}_P \times \boldsymbol{\Omega} \quad (4.8)$$

$$\rho \left\{ \frac{\partial \mathbf{v}_P}{\partial t} + (\mathbf{v}_T \cdot \nabla) \mathbf{v}_T + (\mathbf{v}_P \cdot \nabla) \mathbf{v}_P \right\} = -\nabla p + \frac{1}{\mu} (\text{rot} \mathbf{B}_P \times \mathbf{B}_P + \text{rot} \mathbf{B}_T \times \mathbf{B}_T) + 2\rho \mathbf{v}_T \times \boldsymbol{\Omega} + (\text{Buoyancy, Gravity, etc.}) \quad (4.9)$$

In the following analysis, we will study the case when the characteristic time scale τ of the dynamo action is sufficiently longer than the diffusion time of the magnetic field in the planetary core. This allows us to

omit the time-derivatives in (4.6) and (4.7). In (4.8) and (4.9) also, the time-derivatives can be neglected since $\tau \gg \Omega^{-1}$. The φ -component of the pressure gradient is omitted in the toroidal component of the equation of motion (4.8), since $\partial/\partial\varphi \equiv 0$ in the axisymmetric configuration.

We now estimate the order of magnitude of each equation. In (4.6), we will first assume that $v_T B_P \gtrsim v_P B_T$. We will also assume the $\alpha\omega$ -mechanism of the dynamo. In the formulation of an $\alpha\omega$ -dynamo problem, the poloidal part of the mean-electromotive force $\alpha \mathbf{B}_P$ is neglected compared with the ω -effect term. We then find from (4.6) that

$$\frac{1}{\ell} v_T B_P \sim \frac{1}{\mu\sigma} \frac{1}{\ell^2} B_T \gg \frac{B_T}{\tau}$$

i.e.

$$B_T \sim \mu\sigma \ell v_T B_P \quad (4.10)$$

All the values are those for the planet's core, and ℓ is the characteristic length of the conducting region of the planetary interior such as the core radius. Similarly, eq. (4.8) leads to

$$B_P B_T \sim 2\mu\rho \ell v_P \Omega \quad (4.11)$$

Let us take the magnetostrophic balance also in the poloidal component of equation of motion (4.9). This yields the following estimation:

$$(\nabla p \lesssim) \left| \frac{1}{\mu} (\text{rot} \mathbf{B}_P \times \mathbf{B}_P + \text{rot} \mathbf{B}_T \times \mathbf{B}_T) \right| \sim 2\rho v_T \Omega \quad (4.12)$$

The range of the Lorentz force can be estimated as:

$$\begin{aligned} & \{ \max(B_T^2, B_P^2) - \min(B_T^2, B_P^2) \} / \mu\ell \\ & \lesssim \left| \frac{1}{\mu} (\text{rot} \mathbf{B}_P \times \mathbf{B}_P + \text{rot} \mathbf{B}_T \times \mathbf{B}_T) \right| \\ & \lesssim \{ \max(B_T^2, B_P^2) + \min(B_T^2, B_P^2) \} / \mu\ell \end{aligned} \quad (4.13)$$

With the average of the left and the right hand sides as the characteristic order of magnitude of the middle hand side, we find

$$\left| \frac{1}{\mu} (\text{rot} \mathbf{B}_P \times \mathbf{B}_P + \text{rot} \mathbf{B}_T \times \mathbf{B}_T) \right| \sim \begin{cases} B_P^2 / \mu\ell & \text{for } (B_P \gg B_T) \\ B_P^2 / \mu\ell \sim B_T^2 / \mu\ell & \text{for } (B_P \sim B_T) \\ B_T^2 / \mu\ell & \text{for } (B_P \ll B_T) \end{cases} \quad (4.14)$$

Hence we obtain

$$2\mu\rho \ell v_T \Omega \sim B_P^2 \quad \text{for } (B_P \gtrsim B_T) \quad (4.15)$$

and

$$2\mu\rho \ell v_T \Omega \sim B_T^2 \quad \text{for } (B_P \lesssim B_T) \quad (4.16)$$

