

Local deformation of polynomials with isolated singularities

Tokyo university, Mutsuo Oka

§ 1. Introduction and statement of results

Let $f(z)$ be a polynomial in $\mathbb{C}^n$ which has an isolated critical point at the origin and $f(0)=0$. Then there is the Milnor fibering for a small number $\varepsilon > 0$: $f|_{|f|}: S_\varepsilon^{2n-1} - K_{f,\varepsilon} \longrightarrow S^1$ where $S_\varepsilon^{2n-1} = \{z \in \mathbb{C}^n \mid \|z\| = \varepsilon\}$ and $K_{f,\varepsilon} = S_\varepsilon^{2n-1} \cap f^{-1}(0)$ ([3]). In this paper, we shall prove that the Milnor fibering is invariant under a certain "deformation" (See Definition 1 and Theorem 1). By applying the above result, we shall prove that the Milnor fibering of a weighted homogeneous polynomial is determined by its weight (see Theorem 2) and the Milnor fibering of a polynomial which has a "weighted homogeneous principal part" has a periodic monodromy (see Defi-

nition 2 and Theorem 3).

First we define some notations. A subset $\mathcal{M}(n)$ of $\mathbb{C}[z_1, \dots, z_n]$ is defined by $\mathcal{M}(n) = \{f \in \mathbb{C}[z_1, \dots, z_n] \mid f \text{ has an isolated critical point at the origin}\}$. $\mathcal{W}(n)$ is a subset of $\mathcal{M}(n)$ defined by $\mathcal{W}(n) = \{f \in \mathcal{M}(n) \mid f : \text{weighted homogeneous}\}$.

Definition 1. Let f and g be polynomials in $\mathcal{M}(n)$. A family $\{f_t\}_{t \in I}$ of polynomials in $\mathcal{M}(n)$ is called a deformation from f to g if and only if (i) $f_0 = f$ and $f_1 = g$. (ii) (Piecewise Analyticity) There exists a finite sequence $0 = t_0 < t_1 < \dots < t_N = 1$ such that for each $k = 0, 1, \dots, N-1$, $f_t(\mathbb{Z})$ is complex analytic in $\mathbb{Z}$ and t for $(\mathbb{Z}, t) \in \mathbb{C}^n \times [t_k, t_{k+1}]$ i.e. there exists an analytic function f_k in $\mathbb{C}^n \times \mathbb{C}$ such that $f_k|_{\mathbb{C}^n \times [t_k, t_{k+1}]} = f_t|_{\mathbb{C}^n \times [t_k, t_{k+1}]}$. (iii) (Equi-isolatedness) There exists a positive number ε such that $\text{grad } f_t(\mathbb{Z}) = (\frac{\partial f_t}{\partial z_1}(\mathbb{Z}), \dots, \frac{\partial f_t}{\partial z_n}(\mathbb{Z}))$ is a non-zero vector for each $t \in I$ and t satisfying $0 < \|\mathbb{Z}\| \leq \varepsilon$ and S_ε^{2n-1} is transverse to $f_t^{-1}(0)$ for each $t \in I$.

It is clear that the above deformation defines an equivalence relation in $\mathcal{M}(n)$ and we

denote the corresponding equivalence class of f by $[f]$ and the quotient set by $\mathcal{M}(n)$.

Definition 2. Let f be a polynomial in $\mathcal{M}(n)$. Assume that f can be expressed as $f(z) = f_0(z) + g(z)$ where f_0 is a polynomial in $W(n)$ with I -weight $(q_1, q_2, \dots, q_n | d)$ (i.e. $f_0(z_1^{q_1}, \dots, z_n^{q_n})$ is a homogeneous polynomial of degree d) and $g(z)$ is a polynomial such that each monomial $z_1^{v_1} z_2^{v_2} \dots z_n^{v_n}$ of $g(z)$ satisfies $\sum_{k=1}^n q_k v_k > d$. Then we call f_0 a weighted homogeneous principal part of f .

Now, our results are as follows.

