

On embedding lifts over a Morse function on a circle

Dedicated to Professor Shyuich Izumiya on the occasion of his 60th birthday

By

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Abstract

Let $f : S^1 \rightarrow \mathbb{R}$ be a Morse function and $\Pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ an orthogonal projection. In [4], Saeki and Takase posed the following problem: “Determine those Morse functions $f : S^1 \rightarrow \mathbb{R}$ which have an embedding $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ such that $\Pi \circ \tilde{f} = f$ ”. In this paper, we give a complete answer to this problem and give an application to the existence problem of embedding lifts for fold maps.

§ 1. Introduction

Throughout the paper, all manifolds and maps are differentiable of class C^∞ . Let $f : S^1 \rightarrow \mathbb{R}$ be a function on an oriented circle. A point $q \in S^1$ is a *singular point* of f if the differential df_q at q vanishes. We denote by $S(f)$ the set of singular points of f . For $q \in S(f)$, we call $f(q)$ a *singular value* of f . A function $f : S^1 \rightarrow \mathbb{R}$ is a *Morse function* if for each $q \in S(f)$, there exist local coordinates x around $q \in S^1$ and y around $f(q) \in \mathbb{R}$ such that f has the form $y \circ f = \pm x^2$. If the singular values of a Morse function f are all distinct, then we call f a *stable Morse function*.

Let $f : S^1 \rightarrow \mathbb{R}$ be a Morse function and $\Pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ the orthogonal projection defined by $\Pi(y_1, y_2) = y_1$. Saeki and Takase [4] pose the following problem: Determine those Morse functions $f : S^1 \rightarrow \mathbb{R}$ which have an embedding $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ such that $\Pi \circ \tilde{f} = f$. We call such a map $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ an *embedding lift* over f . In this paper, we give a complete answer to this problem.

This paper is organized as follows. In Section 2, we give a necessary and sufficient condition for the existence of an embedding lift $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ over a given Morse function

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$f : S^1 \rightarrow \mathbb{R}$. In Section 3, as an application, for each interger $n > 1$, we give an example of a stable fold map $f : S^n \rightarrow \mathbb{R}^n$ of the n -dimensional sphere which has an immersion lift $f' : S^n \rightarrow \mathbb{R}^{n+1}$, but which does not have an embedding lift into $\mathbb{R}^{n+1}$.

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§ 2. Embedding lift over a Morse function

In this section, we study the existence of an embedding lift $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ over a given Morse function $f : S^1 \rightarrow \mathbb{R}$.

Definition 2.1. Let $f : S^1 \rightarrow \mathbb{R}$ be a Morse function on an oriented circle. Let $\{p_1, p_2, \dots, p_k\}$, $k \geq 1$ (resp. $\{q_1, q_2, \dots, q_l\}$, $l \geq 1$) be the set of those points in S^1 where f takes its minimum (resp. maximum). If, by using the orientation of S^1 , $p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_l$ are arranged as $p_1 < p_2 < \dots < p_k < q_1 < q_2 < \dots < q_l$ in a cyclic sense on S^1 (this means that if we start p_1 and go along S^1 in the direction given by the orientation, we encounter $p_2, p_3, \dots, p_k, q_1, q_2, \dots, q_l$ in this order and then p_1), we say that $\{p_1, p_2, \dots, p_k\}$ and $\{q_1, q_2, \dots, q_l\}$ are *separated*.

Then, we have the following theorem which answers the problem posed in [4].

Theorem 2.2. Let $f : S^1 \rightarrow \mathbb{R}$ be a Morse function on an oriented circle. Let $\{p_1, p_2, \dots, p_k\}$, $k \geq 1$ (resp. $\{q_1, q_2, \dots, q_l\}$, $l \geq 1$) be the set of those points in S^1 where f takes its minimum (resp. maximum). Then there exists an embedding lift $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ over f if and only if $\{p_1, p_2, \dots, p_k\}$ and $\{q_1, q_2, \dots, q_l\}$ are separated.

Proof. Suppose that $\{p_1, p_2, \dots, p_k\}$ and $\{q_1, q_2, \dots, q_l\}$ are separated. We may assume that by using the orientation of S^1 , $p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_l$ are arranged as $p_1 < p_2 < \dots < p_k < q_1 < q_2 < \dots < q_l$ in a cyclic sense on S^1 . We cut S^1 at p_1 and q_1 . Let A_1 (resp. A_2) be the oriented arc in S^1 whose initial point is p_1 (resp. q_1) and whose terminal point is q_1 (resp. p_1). First, we can construct an embedding lift $\tilde{f}_1 : A_1 \rightarrow \mathbb{R}^2$ over $f|_{A_1}$ which satisfies the following properties (see Figure 1(1)).

1. $\tilde{f}_1(A_1) \subset \mathbb{R} \times [0, \infty)$.
2. $\tilde{f}_1(p_1) = (f(p_1), 0) \in \mathbb{R}^2$, $\tilde{f}_1(q_1) = (f(q_1), 0) \in \mathbb{R}^2$.
3. If we put the critical points of $f|_{A_1}$ as $p_1 = \alpha_0 < \alpha_1 < \dots < \alpha_{a-1} < \alpha_a = q_1$, then $\tilde{f}_1(\alpha_i) = (f(\alpha_i), i) \in \mathbb{R}^2$ holds ($i = 0, 1, \dots, a-1$).