Although we do not totally exclude the possibility of the case $B_P \gg B_T$, we think it is unlikely, for it requires an extremely efficient α -effect to maintain a steady magnetic field. With the more probable cases of $B_P \sim B_T$ and $B_P \ll B_T$ (which can be unified as $B_P \lesssim B_T$), we find from equations (4.10), (4.11), and (4.16) that

$$v_T \sim \frac{2\rho\Omega}{\mu\ell(\sigma B_P)^2} \quad (4.17)$$

$$B_T \sim \frac{2\rho\Omega}{\sigma B_P} \quad (4.18)$$

$$v_P \sim \frac{1}{\mu\sigma\ell} \quad (4.19)$$

Expressions (4.17) and (4.19) lead to

$$\frac{B_T}{B_P} \sim \frac{v_T}{v_P} \left(\sim \frac{2\rho\Omega}{\sigma B_P^2} \right) \quad (4.20)$$

On the other hand, if we assume that $v_T B_P \lesssim v_P B_T$ in eq. (4.6), eq. (4.10) should be replaced by

$$B_T \sim \mu\sigma\ell v_P B_T \quad (4.21)$$

which again leads to the same v_P as given in (4.19), and hence to the same v_T and B_T as derived from eq. (4.6). Thus we find that $v_T B_P \sim v_P B_T$ holds with the expressions of v_T , B_T , and v_P given by (4.17), (4.18), and (4.19), respectively. The result (4.19) implies that the poloidal (i.e. convective) velocity in the planetary core remains of the order of the diffusion velocity of magnetic field. This is due to the back-reaction from the magnetic field; if the poloidal velocity field enhances by chance, it will induce larger toroidal magnetic field which, through back-reaction, acts to squeeze the increased poloidal velocity field. In the steady state, therefore, a poloidal velocity field of the order of the magnetic diffusion velocity shall be realized.

The ratio of Lorentz force to Coriolis force is measured by the Elsasser number (ELSASSER, 1946) defined by

$$El = \frac{\sigma B^2}{\rho\Omega} \quad (4.22)$$

In the same manner we can introduce a dimensionless number

$$El_P = \frac{\sigma B_P^2}{\rho\Omega} \sim \frac{|\sigma(\mathbf{v}_P \times \mathbf{B}_P) \times \mathbf{B}_P|}{|\rho\mathbf{v}_P \times \Omega|} \sim \frac{|\mathbf{j}_T \times \mathbf{B}_P|}{|\rho\mathbf{v}_P \times \Omega|} \quad (4.23)$$

which may be termed as the *poloidal* Elsasser number. Using this dimensionless number, the ratio B_T/B_P is expressed as $B_T/B_P \sim v_T/v_P \sim 2/El_P$, and the condition $B_P \lesssim B_T$ can be written as $El_P \lesssim 1$ (the factor of 2 was omitted).

For the Earth we use the following values for the parameters:

$$\begin{aligned} \mu &\sim 1.3 \times 10^{-6} [\text{NA}^{-2}], & \sigma &\sim 5 \times 10^5 [\text{Sm}^{-1}], \\ \rho &\sim 1 \times 10^4 [\text{kgm}^{-3}], & \Omega &\sim 7.3 \times 10^{-5} [\text{s}^{-1}], \\ \ell &\sim 3.5 \times 10^6 [\text{m}], & B_P &\sim 3 \times 10^{-4} [\text{T}] \end{aligned} \quad (4.24)$$

A little higher value of the electric conductivity of the core was used than was given in HONKURA and MATSUSHIMA (1988). The estimated values are

$$\begin{aligned} v_P &\sim 4 \times 10^{-7} [\text{ms}^{-1}], & v_T &\sim 1 \times 10^{-5} [\text{ms}^{-1}], \\ B_T &\sim 1 \times 10^{-2} [\text{T}], & El_P &\sim 0.062, & \frac{B_T}{B_P} &\sim 32 \end{aligned} \quad (4.25)$$

The toroidal magnetic field intensity in the Earth's core, estimated above, is by one order of magnitude larger than that of the poloidal magnetic field. To be noted here is that the velocity field derived above are somewhat smaller than the values presented in the previous works both based on frozen-flux approximation (MUTH and BENTON, 1981; MADDEN and LEMOUEL, 1982; GIRE, *et al.*, 1986; VOORHIES, 1986) and based on $\alpha\omega$ -mechanism (RIKITAKE, 1967; MATSUSHIMA and HONKURA, 1988, 1989). One of the reasons for this disagreement may be the incorporation of the back-reaction of the magnetic field which we take into account. Also the assumption of steady solution may have led to underestimate.