Theorem 1. Let $\{f_t\}_{t \in I}$ be a deformation from f to g ($f, g \in \mathcal{M}(n)$). Then there exists a positive number ε such that (i) $K_{f, \varepsilon} = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ and $K_{g, \varepsilon} = g^{-1}(0) \cap S_\varepsilon^{2n-1}$ are isotopic i.e. there exists an isotopy $h_t : S_\varepsilon^{2n-1} \rightarrow S_\varepsilon^{2n-1}$ such that $h_0 = \text{id}$. and $h_1(K_{f, \varepsilon}) = K_{g, \varepsilon}$. (ii) There is a fibre-preserving diffeomorphism $\psi : S_\varepsilon^{2n-1} - K_{f, \varepsilon} \rightarrow S_\varepsilon^{2n-1} - K_{g, \varepsilon}$. That is, the following diagram commutes.

$$\begin{array}{ccc} S_\varepsilon^{2n-1} - K_{f, \varepsilon} & \xrightarrow{\psi} & S_\varepsilon^{2n-1} - K_{g, \varepsilon} \\ & \searrow f/|f| & \swarrow g/|g| \\ & S^1 & \end{array}$$

Theorem 2. If two polynomials f and g in $W(n)$ have the same weight, then $[f] = [g]$.

Theorem 3. Assume that a polynomial f in $W(n)$ has a weighted homogeneous principal part f_0 . Then the monodromy of f is periodic.

The following corollaries are immediate consequences of Theorem 2.

Corollary 1. Let H be a non-singular hypersurface in $\mathbb{CP}^n$. Then H is isotopic to the standard hypersurface X^d where d is the degree of H and $X^d = \{z \in \mathbb{CP}^n \mid z_0^d + z_1^d + \dots + z_n^d = 0\}$.

Corollary 2. Let f be a polynomial in $W(n)$, having a weight $w = (w_1, \dots, w_n)$. Assume that each w_k is an integer ≥ 1 . Then f is equivalent to $g(z) = z_1^{w_1} + \dots + z_n^{w_n}$.

§2 Proof of Theorem 1

Let $f_t(z)$ be a family of polynomials satisfying (i), (ii) and the first part of (iii) of the definition of deformations. The following lemma is due to

Ramalyan.

Lemma 1. Let $\{f_t(z)\}_{t \in I}$ be as above. Then the monodromies of f_0 and f_1 coincide. If further $n \neq 3$, $K_{0,\varepsilon} = f_0^{-1}(0) \cap S_\varepsilon^{2n-1}$ and $K_{1,\varepsilon} = f_1^{-1}(0) \cap S_\varepsilon^{2n-1}$ are diffeomorphic.

proof: Let ε_0 be the stable radius of f_0 . Then there is a $\eta > 0$ such that $f_t^{-1}(0)$ and $S_{\varepsilon_0}^{2n-1}$ are transverse for each $t \in [0, \eta]$. Let W be $\{(z, t) \in S_{\varepsilon_0}^{2n-1} \times [0, \eta] \mid |f_t(z)| = \delta\}$ where δ is a small number such that $f_t^{-1}(\delta \cdot e^{i\theta})$ and S_ε^{2n-1} are transverse for each $t \in [0, \eta]$ and $\theta \in [0, 2\pi]$. We consider the fibering $\varphi: W \rightarrow S'_\delta = S'$ defined by $\varphi(z, t) = f_t(z)$ and the projection $\pi: W \rightarrow [0, \eta]$. It is easy to see that π is non-degenerate even if π is restricted to each fibre $F_\theta = \varphi^{-1}(\theta)$. Therefore, by using a connection vector field for π which is tangent to each fibres, we have a fibre-preserving diffeomorphism:

$$\begin{array}{ccc} D_{\varepsilon_0}^{2n} \cap f_0^{-1}(S'_\delta) & \xrightarrow{\exists \psi} & f_\eta^{-1}(S'_\delta) \cap D_{\varepsilon_0}^{2n} \\ & \searrow f_0 & \swarrow f_\eta \\ & S' & \end{array}$$

Let ε_η be the stable radius of f_η ($\varepsilon_\eta \leq \varepsilon_0$). Then we have the following commutative diagram of fibration.

$$\begin{array}{ccc} f_\eta^{-1}(S'_\eta) \cap D_{\varepsilon_0}^{2n} & \xleftarrow{\quad} & f_\eta^{-1}(S'_\eta) \cap D_{\varepsilon_\eta}^{2n} \\ & \searrow f_\eta \quad \swarrow f_\eta & \\ & S'_\eta & \end{array}$$

