

F -algebra M of holomorphic functions

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1. Introduction.

Let U be the unit disc $\{|z| < 1\}$ in $\mathbb{C}$. A function f holomorphic in U is said to belong to the class M if

$$\rho(f) \equiv \int_0^{2\pi} \log(1 + Mf(\theta)) d\theta < \infty$$

where $Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|$ and $\log^+ x = \max(\log x, 0)$, $x > 0$. The class M was introduced and studied in [1, 2, 3, 4, 5]. The class M is related to the usual Hardy space H^p ($p > 0$) and the Nevanlinna class N^+ as

$$\bigcup_{p>0} H^p \subsetneq M \subsetneq N^+$$

The class M with the metric $d(f, g) = \rho(f - g)$ is an F -algebra, i.e, a topological vector space with a complete translation invariant metric in which multiplication is continuous. The class M has many similarities with N^+ , but it is not fully studied as N^+ . In this report we wish to summarize the works on the class M [1, 2, 3, 4, 5] and some open problems. We refer to [7] for the Hardy space and the Smirnov class.

2. M as a class of functions.

For a real-valued function h in $L^1(\partial U)$, we let

$$f(z) = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h(e^{it}) dt \right).$$

We have

2.1 Theorem. If $P[h^+] \in \text{Re } H^1$, then $f \in M$ where $P[h^+]$ is the Poisson integral of $h^+ = \max(h, 0)$. The converse is false.

2.2 Problem. Find a necessary and sufficient condition on h in order that $f \in M$. That is, characterize those outer functions in M .

Unlike N^+ , the inner factor cannot be cancelled in M as in the following theorem.

2.3 Theorem. [1] There exists an f in M whose outer factor F is not in M . It is easy to see that a finite Blaschke factor of $f \in M$ can be cancelled in M but we do not know whether an infinite Blaschke factor of f can be cancelled in M or not.

2.4 For $\alpha > 1$, we define

$$M_\alpha f(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma_\alpha(e^{i\theta})\}$$

where $\Gamma_\alpha(e^{i\theta})$ is the nontangential region at $e^{i\theta}$ defined as

$$\Gamma_\alpha(e^{i\theta}) = \{z \in U : |e^{i\theta} - z| < \frac{\alpha}{2}(1 - |z|^2)\}$$

In the definition of M , the radial maximal function $Mf(e^{i\theta})$ can be replaced by the nontangential maximal function $M_\alpha f(e^{i\theta})$. Precisely we have

2.5 Theorem. There exists a positive constant C_α such that

$$\int_0^{2\pi} \log(1 + Mf(e^{i\theta})) d\theta \leq C_\alpha \int_0^{2\pi} \log(1 + M_\alpha f(e^{i\theta})) d\theta$$

2.6 Corollary. The class M is invariant under the composition of automorphisms of the unit disc U . More precisely, if $M \in M$ then $f \circ \varphi \in M$ for any $\varphi \in \text{Aut}(U)$.

2.7 Problem. Is M invariant under the composition of inner functions? Recall that N^+ is invariant under the composition of inner functions.

For the boundary values of functions in M , the following is proved in [5].

2.8 Theorem. [5] A measurable function $g(e^{i\theta})$ on ∂U coincides with the angular boundary value of some function f in M if and only if there exists a sequence of polynomials p_n with properties :

- (a) $p_n(e^{i\theta}) \rightarrow g(e^{i\theta})$ a.e. on ∂U and
- (b) $\overline{\lim}_{n \rightarrow \infty} \int_0^{2\pi} \log(1 + Mp_n(\theta)) d\theta < \infty$.

3. M as an F -space

It is proved in [1] that M with the metric $d(f, g) = \rho(f - g)$ is a separable F -space. The space M has many similarities as N^+ as F -spaces.

3.1 Theorem. M is not locally bounded.

3.2 Theorem. If Λ is a continuous linear functional on M , then there exists a $g \in A^\infty(U)$ (i.e., g is analytic in U and C^∞ on $\bar{U}$) such that

$$\Lambda f = \lim_{r \rightarrow 1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(e^{i\theta})} d\theta, \quad f \in M.$$

Conversely, if $g \in A^\infty(U)$ and if

$$\Lambda f = \lim_{r \nearrow 1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(e^{i\theta})} d\theta$$

exists for all $f \in M$, then Λ defines a continuous linear functional on M .

3.3 Problem. Describe $g \in A^\infty(U)$ more precisely in the above theorem.

3.4 Theorem. M is not locally convex.

4. M as an F -algebra

As an F -algebra M , the invertible elements, multiplicative linear functionals, closed maximal ideals and onto algebra endomorphisms of M are determined as we see in the following theorems.

4.1 Theorem. The only invertible elements of M are those outer function f with $\log |f| \in \text{Re } H^1$.

4.2 Theorem. γ is a nontrivial multiplicative linear functional on M if and only if $\gamma(f) = f(\lambda)$, $f \in M$, for some $\lambda \in U$. Therefore, every nontrivial multiplicative linear functional is continuous.

4.3 Theorem. Every closed maximal ideal of M is the kernel of a multiplicative linear functional.

4.4 Theorem. There exists a maximal ideal M which is not the kernel of a multiplicative linear functional.

4.5 Theorem. $\Gamma : M \rightarrow M$ is an onto algebra endomorphism if and only if $\Gamma(f) = f \circ \varphi$, $f \in M$, for some automorphism φ of U . In particular, Γ is invertible.

References

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