4-3 Scaling law for the planetary magnetism

We have so far dealt with B_P as one of the known parameters, and have expressed v_T , v_P , and B_T using basic parameters including B_P . Next, we try to give a scaling law for B_P of the planets. From the remaining equation (4.7) we obtain

$$B_P \sim \mu\sigma\ell\alpha B_T \quad (4.26)$$

The term $\mathbf{v}_P \times \mathbf{B}_P$ is omitted since this term is known to make no contribution to the dynamo action in the axisymmetric configuration (COWLING, 1934; BRAGINSKII 1964). With the relation (4.10) and the B_T given by (4.16), we find

$$\alpha \sim \frac{1}{v_T(\mu\sigma\ell)^2} \sim \frac{B_P^2}{2\mu\rho\ell\Omega} \quad (4.27)$$

By substitution of appropriate numerical values for the Earth we estimate that $\alpha \sim 1 \times 10^{-8} [\text{ms}^{-1}]$ in the Earth's core. Since the coefficient α reflects the characteristics of turbulent, small-scale motions in the core, it is assumed here that α is not directly dependent on the large-scale parameters such as ℓ , v_T , v_P , except on Ω which, through Coriolis force acting on the small-scale motions, may give rise to non-vanishing helicity which would generate the mean electromotive force of the form $\alpha\mathbf{B}_T$ (PARKER, 1955; STEENBECK and KRAUSE, 1969; see also MOFFATT, 1978). And since we have obtained by now no direct information on the characteristic scales (spatial and temporal) of the turbulent motions in the planetary core, we have nothing but to assume also that those scales coincide for every planet. If we always take Ω as positive, we can take α as a function of Ω like $\alpha = \alpha(\Omega)$, rather than a function of $|\Omega|$. And if $\alpha(\Omega)$ vanishes at $\Omega = 0$, the leading term in the Taylor-series expansion of α with respect to Ω leads to $\alpha \propto \Omega$, i.e.

$$\frac{\alpha^P}{\alpha^E} = \frac{\Omega^P}{\Omega^E} \quad (4.28)$$

where the superscripts P and E refer to the planetary and the terrestrial values, respectively. For the characteristic dipole moment M defined by

$$M \sim B_P \cdot \ell^3 \sim \ell^3 \sqrt{2\mu\rho\ell\alpha\Omega} \quad (4.29)$$

we derive a scaling law:

$$\frac{M^P}{M^E} = \left(\frac{\ell^P}{\ell^E} \right)^{7/2} \left(\frac{\rho^P}{\rho^E} \right)^{1/2} \left(\frac{\Omega^P}{\Omega^E} \right) \quad (4.30)$$

Since the expression (4.27) does not involve the electric conductivity σ of the planet's core, the scaling law (4.30) also becomes independent of σ , provided that the system can be regarded as being steady in time, and that the relation (4.20) holds. Of course the result should be modified if the mean electromotive force α depends on the electric conductivity σ . However, KRAUSE and RÄDLER (1980) have shown that the coefficient α does not depend on σ when the diffusion time is sufficiently longer than the correlation time of the small-scale turbulence. Therefore we regard that the coefficient α is approximately independent of the electric conductivity in the planet's core.

4-4 Discussion

To compare the dipole moments of the planets, it is necessary to know the values of the characteristic length ℓ and the mean density ρ for the conducting region of each planet. For the terrestrial planets we assume that the core mean density is the same as that of the Earth. For the giant planets, we take the mean density in the conducting region as 4.0, 2.5, 4.0, and $5.0 [\times 10^3 \text{kgm}^{-3}]$ respectively for Jupiter, Saturn, Uranus, and Neptune (NESS, 1978; PHILLIPS and MALIN, 1983; TSEKLEVICH *et al.*, 1983; STEVENSON, 1982; GUDKOVA *et al.*, 1988; HUBBERD *et al.*, 1991). For the terrestrial planets we take ℓ equal to the core radius. The giant planets have central rocky core which, as opposed to the Earth's inner core, is thought to be electrically insulating. In this case the scaling law should be

$$M \sim (\ell_{\text{out}} - \ell_{\text{in}})^{1/2} \ell_{\text{out}}^3 \rho^{1/2} \Omega \quad (4.31)$$

	Me	V	E	Ma	J	S	U	N
Radius of rocky core	—	—	—	—	4.1	4.3	2.2	1.4
Core radius	0.53	0.93	1.00	0.47	16	8.7	5.3	4.3
Core mean density	1.0	1.0	1.0	0.40	0.25	0.40	0.40	0.50
Angular velocity	0.017	0.004	1.000	0.972	2.409	2.246	1.537	1.299
El_P	0.002	1.4×10^{-4}	0.062	1.5×10^{-6}	5.5	0.22	0.2	0.03
$\gamma \equiv 2/El_P$	1000	1.4×10^4	32	1.3×10^6	0.36	9.0	10	62
$\log M$ (observed)	-3.30	-2.62†	0.00	-3.60†	4.27	2.76	1.69	1.40
$\log M$ (predicted)	-2.8	-2.5	0.0	-1.1	4.3	3.2	2.4	2.1