Similarly, we can construct an embedding lift $\tilde{f}_2 : A_2 \rightarrow \mathbb{R}^2$ over $f|_{A_2}$ which satisfies the following properties (see Figure 1(2)).

1. $\tilde{f}_2(A_2) \subset \mathbb{R} \times (-\infty, 0]$.
2. $\tilde{f}_2(p_1) = (f(p_1), 0) \in \mathbb{R}^2, \tilde{f}_2(q_1) = (f(q_1), 0) \in \mathbb{R}^2$.
3. If we put the critical points of $f|_{A_2}$ as $q_1 = \beta_0 < \beta_1 < \cdots < \beta_{b-1} < \beta_b = p_1$, then $\tilde{f}_2(\beta_j) = (f(\beta_j), -j) \in \mathbb{R}^2$ holds ($j = 0, 1, \dots, b-1$).

Then, by attaching the two maps $\tilde{f}_1$ and $\tilde{f}_2$, we have an embedding $\tilde{f} = \tilde{f}_1 \cup \tilde{f}_2 : S^1 = A_1 \cup A_2 \rightarrow \mathbb{R}^2$. This $\tilde{f}$ is the desired embedding lift over f .

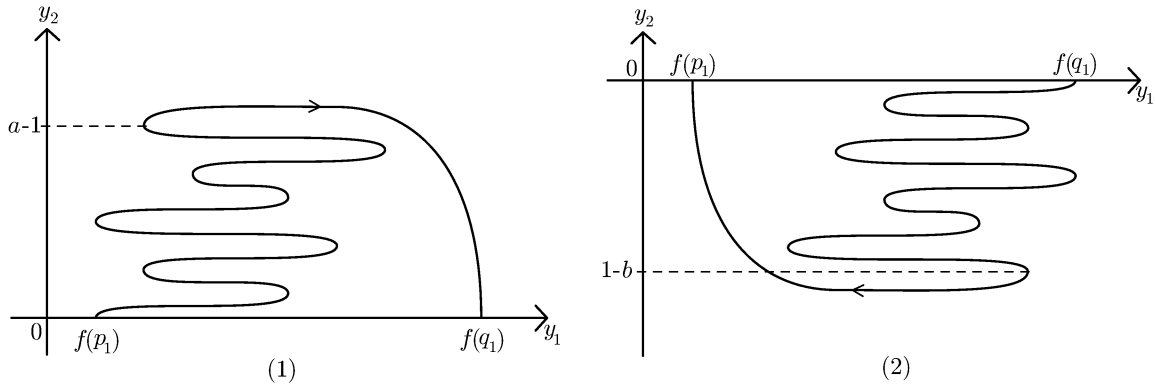


Figure 1. Embedding lifts (1) $\tilde{f}_1 : A_1 \rightarrow \mathbb{R}^2$ over $f|_{A_1}$ and (2) $\tilde{f}_2 : A_2 \rightarrow \mathbb{R}^2$ over $f|_{A_2}$.

Conversely, suppose that there exists an embedding lift $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$ over the given Morse function f . By the Jordan curve theorem, $\mathbb{R}^2 \setminus \tilde{f}(S^1)$ is the union of two disjoint connected open sets U_1 and U_2 . Let U_1 be the bounded set. We assume that the orientation of S^1 satisfies the following: when we walk along $\tilde{f}(S^1)$, we have U_1 to our right. Let $\{q_1, q_2, \dots, q_l\}$, $l \geq 1$, be the set of those points in S^1 where f takes its maximum and K_i a sufficiently small oriented arc around $\tilde{f}(q_i)$ in $\tilde{f}(S^1)$ such that the initial point of K_i is s_i and the terminal point of K_i is t_i ($i = 1, 2, \dots, l$). Since $U_1 \subset \Pi^{-1}((-\infty, f(q_1)])$ must be satisfied, by renumbering the indices we may assume the following properties (see Figure 2).

1. $f(s_1) = f(t_1) = f(s_2) = f(t_2) = \cdots = f(s_l) = f(t_l)$.
2. $(y_2\text{-coordinate of } s_1) > (y_2\text{-coordinate of } \tilde{f}(q_1)) > (y_2\text{-coordinate of } t_1) > \cdots > (y_2\text{-coordinate of } s_l) > (y_2\text{-coordinate of } \tilde{f}(q_l)) > (y_2\text{-coordinate of } t_l)$.

Then, we have the following lemma.

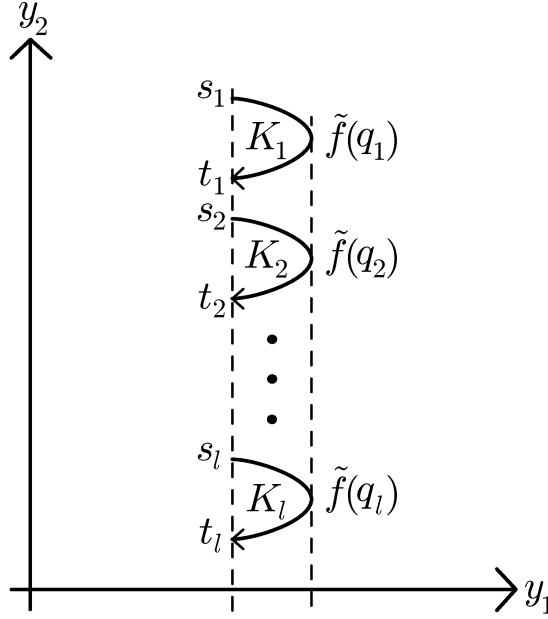


Figure 2. Arrangement of $\tilde{f}(q_i)$ and K_i ($i = 1, 2, \dots, l$) in $\mathbb{R}^2$.