Let $\tilde{F} = f_\eta^{-1}(\partial) \cap D_{\varepsilon_0}^{2n}$ and $F = f_\eta^{-1}(\partial) \cap D_{\varepsilon_\eta}^{2n}$. Because the local degree of f_η is constant and by [3], we know that $\tilde{F}$ and F is homotopic to a bouquet $S^{n-1} \vee \dots \vee S^{n-1}$, therefore to prove that $F \hookrightarrow \tilde{F}$ is homotopy equivalent, it suffices to prove that $H_{n-1}(F) \rightarrow H_{n-1}(\tilde{F})$ is a bijection. By using a suitable Morse function, $\tilde{F}$ is obtained by attaching handles of index $\leq n-1$ to F . Therefore $H_n(\tilde{F}, F) = 0$ and $H_{n+1}(\tilde{F}, F)$ is torsion free. Then it is clear that $H_{n-1}(F) \rightarrow H_{n-1}(\tilde{F})$ is a bijection from the homology exact sequence of the couple $(\tilde{F}, F)$. The latter part of the Lemma is the consequence of h -cobordism theorem. Q.E.D.

Proof of the Theorem 1. Let $\{f_t\}_{t \in I}$ be a deformation from f to g and let $\widehat{K} = \{(z, t) \in S_\varepsilon^{2n-1} \times I; f_t(z) = 0\}$. Let $\pi: S_\varepsilon^{2n-1} \times I \rightarrow I$ be the projection, then by the assumption of the deformation, $\pi: (S_\varepsilon^{2n-1} \times I, \widehat{K}) \rightarrow I$ is non-degenerate. Therefore by constructing a connection vector field for π , we have a desired isotopy from $K_{f, \varepsilon} = f^{-1}(0) \cap S_\varepsilon^{2n-1}$ and $K_{g, \varepsilon} = g^{-1}(0) \cap S_\varepsilon^{2n-1}$ by the 1-parameter group of the connection. This completes the proof of the first part. Let δ be a small positive number such that $f_t^{-1}(\eta)$ and S_ε^{2n-1} are transverse for each $|\eta| = \delta$.

Let $E = \{(z, t) \in D_\varepsilon^{2n} \times I \mid |f_t(z)| = \delta\}$ and $\varphi: E \rightarrow S'_\delta = S'$ be defined by $\varphi(z, t) = f_t(z)$. Then φ is a fiber bundle. Let $\widehat{\pi}$ be the projection $W \rightarrow I$. Then $\widehat{\pi}$ is non-degenerate on each fibre of φ . Thus we have a fibre preserving diffeomorphism ψ

$$\begin{array}{ccc} f_0^{-1}(S'_\delta) \cap D_\varepsilon^{2n} & \xrightarrow{\psi} & f_1^{-1}(S'_\delta) \cap D_\varepsilon^{2n} \\ & \searrow f = f_0 & \swarrow f_1 = g \\ & S'_\delta & \end{array}$$

By Theorem 5.11 of [3], there is a fibre-preserving diffeomorphism φ_i ($i=0, 1$)

$$\begin{array}{ccc} f_i^{-1}(S'_0) \cap D_\epsilon^{2n} & \xrightarrow{\varphi_i} & S_\epsilon^{2n-1} - f_i^{-1}(\text{Int } D_\delta^2) \\ & \searrow f_i & \downarrow f_i/|f_i| \\ & & S'_0 = S' \end{array}$$

Combining these diffeomorphisms, we obtain a fibre preserving diffeomorphism ψ :

$$\begin{array}{ccc} S_\epsilon^{2n-1} - K_{f,\epsilon} & \xrightarrow{\quad} & S_\epsilon^{2n-1} - K_{g,\epsilon} \\ & \searrow f/|f| & \downarrow g/|g| \\ & & S' \end{array}$$

Q. E. D.

§ 3. Proof of the theorem 2.

Let $f(z)$ be a polynomial in $\mathbb{C}^n$. $f(z)$ is called a weighted homogeneous polynomial of type $w=(w_1, \dots, w_n)$

where each w_k is a positive rational number, if and only if each monomial $z_1^{v_1} z_2^{v_2} \dots z_n^{v_n}$ of f satisfies $\sum_{k=1}^n \frac{v_k}{w_k} = 1$.