(† upper limit)

Table 5. The planetary parameters normalized to the Earth's values, and values of the dimensionless number El_P and $\gamma \equiv 2/El_P$. Radii of rocky core of the giant planets are normalized by the Earth's core radius.

where ℓ_{out} is the core radius and ℓ_{in} the radius of the central rocky core, $\ell_{out} - \ell_{in}$ being the radial extent of the metallic hydrogen/ice region. The values of the fundamental parameters normalized to the Earth's values are listed in Table 5 together with the values of the non-dimensional parameter El_P . From the top, radius of the central rocky core ℓ_{in} , core radius ℓ_{out} , core mean density ρ , angular velocity Ω , dimensionless number El_P , toroidal/poloidal ratio $\gamma \equiv 2/El_P$, logarithm of the dipole moments of the planetary magnetic field based on observations, and based on the present scaling law, respectively. A common σ of 5×10^5 [Sm⁻¹] for every planet was assumed during the calculation of El_P in this table. Using these parameters, the predicted dipole moments of the planets are plotted against the dipole moments determined from observations in Figure 7.

The dipole moments of the planets are based on the spacecraft observations (NESS, 1979; RUSSELL *et al.*, 1980; CONNERNEY, 1981; CONNERNEY *et al.*, 1982; CONNERNEY and ACUÑA, 1987; NESS *et al.*, 1989). Upper limits are shown for Venus and Mars. The thick line of 45° declination shows predictions coinciding with observations. We recognize in the figure that the scaling law gives satisfactory agreement with the observations in spite of the present uncertainty on the internal structure of the planets other than the Earth. One exception is the case of Mars, which is supposed to be inactive in dynamo operation (CURTIS and NESS, 1986). Although Venus is usually regarded as having no dynamo, the predicted dipole moment of Venus becomes close to the upper limit of its inferred intrinsic field. If the coefficient α depends directly on Ω as is assumed in the present study, the extremely weak intrinsic field of Venus might be attributed to its very slow rotation, and the possibility of dynamo action at Venus cannot be excluded yet. If this is the case, Venus is found to be a planet which presumably possesses an intrinsic toroidal magnetic field inside the core, but presenting almost unobservably weak poloidal magnetic field. In fact, the poloidal magnetic field at Venus is estimated as 14000 times weaker than the toroidal field, as one can see in Table 5. Although the assumption of axisymmetry seems to be too crude for the very eccentric dipoles at Uranus and Neptune, the predicted values are still in good agreement with the observations. However, a closer inspection of the figure shows that there is a tendency of overestimation for the predictions of the scaling law compared with the observations, which is probably because the electric currents do not necessarily extend over the whole planetary core in the strict sense and we may overestimate the value of $\ell_{out} - \ell_{in}$. This tendency seems more relevant for Uranus and Neptune, which may be due to their eccentricity of the dipole locations.

There has been a criticism against the existence of scaling laws that a scaling law which is not explicitly dependent on the strength of the driving force is questionable from a physical point of view, for the activity of a dynamo should be inevitably related to some kind of energy supply. The scaling law of CURTIS and NESS (1986) may have been evaluated with this criticism taken into account. The present analysis is based on an idea that a scaling law which does not depend explicitly on the strength of the driving force may be also possible for non-linear MHD dynamos. The energy supply should exceed the required minimum