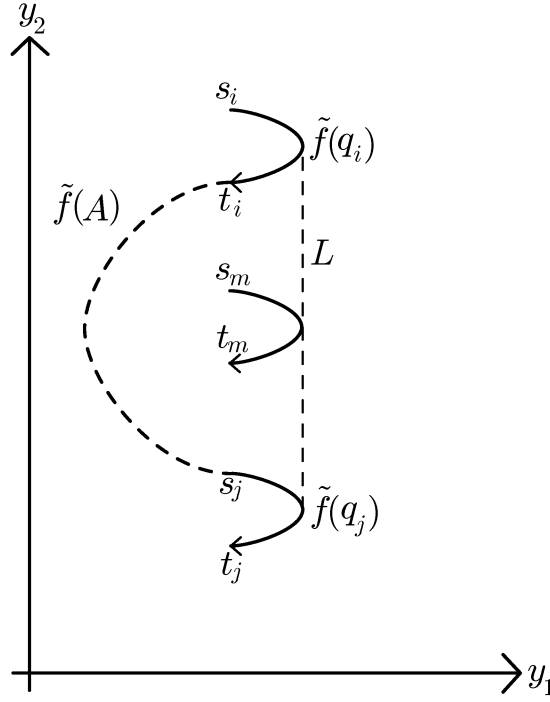
Lemma 2.3. *The maximum points of f necessarily satisfy $q_1 < q_2 < \dots < q_l$ in a cyclic sense on S^1 .*

Proof. Let i be an integer such that $1 \leq i < l$. Suppose that q_j is the first maximum point that we encounter after leaving q_i in the direction given by the orientation of S^1 and suppose that $j > i + 1$.

Let A be the oriented arc in S^1 whose initial point is q_i and whose terminal point is q_j and L a line segment in $\mathbb{R}^2$ which connects $\tilde{f}(q_i)$ and $\tilde{f}(q_j)$. Note that $\tilde{f}(A) \cup L$ is a simple closed curve, and let V be the open region bounded by $\tilde{f}(A) \cup L$. Then U_1 and V must satisfy $U_1 \cap V = \emptyset$: however, V contains the small oriented arc K_m ($i < m < j$) (see Figure 3). This is a contradiction. Therefore, $j \leq i + 1$ holds.

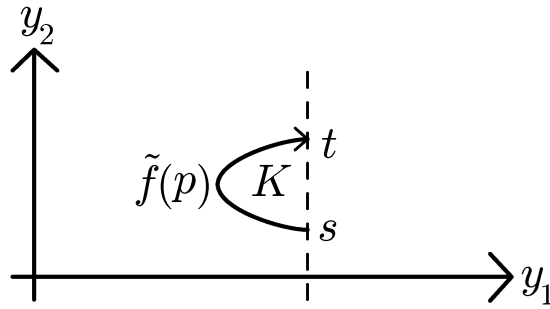
Let us consider the case $i = 1$. Since $j \leq 2$ and $q_1 \neq q_j$, we have $j = 2$. Then, for $i = 2$, since $j \leq 3$ and $q_2 \neq q_j$, we have $j = 1$ or $j = 3$. If $j = 1$, then we have exactly two maximum points on S^1 : therefore, $l = 2$, but in the first paragraph of the proof, we assumed that $i = 2 < l$. Then we have $j = 3$. We can repeat this argument until $i = l - 1$. This completes the proof of the lemma. $\square$

Now we go back to the proof of Theorem 2.2. Suppose that between two maximum points q_i and q_{i+1} , there exists a point p in S^1 where f takes its minimum and that q_i, q_{i+1}, p satisfy $q_i < p < q_{i+1}$ in a cyclic sense on S^1 for some $i = 1, 2, \dots, l - 1$. Suppose that p is the first minimum point that we encounter after leaving q_i in the

Figure 3. Arrangements of $\tilde{f}(A)$ and L .

direction given by the orientation. Let K be a sufficiently small oriented arc around $\tilde{f}(p)$ in $\tilde{f}(S^1)$ such that the initial point of K is s and the terminal point of K is t . Since $\Pi^{-1}([f(p), \infty)) \supset U_1$ must be satisfied, we may assume the following properties (see Figure 4).

1. $f(s) = f(t)$.
2. $(y_2\text{-coordinate of } s) < (y_2\text{-coordinate of } \tilde{f}(p)) < (y_2\text{-coordinate of } t)$.

Figure 4. Arrangement of $\tilde{f}(p)$ and K in $\mathbb{R}^2$.

Let $L_1 = \{(f(p), y_2) \in \mathbb{R}^2 \mid y_2 \geq (y_2\text{-coordinate of } \tilde{f}(p))\}$ and $L_2 = \{(f(q_i), y_2) \in \mathbb{R}^2 \mid y_2 \geq (y_2\text{-coordinate of } \tilde{f}(q_i))\}$ be half lines and A the oriented arc in S^1 whose initial point is q_i and whose terminal point is p . The proper arc $L_1 \cup \tilde{f}(A) \cup L_2$ separates $\mathbb{R}^2$ into two open regions. Since the point t is in one open region and the point s_{i+1} is in the other open region, the arc connecting t and s_{i+1} must cross the proper arc $L_1 \cup \tilde{f}(A) \cup L_2$ (see Figure 5). This is a contradiction.