We call w the weight of f . Expressing $w_k = \frac{u_k}{v_k}$, we define positive integers $q_1, \dots, q_n$ by $d = [u_1, \dots, u_n]$ (the least common multiple) and $q_k = d/u_k$. We call $(q_1, \dots, q_n)$ I-weight of $f(z)$. Then from the definition, $g(z) = f(z_1^{q_1}, \dots, z_n^{q_n})$

is a homogeneous polynomial of degree d . More generally we consider $(Q^+)^n = Q^+ \times \dots \times Q^+$ where Q^+ is the set of positive rational numbers and we call $w \in (Q^+)^n$ an abstract weight. For a given $w = (w_1, \dots, w_n) \in (Q^+)^n$,

let $S(w) = \{v = (v_1, \dots, v_n) \in \mathbb{N}^n \mid \sum_{k=1}^n \frac{v_k}{w_k} = 1\}$. ($\mathbb{N}$: natural numbers). Clearly $S(w)$ is a finite set.

Definition 3. Let $f(w)$ be the number of elements of $S(w)$. For each point $t = (t_v)_{v \in S(w)} \in \mathbb{C}^{f(w)}$, we define a polynomial W_t by $W_t(z) = \sum_{v \in S(w)} t_v z_1^{v_1} z_2^{v_2} \dots z_n^{v_n}$.

A subset $V(w)$ of $\mathbb{C}^{f(w)}$ is defined by

$$V(w) = \{t = (t_v) \in \mathbb{C}^{f(w)} \mid W_t(z) \notin w(n)\}.$$

We call $V(w)$ the characteristic set of w .

Lemma 6. For each weight $w = (w_1, w_2, \dots, w_n) \in (Q^+)^n$,

$V(w)$ is an algebraic set defined by homogeneous polynomials.

Proof: We consider the branched covering $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}^n$ defined by $\varphi(x_1, \dots, x_n) = (x_1^{q_1}, \dots, x_n^{q_n})$ where $(q_1, \dots, q_n; d)$ is the I -weight of w . Then $w_i(\varphi(x))$ and $\frac{\partial w_i}{\partial x_k}(\varphi(x))$ are homogeneous polynomials in x and their degrees are d and $d - q_k$ respectively. Let $V(w) = \{(x_1, \dots, x_n, t) \in \mathbb{CP}^{n-1} \times \mathbb{C}^{P(w)} \mid \frac{\partial w_i}{\partial x_k}(\varphi(x)) = 0, k=1, 2, \dots, n\}$ and $\pi: \mathbb{CP}^{n-1} \times \mathbb{C}^{P(w)} \rightarrow \mathbb{C}^{P(w)}$ be the projection. Then clearly $V(w) = \pi^{-1}(V(w))$, therefore by the Remmert's proper mapping theorem ([2], p. 162), $V(w)$ is an analytic set. But $V(w)$ is clearly a cone at the origin, and therefore $V(w)$ is an algebraic set defined by homogeneous polynomials by the Chow's theorem (see for example Satz 14 of [6]). Q.E.D.

Remark: Lemma 6 can also be proved by a purely algebraic method. (see [8], §§ 80, 81).

Proof of Theorem 2: Let f and g be polynomials in $W(n)$, having the same weight $w = (w_1, \dots, w_n)$. We consider the associated characteristic set $V(w)$. Let $t(f)$ and $t(g)$ be points in $\mathbb{C}^{P(w)}$ corresponding to f and g i.e. $W_{t(f)} = f$ and $W_{t(g)} = g$.

Then by the assumption, $t(f) \notin V(w)$ and $t(g) \notin V(w)$. Thus $V(w)$ is a proper algebraic set of $\mathbb{C}^{P(w)}$ and there is a piecewise linear path $P: I \rightarrow \mathbb{C}^{P(w)} - V(w)$ from $t(f)$ to $t(g)$. We define $f_t = w_{P(t)}$ for each $t \in I$. Then it is clear that $\{f_t\}$ satisfies (i), (ii) and the first part of (iii). Assume $\lambda z = \text{grad} f_t(z)$ for some $z \neq 0$ and $\lambda \in \mathbb{C}^*$ where $f_t(z) = 0$. Because $f_t(z)$ satisfies the equation:

$$f_t(z) = \sum_{k=1}^m \frac{z_k}{w_k} \frac{\partial f_t}{\partial z_k}(z)$$

This means $\sum \frac{\lambda}{w_k} |z_k|^2 = 0$. This is a contradiction. Q. E. D.