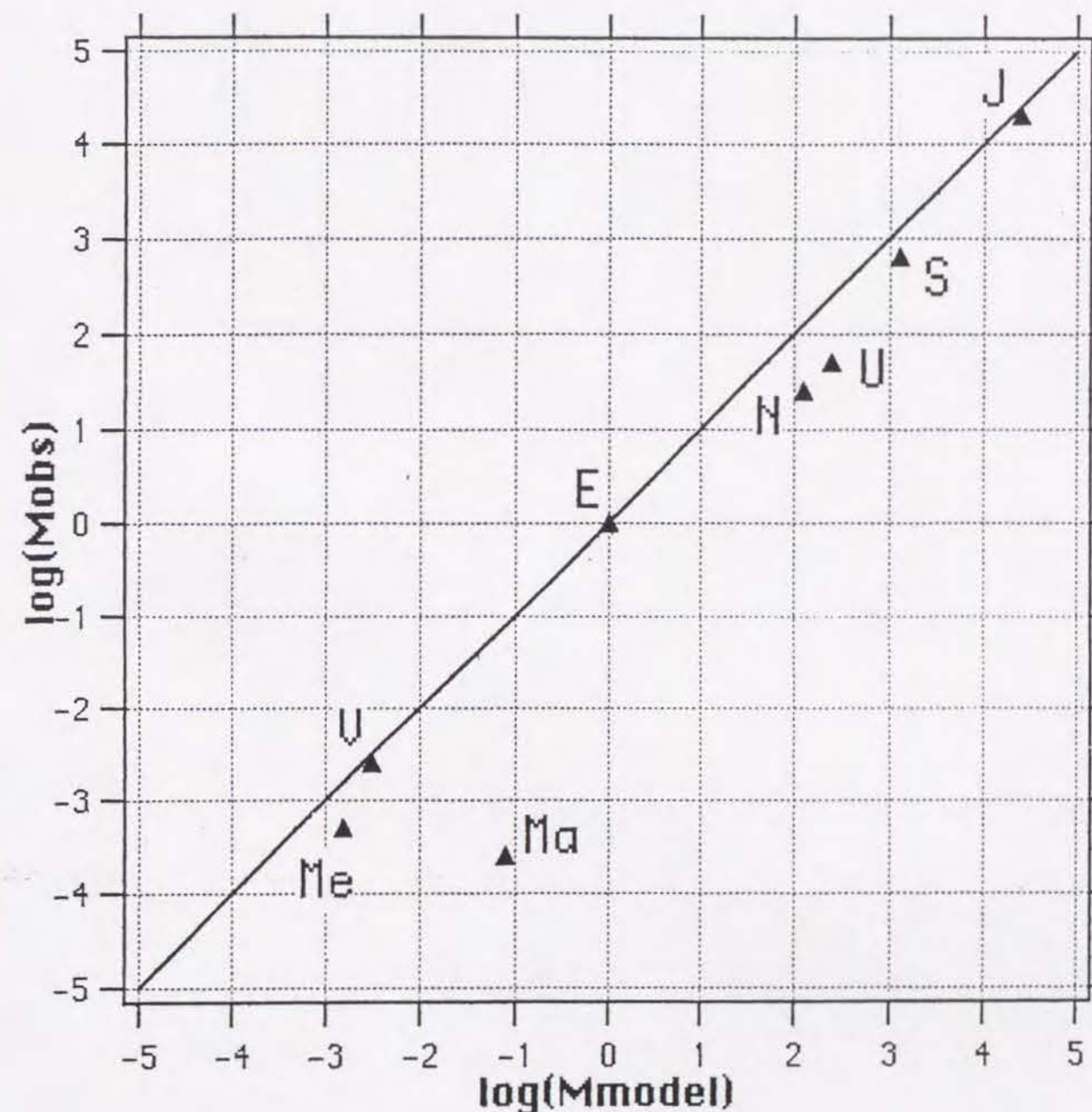


Fig. 7. The predicted planetary dipole moments against the dipole moments determined from observations. Abbreviations used are Me:Mercury, V:Venus, E:Earth, Ma:Mars, J:Jupiter, S:Saturn, U:Uranus, and N:Neptune. Upper limits are shown for Venus and Mars.

energy to maintain a dynamo against ohmic loss. The magnetic field, in the author's opinion, can grow only until some saturation level if the dynamo is of non-linear MHD type. And if the rate of energy supply exceeds the Joule loss rate only by a small amount, this saturation level may change sensitively with the change of the energy supply. This means that the Lorentz force and the driving force nearly balances in this case. However, if the rate of energy supply is sufficiently greater than the rate of energy consumption at dynamo, the saturation level may be set at a level which is not critically dependent on the energy supply any more. Only a part of energy input is converted to the energy of magnetic field; other remaining part of the energy input has nothing to do with the activity of dynamo. Presumably, what balances the Lorentz force is not the driving force itself, but Coriolis force for this type of dynamo. If this is the case, a scaling law may be acceptable which does not involve some parameters concerning the energy input explicitly.

As is listed in Table 1, the condition $El_P \lesssim 1$ seems to be true for the planets other than Jupiter. It is, however, unlikely that this condition holds in the case of Jupiter which, if $\sigma \gtrsim 1 \times 10^5 [\text{Sm}^{-1}]$, gives $El_P \gtrsim 1$, and contradicts the assumption $B_P \lesssim B_T$. One possibility is that $B_P \gtrsim B_T$ is realized in the Jovian core. Another (and maybe more probable) possibility is that other forces such as buoyancy or pressure gradient become important in the Jovian core as well as the Lorentz and the Coriolis forces (may be termed as a 'supermagnetostrophic balance').