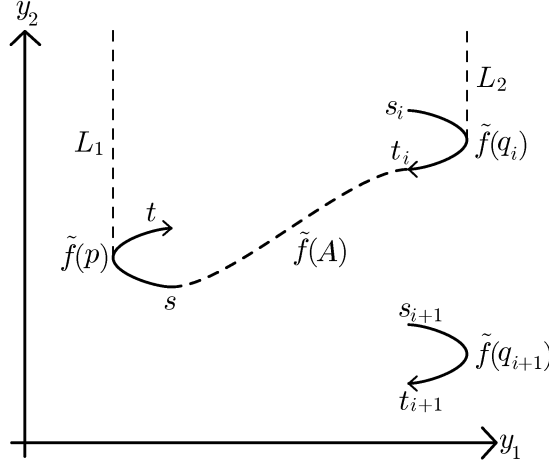


Figure 5. Arrangements of L_1, L_2 and $\tilde{f}(A)$.

Therefore, in the oriented arc in S^1 whose initial point is q_1 and whose terminal point is q_l , there does not exist a point where f takes its minimum. This means that the set of points in S^1 where f takes its minimum and the set of points in S^1 where f takes its maximum are separated. This completes the proof of Theorem 2.2. $\square$

Remark 2.4. If a Morse function $f : S^1 \rightarrow \mathbb{R}$ has exactly one minimum or exactly one maximum, then f has an embedding lift $\tilde{f} : S^1 \rightarrow \mathbb{R}^2$. In particular, every stable Morse function has an embedding lift.

§ 3. Application

In this section, as an application of Theorem 2.2, we study the existence of an embedding lift over a stable fold map of a closed n -dimensional manifold into $\mathbb{R}^n$ ($n > 1$).

Let M be a closed n -dimensional manifold and $f : M \rightarrow \mathbb{R}^n$ a smooth map. A point $q \in M$ is a *singular point* of f if the rank of the differential df_q at q is strictly smaller than n . We denote by $S(f)$ the set of singular points of f . A smooth map $f : M \rightarrow \mathbb{R}^n$ is a *fold map* if for each $q \in S(f)$, there exist local coordinates $(x_1, x_2, \dots, x_n)$ around

$q \in M$ and $(y_1, y_2, \dots, y_n)$ around $f(q) \in \mathbb{R}^n$ such that f has the form

$$\begin{cases} y_i \circ f = x_i & (i = 1, 2, \dots, n-1), \\ y_n \circ f = x_n^2. \end{cases}$$

We remark that for a fold map $f : M \rightarrow \mathbb{R}^n$, $S(f)$ is an $(n-1)$ -dimensional closed submanifold of M and $f|_{S(f)}$ is an immersion. We mention that the fold maps that appear here are exactly the same as the special generic maps as defined, for example, in [4]. If a fold map $f : M \rightarrow \mathbb{R}^n$ satisfies that $f|_{S(f)}$ is an immersion with normal crossings, then we call f a *stable fold map* (see [1]).

Let $\Pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be an orthogonal projection and $f : M \rightarrow \mathbb{R}^n$ a stable fold map. If there exists an embedding $\tilde{f} : M \rightarrow \mathbb{R}^{n+1}$ such that $\Pi \circ \tilde{f} = f$, then we say that f has an *embedding lift* $\tilde{f}$. Note that if M is orientable, any fold map $f : M \rightarrow \mathbb{R}^n$, has an immersion $f' : M \rightarrow \mathbb{R}^{n+1}$ such that $\Pi \circ f' = f$ (see [5]).

Let $f : M \rightarrow \mathbb{R}^n$ be a stable fold map and $l \subset \mathbb{R}^n$ an embedded arc such that $f|_{S(f)}$ is transverse to l , $l \cap f(M) \neq \emptyset$ and $f^{-1}(\partial l) = \emptyset$. We call such an arc l a *t-arc* for f . Let $\tilde{\Pi} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be a submersion such that $\tilde{\Pi}(l)$ is a point in $\mathbb{R}^{n-1}$. Since $f|_{S(f)}$ is transverse to l , if we restrict $\tilde{\Pi} \circ f$ to the interior of a sufficiently small tubular neighborhood $N(f^{-1}(l))$ of $f^{-1}(l)$, $\tilde{\Pi}(l)$ is a regular value of the restricted map $\tilde{\Pi} \circ f|_{\text{Int}(N(f^{-1}(l)))} : \text{Int}(N(f^{-1}(l))) \rightarrow \mathbb{R}^{n-1}$. Thus, $f^{-1}(l) \subset (\tilde{\Pi} \circ f)^{-1}(\tilde{\Pi}(l))$ is a finite disjoint union of circles and $f|_{f^{-1}(l)} : f^{-1}(l) \rightarrow l$ is a Morse function.

Then we have the following observation.

Observation 3.1. *Let M be a closed n -dimensional manifold and $f : M \rightarrow \mathbb{R}^n$ a stable fold map ($n > 1$). If f has an embedding lift $\tilde{f} : M \rightarrow \mathbb{R}^{n+1}$, then for any t-arc l for f , $\tilde{f}|_{f^{-1}(l)} : f^{-1}(l) \rightarrow \Pi^{-1}(l)$ is an embedding lift over the Morse function $f|_{f^{-1}(l)} : f^{-1}(l) \rightarrow l$.*

Using Observation 3.1, we construct stable fold maps $f : M \rightarrow \mathbb{R}^n$ which do not have an embedding lift.