§ 4. Proof of Theorem 3.

Let f be a polynomial in $\mathcal{M}(n)$ and assume that f has a weighted homogeneous principal part f_0 . Let $(\gamma_1, \dots, \gamma_n; d)$ be the I -weight of f_0 and consider the mapping $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ($\varphi(x_1, \dots, x_n) = (x_1^{z_1}, \dots, x_n^{z_n})$). Then $f_0(\varphi(x))$ and $\frac{\partial f_0}{\partial z_k}(\varphi(x))$ are homogeneous polynomials of degree d and $d - \gamma_k$ respec-

tively. We prove that $f_t(z) = f_0(z) + t g(z)$, where $g(z) = f(z) - f_0(z)$, satisfies the condition of the Lemma 1. Let $M = \min_{x \in S^{2n-1}} \|\text{grad } f_0(\varphi(x))\|$ where $\text{grad } f_0(\varphi(x)) = (\frac{\partial f_0}{\partial z_1}(\varphi(x)), \dots, \frac{\partial f_0}{\partial z_n}(\varphi(x)))$. Then M is a positive number. Let $U_k = \{x \in S^{2n-1} \mid |\frac{\partial f_0}{\partial z_k}(\varphi(x))| > M/2\sqrt{n}\}$, and $CU'_k = \{rx \mid 0 < r \leq 1, x \in U_k\}$. Then $\bigcup_k U_k = S^{2n-1}$. For each fixed k , let $x \in CU'_k$ and let $\eta = \|x\|$. Then by the above observation $|\frac{\partial f_0}{\partial z_k}(\varphi(x))| > M/2\sqrt{n} \cdot \eta^{d-\delta_k}$, while $|\frac{\partial g}{\partial z_k}(\varphi(x))|/\eta^{d-\delta_k+1}$ is bounded. Therefore we can find $\varepsilon_k > 0$ so that $|\frac{\partial f_0}{\partial z_k}(\varphi(x))| > 2 \cdot |\frac{\partial g}{\partial z_k}(\varphi(x))|$ for each $x \in CU'_k \cap D_{\varepsilon_k}^{2n}$. Let $\varepsilon' = \min_{1 \leq k \leq n} \varepsilon_k$ and choose ε such that $D_{\varepsilon}^{2n} \cap \{z \in \mathbb{C}^n \mid \|z\| \leq \varepsilon\} = \{z \in \mathbb{C}^n \mid \|z\| \leq \varepsilon\}$ is contained in $\varphi(D_{\varepsilon'}^{2n}) = \{z \in \mathbb{C}^n \mid |z_1|^{\frac{2}{p}} + \dots + |z_n|^{\frac{2}{q}} \leq \varepsilon'^2\}$. It is obvious that ε satisfies the condition of the Lemma 1. Q.E.D.

Example: Let $f(z) = z_1^p + z_2^q + z_1^r z_2^s$. Then $f \in \mathcal{W}(2)$. Case 1: $r/p + s/q > 1$ and let $f_0(z) = z_1^p + z_2^q$. Then f_0 is a weighted homogeneous principal part of f . Case 2: $r/p + s/q = 1$.

Then $f \in W(\mathbb{C})$ and $f_0 \in W(\mathbb{C})$ and $[f] = [f_0]$.

Case 3. $\frac{1}{p} + \frac{s}{q} < 1$ and $r, s \geq 2$. Then f has ~~no~~ no weighted homogeneous principal part.

References

- [1] Brieskorn, E: Die monodromie der isolierten singularitäten von hyperflächen, Manuscripta Math. (1970)
- [2] Gunning and Rossi: Analytic functions of several complex variables: Prentice Hall (1965)
- [3] Milnor, J: Singular points of complex hypersurfaces. Study 61, (1968)
- [4] Oka, M: On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials: (to appear)
- [5] Orlik, P and Wagreich, P: Isolated singularities of algebraic surfaces with $\mathbb{C}^*$ -action. Ann. Math (1971)
- [6] Remmert, R and Stein, K: Über die wesentlichen singularitäten analytischer mengen: Math. Annalen. Bd 126 (1953)
- [8] Van der Waerden: Moderne Algebra, Springer Verlag (1937)