Thus, El_P is found to be one of the factors which determine the kind of dynamo — MHD or kinematic. From the definition of El_P (Lorentz force/Coriolis force), it might be thought that a non-linear MHD dynamo would be set up when $El_P > 1$. We, however, take the opposite interpretation; Lorentz force dominates Coriolis force since the dynamo is kinematic. Some kind of driving mechanism such as thermal convection is required to maintain a dynamo. If the driving force dominates over both the Lorentz and the Coriolis forces (a kinematic dynamo), then the poloidal velocity will exceed the diffusion velocity of the magnetic field, will twist the magnetic field lines strongly beyond the equilibrium between the Lorentz and the Coriolis forces, and will generate stronger magnetic fields than in the MHD case. For those kinematic dynamos, the generated magnetic field is approximately a linear functional of the velocity field. So, if the Lorentz force is a square functional of the velocity field, El_P giving the ratio of Lorentz force to Coriolis force becomes approximately proportional to the magnitude of the velocity field. This implies a relation $|\text{driving force}| > |\text{Lorentz force}| \gtrsim |\text{Coriolis force}|$, which leads to $El_P \gtrsim 1$ for kinematic dynamos of this kind. In the opposite case $|\text{driving force}| < |\text{Coriolis force}|$, the magnetic field can grow only until the Lorentz force balances the Coriolis force, so we will have $El_P \lesssim 1$. This means that the magnetic field settles to its saturation level in a nonlinear MHD case. The poloidal velocity field is suppressed by the magnetic field to the order of magnitude of the diffusion velocity of the magnetic field. For these dynamos, the power of Coriolis force integrated over the source region is of the same order of magnitude as the Joule loss rate of the energy of the magnetic field:

$$2\rho v_T v_P \Omega \cdot \frac{4\pi}{3} \ell^3 \sim \frac{\frac{1}{2\mu} B_T^2 \times \frac{4\pi}{3} \ell^3}{\mu \sigma \ell^2} \quad (4.32)$$

which is estimated as $\sim 8 \times 10^8 [\text{W}]$ for the Earth.

In the kinematic case where $El_P \gtrsim 1$, the driving forces such as buoyancy may need an explicit treatment as opposed to the previous MHD case. In such a kinematic case, the velocity field is first determined from the equation of motion independent of the magnetic field, and then the magnetic field is solved regarding the velocity field as given. Dynamos of such kind are investigated by a lot of workers (for detail, see MOFFATT, 1978; KRAUSE and RÄDLER, 1980). The regular polarity reversals of the solar magnetic field (MAUNDER, 1914; HALE and NICHOLSON, 1925) are known to be well reproduced by the $\alpha\omega$ -dynamo model (STEENBECK and KRAUSE, 1969). On the other hand, the two-disk dynamo model (RIKITAKE, 1958) and the associated models (e.g. SHIMIZU and HONKURA, 1986) imply that a highly nonlinear feature appears when the effect of Lorentz force is involved. Hence irregular polarity reversals are expected for an MHD dynamo. If the dimensionless number El_P specifies the kind of dynamos, this parameter can be taken as expressing the nature of polarity reversals of the concerned dynamo; for $El_P \gtrsim 1$ regular reversals, and

BLACKETT (1947)	$M \sim L$
BUSSE (1976)	$M \sim \rho^{1/2} \ell^4 \Omega$
STEVENSON (1983)	$M \sim \begin{cases} \rho^{1/2} \ell^3 \Omega^{1/2} \sigma^{-1/2} \\ \rho^{1/2} v \ell^3 R_O^{-1/2} \end{cases}$
CURTIS and NESS (1986)	$M \sim \rho^{1/3} \ell^{7/2} \Omega^{1/2} E^{1/6}$
ISAS-1 (1992)	$M \sim \rho^{1/2} \ell^{7/2} \Omega^{3/4} \sigma^{-1/4}$
ISAS-2 (1992)	$M \sim \rho^{1/2} \ell^3 \Omega^{1/2} \sigma^{-1/2}$
Present study	$M \sim \rho^{1/2} \ell^{7/2} \Omega$

Table 6. Comparison between the several major scaling laws. M denotes the dipole moment, L planetary angular momentum, ρ core mean density, ℓ core radius, Ω angular velocity of the planet, E latent heat flux, σ electric conductivity of the core, v characteristic velocity in the core, and R_O Rossby number.

for $El_P \lesssim 1$ irregular reversals. It may be possible in this sense that the magnetic field at Jupiter would experience rather regular polarity reversals while irregular reversals for other planets.