Example 3.2. Let f_1 and $f_2 : S^1 \times D^1 \rightarrow \mathbb{R}^2$ be orientation preserving immersions such that $f_1|_{S^1 \times \partial D^1} = f_2|_{S^1 \times \partial D^1}$ and that their images are as shown in Figure 6.

By attaching f_1 and f_2 along $S^1 \times \partial D^1$ and changing the orientation of the source $S^1 \times D^1$ of f_2 , we have a stable fold map of a torus T^2 into $\mathbb{R}^2$. We denote by $f : T^2 \rightarrow \mathbb{R}^2$ this stable fold map. Let l be a t-arc, as shown in Figure 7, which passes through the two normal crossings of $f(S(f))$. Then, $f^{-1}(l)$ is a circle and the Morse function $f|_{f^{-1}(l)} : f^{-1}(l) \rightarrow l$ has two minimum p_1, p_2 and two maximum q_1, q_2 which are not separated in the sense of Definition 2.1. Therefore, f does not have an embedding lift by Theorem 2.2 and Observation 3.1. See Figure 7.

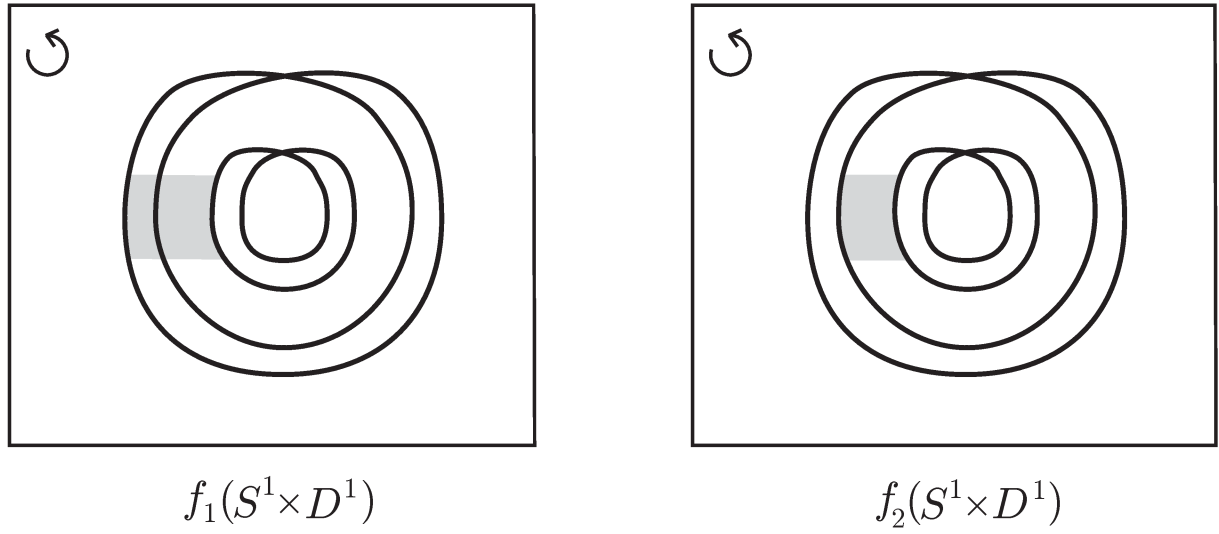


Figure 6. Orientation preserving immersions of $S^1 \times D^1$ into $\mathbb{R}^2$. The black curves represent the images $f_i(S^1 \times \partial D^1)$ and the gray strips represent parts of $f_i(S^1 \times D^1)$ ($i = 1, 2$).

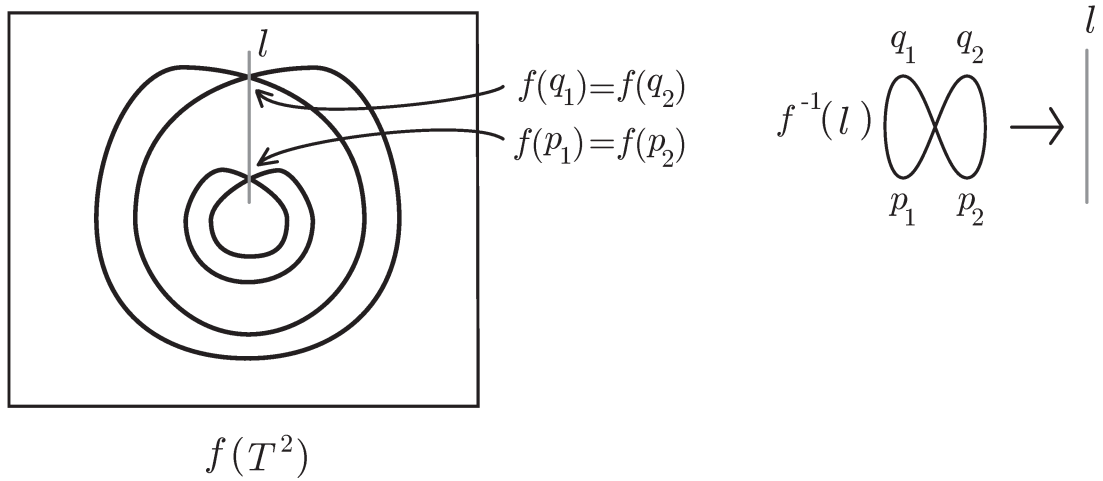


Figure 7. The t-arc l for $f : T^2 \rightarrow \mathbb{R}^2$ such that $f|_{f^{-1}(l)}$ does not have an embedding lift.

Example 3.3. Let g_1 and $g_2 : D^2 \rightarrow \mathbb{R}^2$ be orientation preserving immersions such that $g_1|_{\partial D^2} = g_2|_{\partial D^2}$ and that their images are as shown in Figure 8. These immersions are known as Milnor's example (see [3]).

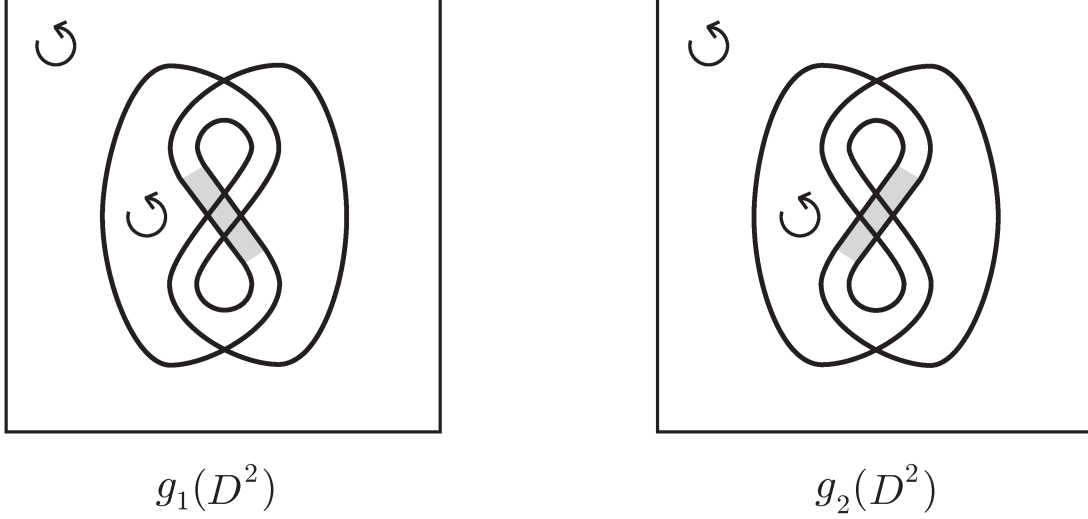


Figure 8. Orientation preserving immersions of D^2 into $\mathbb{R}^2$ which are known as Milnor's example. The black curves represent the images $g_i(\partial D^2)$ and the gray strips represent parts of $g_i(D^2)$ ($i = 1, 2$).

By attaching g_1 and g_2 along ∂D^2 and changing the orientation of the source D^2 of g_2 , we have a stable fold map $g : S^2 \rightarrow \mathbb{R}^2$. Let l be a t-arc, as shown in Figure 9, which passes through the two outermost normal crossings of $g(S(g))$. Then, $g^{-1}(l)$ is a circle and the Morse function $g|_{g^{-1}(l)} : g^{-1}(l) \rightarrow l$ has two minimum p_1, p_2 and two maximum q_1, q_2 which are not separated in the sense of Definition 2.1. Therefore, g does not have an embedding lift by Theorem 2.2 and Observation 3.1. See Figure 9.

Example 3.4. Using the stable fold map $g : S^2 \rightarrow \mathbb{R}^2$ which is constructed in Example 3.3, we can construct a family of stable fold maps $h_n : S^n \rightarrow \mathbb{R}^n$, $n \geq 2$, which have an immersion lift but which do not have an embedding lift into $\mathbb{R}^{n+1}$ as follows.

Let $\Pi' : \mathbb{R}^n \rightarrow \mathbb{R}$ be the orthogonal projection defined by $\Pi'(y_1, y_2, \dots, y_n) = y_1$. Suppose that a stable fold map $h_n : S^n \rightarrow \mathbb{R}^n$ satisfies the following conditions.

$$(1_n) \quad h_n(S^n) \subset [1, 3] \times \mathbb{R}^{n-1}.$$

$$(2_n) \quad \{(2, y_2, 0, \dots, 0) \mid y_2 \in \mathbb{R}\} \subset \mathbb{R}^n \text{ contains a t-arc } l_n \text{ for } h_n \text{ such that}$$

$$h_n|_{h_n^{-1}(l_n)} : h_n^{-1}(l_n) \rightarrow l_n$$

does not have an embedding lift.