Table 6 lists several major scaling laws proposed by far. Most of the scaling laws apply to the observations to a certain extent, but not completely. Unfortunately, it seems difficult to judge between those scaling laws due to the present uncertainties on the internal structure of the planetary cores. Ω is very precisely known for every planet, but other parameters still suffer great uncertainties. Even the most "primitive" scaling law, (dipole moment) versus (angular momentum), roughly gives a proportionality. Close to the present scaling law are scaling law of Busse (1976) $M \propto \rho^{1/2} \ell^4 \Omega$ and ISAS-1 scaling law $M \propto \rho^{1/2} \ell^{7/2} \Omega \sigma^{-1/4}$ if we assume the same electric conductivity for every planet. Busse's scaling law gives a good fit to the fields of the terrestrial planets except Mars (in any scaling law model Mars does not seem to agree with prediction), but it would give a considerable overestimation to the fields of the giant planets even if the scaling law were modified as $M \propto (\ell_{\text{out}} - \ell_{\text{in}}) \ell_{\text{out}}^3 \Omega$ from the original scaling law. ISAS-1 scaling law does not apply to the fields of Mercury and Venus (and Mars) although it well predicts the fields of the giant planets provided that the electric conductivity of the core coincides for every (giant) planet. Present scaling law gives on the average a good fit both to the fields of the terrestrial and of the giant planets.

4-5 Conclusion

A scaling law for the planetary magnetism is proposed. We adopt a steady $\alpha\omega$ -dynamo model, and assume the magnetostrophic balance both for the poloidal and the toroidal parts of equation of motion. We also assume that the mean electromotive force arising from turbulence in the planetary core is dependent on the angular velocity Ω of the planet's rotation through a relation $\alpha \propto \Omega$. Based on the present scaling law, the dipole moment M of a planet is given by the combination of three basic parameters ρ , ℓ , and Ω ; the planetary dipole moment scales as $M \propto \ell^{7/2} \rho^{1/2} \Omega$. The predictions based on the scaling law agree well with the observations except Mars. It is also found that the state of the whole system is specified by the value of a dimensionless parameter El_P defined by $El_P \equiv \sigma B_P^2 / \rho \Omega$. When $El_P \lesssim 1$, the system is in a magnetostrophic balance, i.e. the balance between Lorentz and Coriolis forces would hold. In this case, the ratio of the toroidal magnetic field to the poloidal magnetic field is estimated to be $2El_P^{-1}$, which is also roughly equal to the ratio of the toroidal velocity field to the poloidal velocity field. In case $El_P \gtrsim 1$, the system may be in a supermagnetostrophic balance in which Coriolis force alone cannot sustain Lorentz force. The scaling law suggests a possibility of supermagnetostrophic balance in the Jovian core.

4-6 Appendix

Let $\mathbf{B}$ be a three dimensional vector field of position and time satisfying $\text{div } \mathbf{B} = 0$. Any of its vector potentials be denoted by $\mathbf{A}_0$. On the spherical polar coordinates (r, θ, φ) , $\mathbf{A}_0$ is expressed as

$$\mathbf{A}_0 = A_{0r}\mathbf{e}_r + A_{0\theta}\mathbf{e}_\theta + A_{0\varphi}\mathbf{e}_\varphi \quad (\text{A.1})$$

where $\mathbf{e}_i$, ($i = r, \theta, \varphi$) denotes the unit vector in the direction of increasing i . Toroidal field and poloidal field are the vector fields of the form of $\text{rot } (\Psi\mathbf{r})$ and $\text{rot rot } (\Phi\mathbf{r})$, respectively. This implies that the vector potential can be taken as having the form of

$$\mathbf{A} = \Psi\mathbf{r} + \text{rot } (\Phi\mathbf{r}) = (\Psi r)\mathbf{e}_r + \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \varphi} \mathbf{e}_\theta - \frac{\partial \Phi}{\partial \theta} \mathbf{e}_\varphi \quad (\text{A.2})$$

However, $\mathbf{A}_0$, taken arbitrary, does not generally have the form of (A.2).