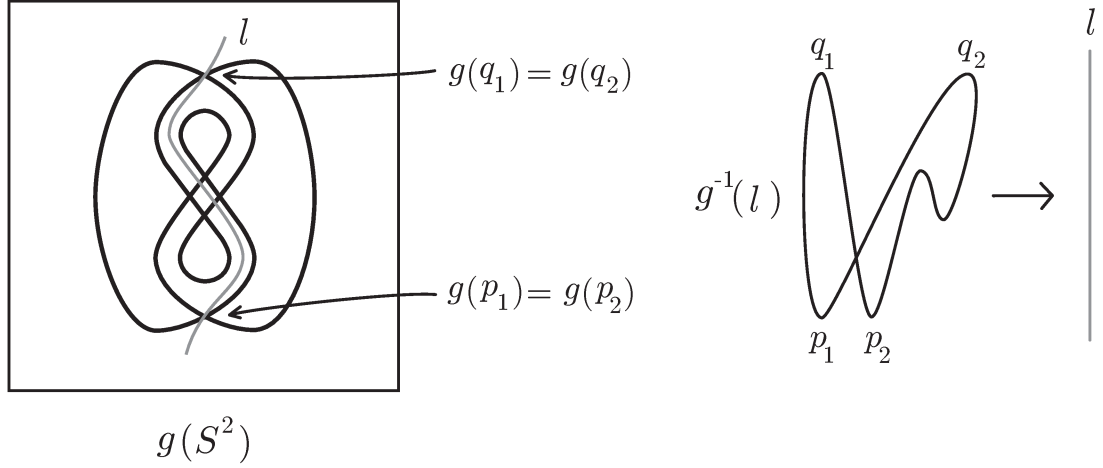


Figure 9. The t-arc l for $g : S^2 \rightarrow \mathbb{R}^2$ such that $g|_{g^{-1}(l)}$ does not have an embedding lift.

- (3_n) The set $(\Pi' \circ h_n)^{-1}(\{1\})$ (resp. $(\Pi' \circ h_n)^{-1}(\{3\})$) consists exactly of one point in S^n and we denote this point by p^n (resp. q^n). Furthermore, $\Pi' \circ h_n$ takes its minimum (resp. maximum) at p^n (resp. q^n). Both p^n and q^n are non-degenerate critical points of $\Pi' \circ h_n$.
- (4_n) For a sufficiently small real number ε with $0 < \varepsilon < 1$, $h_n|_{S(h_n)}$ is transverse to $\Pi'^{-1}(\{1 + \varepsilon\})$ and $B_n = (\Pi' \circ h_n)^{-1}([1, 1 + \varepsilon])$ is an n -ball around p^n such that $\partial B_n = (\Pi' \circ h_n)^{-1}(\{1 + \varepsilon\})$ and $\partial B_n \cap S(h_n)$ consists of an $(n - 2)$ -dimensional sphere.
- (5_n) $h_n|_{\partial B_n} : \partial B_n \rightarrow \Pi'^{-1}(\{1 + \varepsilon\})$ is a stable fold map such that $\partial B_n \cap S(h_n) = S(h_n|_{\partial B_n})$.
- (6_n) $B_n \cap S(h_n)$ is an $(n - 1)$ -ball around p^n and $h_n|(B_n \cap S(h_n)) : B_n \cap S(h_n) \rightarrow h_n(S(h_n))$ is an embedding.

By composing $g : S^2 \rightarrow \mathbb{R}^2$ in Example 3.3 with a suitable diffeomorphism of $\mathbb{R}^2$, we get a stable fold map $h_2 : S^2 \rightarrow \mathbb{R}^2$ which satisfies the conditions (1₂)–(6₂). See Figure 10.

Let us now construct h_{n+1} , $n \geq 2$, inductively as follows. Consider the stable fold map $h_{n+1}^1 : S^n \times S^1 \rightarrow \mathbb{R}^{n+1}$ defined by

$$h_{n+1}^1(x, (\cos \theta, \sin \theta)) = ((y_1 \circ h_n(x)) \cos \theta, y_2 \circ h_n(x), \dots, y_n \circ h_n(x), (y_1 \circ h_n(x)) \sin \theta).$$

Here, S^1 is identified with the boundary of the unit disk $D^2 = \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$, $(\cos \theta, \sin \theta) \in S^1 = \partial D^2$ and $x \in S^n$. Let $h_{n+1}^2 :$

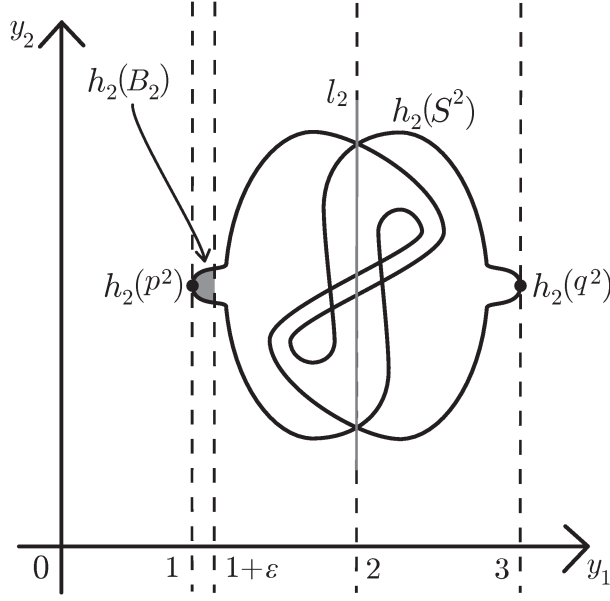


Figure 10. Positions of the t-arc l_2 , $h_2(p^2)$, $h_2(q^2)$ and $h_2(B_2)$ for $h_2 : S^2 \rightarrow \mathbb{R}^2$.