There exists an arbitrariness of the gradient of a scalar function f for the vector potential. We then introduce a new vector potential $\mathbf{A}$ by

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_0 + \nabla f \\ &= (A_{0r} + \frac{\partial f}{\partial r})\mathbf{e}_r + (A_{0\theta} + \frac{1}{r} \frac{\partial f}{\partial \theta})\mathbf{e}_\theta + (A_{0\varphi} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi})\mathbf{e}_\varphi \\ &\equiv A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\varphi\mathbf{e}_\varphi \end{aligned} \quad (\text{A.3})$$

The condition on this vector potential $\mathbf{A}$ to have the the same form as in (A.2) is

$$\frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{\partial}{\partial \varphi} (A_\varphi) = 0 \quad (\text{A.4})$$

which can be rewritten as

$$\Delta_2 f = -\text{div}_2 \mathbf{A}_0 \quad (\text{A.5})$$

where

$$\Delta_2 f \equiv \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta \frac{\partial f}{\partial \theta}) + \frac{1}{\sin \theta} \frac{\partial^2 f}{\partial \varphi^2} \right] \quad (\text{A.6})$$

and

$$\text{div}_2 \mathbf{A}_0 \equiv \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_{0\theta}) + \frac{\partial A_{0\varphi}}{\partial \varphi} \right] \quad (\text{A.7})$$

It is probably without problem to assume that the right hand side of (A.5) is expandable into a series of spherical harmonics $Y_n^m(\theta, \varphi)$ in dealing with the vector potentials of the planetary magnetic field:

$$\text{R.H.S. of (A.5)} \equiv \sum u_n(r) Y_n^m(\theta, \varphi) \quad (\text{A.8})$$

Eq. (A.5) guarantees the existence of the scalar function

$$f = \sum \frac{r u_n(r)}{n(n+1)} Y_n^m(\theta, \varphi) \quad (\text{A.9})$$

as long as the right hand side of (A.5) is expandable into a series of spherical harmonics.

Defining

$$\Phi \equiv - \int_0^\theta \left(a_{0\varphi} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \right) d\theta = \int_0^\varphi \left(a_{0\theta} + \frac{1}{r} \frac{\partial f}{\partial \theta} \right) d\varphi \quad (\text{A.10})$$

and

$$\Psi \equiv \frac{1}{r} \left(a_{0r} + \frac{\partial \Phi}{\partial r} \right) \quad (\text{A.11})$$

the vector potential $\mathbf{A}$ is expressed as

$$\mathbf{A} = \Psi\mathbf{r} + \text{rot } (\Phi\mathbf{r}) \quad (\text{A.12})$$

which gives

$$\mathbf{B} = \text{rot } (\Psi\mathbf{r}) + \text{rot rot } (\Phi\mathbf{r}) \quad (\text{A.13})$$

Chapter V. General Conclusion

5-1 General Conclusion

Magnetic fields of the Earth and of the planets are studied. The concept of the eccentric dipole is re-examined, and a new definition is presented leading to an eccentric dipole which would best fit to all the harmonics of a given distribution of magnetic potential. It is shown that the moment of this dipole should be varied in the process of least squares fitting, although the invariance of dipole moment certainly holds for any distributions of magnetic field. The new definition of an eccentric dipole is applied to the geomagnetic field data between 1945-1985, and the optimum dipoles are determined under several different minimization criteria for the least squares fitting. The locations of the newly determined dipoles are apart from the conventional eccentric dipole position in the longitudinal (4 to 23 degrees), latitudinal (4 to 12 degrees), and radial (less than ± 20 km) directions, but the manner of their secular variations is basically the same as that of the conventional eccentric dipole. This suggests that the higher harmonics above the quadrupole, which are newly included into the definition in this study, mainly affect the spatial (rather than the temporal) characteristics of the geomagnetic field.

Subsequently, planetary magnetic fields are investigated. Since the planetary magnetic fields are not measured in such detail as the geomagnetic field, we restricted ourselves to the centered dipole of the planets, and derived a simple formula (termed as 'a scaling law') which gives relative ratios of the magnitudes of the dipole moments of the planets. During the derivation of this formula, an expression is also derived estimating the toroidal magnetic field intensity which is unobservable from outside the planet's core; the result implies a typical toroidal magnetic field intensity of 100 [G], for example, in the Earth's core. Since the present study is based on an $\alpha\omega$ -dynamo model, the resultant scaling law depends on the efficiency of the α -effect. If we adopt the dependence of the form $\alpha \propto \Omega$ (with Ω the angular velocity of the planet's self-rotation), the magnetic dipole moment M of a planet scales as $M \propto (\text{characteristic length})^{7/2} (\text{mean density})^{1/2} (\text{angular velocity})$. The predictions agree well with the observations except Mars.

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