$\partial B_n \times D^2 \rightarrow \mathbb{R}^{n+1}$ be the smooth map defined by

$$\begin{aligned} h_{n+1}^2(x, (r \cos \theta, r \sin \theta)) \\ = ((y_1 \circ h_n(x))r \cos \theta, y_2 \circ h_n(x), \dots, y_n \circ h_n(x), (y_1 \circ h_n(x))r \sin \theta). \end{aligned}$$

From these constructions, we get the $(n+1)$ -dimensional manifold

$$(3.1) \quad ((S^n \times S^1) \setminus (\text{Int}(B_n) \times S^1)) \cup (\partial B_n \times D^2)$$

which is diffeomorphic to S^{n+1} , and the map $(h_{n+1}^1 | ((S^n \times S^1) \setminus (\text{Int}(B_n) \times S^1))) \cup h_{n+1}^2$. By slightly deforming $h_{n+1}^1 | ((S^n \times S^1) \setminus (\text{Int}(B_n) \times S^1))$ and h_{n+1}^2 around the boundary $\partial B_n \times S^1$, we get a stable fold map h'_{n+1} of S^{n+1} into $\mathbb{R}^{n+1}$. In the following, we will identify the manifold (3.1) with S^{n+1} . By construction, the stable fold map $h'_{n+1} : S^{n+1} \rightarrow \mathbb{R}^{n+1}$ satisfies the following conditions.

$$(1'_{n+1}) \quad h_{n+1}(S^{n+1}) \subset [-3, 3] \times \mathbb{R}^n.$$

$$(2'_{n+1}) \quad \{(2, y_2, 0, \dots, 0) \mid y_2 \in \mathbb{R}\} \subset \mathbb{R}^{n+1} \text{ contains a t-arc } l'_{n+1} \text{ for } h'_{n+1} \text{ such that}$$

$$h'_{n+1}|(h'_{n+1})^{-1}(l'_{n+1}) : (h'_{n+1})^{-1}(l'_{n+1}) \rightarrow l'_{n+1}$$

does not have an embedding lift.

$$(3'_{n+1}) \quad \text{For } (-1, 0) \in S^1 \text{ and } (1, 0) \in S^1, \text{ we define } (p^{n+1})' \in ((S^n \times S^1) \setminus (\text{Int}(B_n) \times S^1)) \subset S^{n+1} \text{ by } (p^{n+1})' = q^n \times \{(-1, 0)\} \text{ and define } (q^{n+1})' \in ((S^n \times S^1) \setminus (\text{Int}(B_n) \times S^1)) \subset S^{n+1}$$

S^{n+1} by $(q^{n+1})' = q^n \times \{(1, 0)\}$. By the definition, we have $(\Pi' \circ h'_{n+1})^{-1}(\{-3\}) = \{(p^{n+1})'\}$ and $(\Pi' \circ h'_{n+1})^{-1}(\{3\}) = \{(q^{n+1})'\}$, and $\Pi' \circ h'_{n+1}$ takes its minimum (resp. maximum) at $(p^{n+1})'$ (resp. $(q^{n+1})'$). Both $(p^{n+1})'$ and $(q^{n+1})'$ are non-degenerate critical points of $\Pi' \circ h'_{n+1}$.

By composing $h'_{n+1} : S^{n+1} \rightarrow \mathbb{R}^{n+1}$ with a suitable diffeomorphism of $\mathbb{R}^{n+1}$, we get a stable fold map $h_{n+1} : S^{n+1} \rightarrow \mathbb{R}^{n+1}$ which satisfies the conditions $(1_{n+1})-(6_{n+1})$. By [5], h_{n+1} has an immersion lift into $\mathbb{R}^{n+2}$. However, by Observation 3.1, h_{n+1} does not have an embedding lift into $\mathbb{R}^{n+2}$.

We end this paper by stating a future problem.

Problem 3.5. If M is a closed orientable n -dimensional manifold, then does the converse of Observation 3.1 hold? That is, let M be a closed orientable n -dimensional manifold and $f : M \rightarrow \mathbb{R}^n$ a stable fold map ($n > 1$). Suppose that for any t-arc l for f , the Morse function $f|_{f^{-1}(l)} : f^{-1}(l) \rightarrow l$ has an embedding lift. Then, does the stable fold map f has an embedding lift?

When M is nonorientable, we have a counterexample as follows. Let $f : M \rightarrow \mathbb{R}^2$ be a stable fold map of the Klein bottle such that $f(S(f))$ consists of two concentric circles and $f(M)$ is an embedded annulus whose boundary is $f(S(f))$. See Figure 11. Since $f(S(f))$ does not have normal crossings, for any t-arc l , $f|_{f^{-1}(l)} : f^{-1}(l) \rightarrow l$ is a stable Morse function and it has an embedding lift (see Remark 2.4). But it is known that the Klein bottle cannot be embedded in $\mathbb{R}^3$. Therefore, f does not have an embedding lift. By Haefliger's theorem [2], f does not have an immersion lift either.

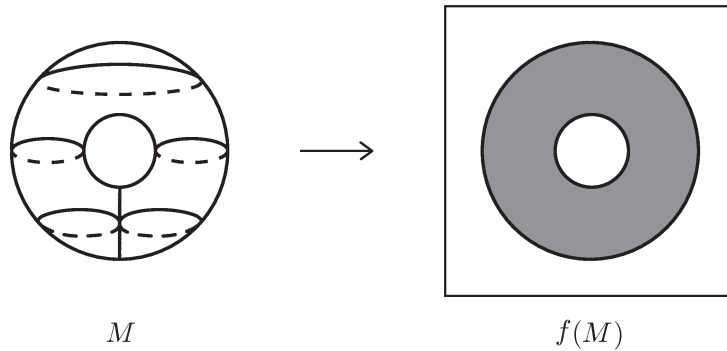


Figure 11. The stable fold map $f : M \rightarrow \mathbb{R}^2$ of the Klein bottle M .

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