

**Absence of Phase Transitions in
Two-dimensional $O(N)$ Spin Models with Large N
—Through the Renormalization Group Flow —**

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Abstract

We Fourier-transform the classical $O(N)$ spin models in two dimensions to obtain a Gaussian system perturbed by a functional determinant. We analyze the system by renormalization group type arguments, and show that there exist no phase transitions if N is sufficiently large, no matter how large β is.

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1. *Introduction.* The existence of the phase transition in two dimensional (2D) Ising model was established by Onsager [1] in the middle of the last century, and the existence of the Kosterlitz-Thouless transition in 2D XY-model was rigorously established by Fröhlich and Spencer [2] three decades ago.

As for non-abelian systems in lower dimensions, however, our knowledge is very poor. Spontaneous mass generations in 2D non-Abelian sigma models (Heisenberg model) and quark confinement in 4D non-Abelian lattice gauge theories have been widely believed [3–5] since the last century, but their proofs still remain to be seen. These models exhibit no phase transitions in the hierarchical model approximations of Wilson-Dyson type or Migdal-Kadanov type [6, 7].

One of the main difficulties in these models is that the field variables are non-abelian objects and block spin transformations break the structures. In some cases, this can be avoided by introducing an auxiliary field ψ [9]. Using this idea, together with the help of the cluster expansion [10], we showed [13, 14] in the 2D $O(N)$ sigma model with large N that

$$\beta_c \geq \text{const} N \log N \quad (1)$$

where $\beta_c(N)$ be the lower bound for the critical inverse temperature of 2d $O(N)$ spin model.

In this Letter, we show our new analysis [16] based on the duality arguments type, and announce some partial results:

Theorem *There exist no phase transitions in the two-dimensional $O(N)$ classical spin model if N is sufficiently large.*

We scale the inverse temperature β by N . The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$\langle f \rangle \equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_i \delta(\phi_i^2 - N\beta) d\phi_i \quad (2)$$

Here Λ is an arbitrarily large square with center at the origin. Moreover $\phi(x) = (\phi(x)^{(1)}, \dots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $\langle 1 \rangle = 1$. The Hamiltonian H_Λ is given by

$$H_\Lambda \equiv -\frac{1}{2} \sum_{|x-y|=1} \phi(x)\phi(y), \quad (3)$$

First substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)] da/2\pi$ into (2) with the

condition that $\text{Im} a_i < -\nu$ [9], we set

$$\text{Im} a_i = -(\nu + \frac{m^2}{2}), \quad \text{Re} a_i = \frac{1}{\sqrt{N}}\psi_i \quad (4)$$

where $m > 0$ is an arbitrary constant. Thus we have

$$\begin{aligned} Z_\Lambda &= c^{|\Lambda|} \int \cdots \int \exp[-W_0(\phi, \psi)] \prod \frac{d\phi_j d\psi_j}{2\pi} \\ &= c^{|\Lambda|} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \end{aligned} \quad (5)$$

where

$$\begin{aligned} W_0(\phi, \psi) &= \frac{1}{2} \langle \phi, (m^2 - \Delta + i\alpha\psi)\phi \rangle - \sum_j i\sqrt{N}\beta\psi_j \\ &= \frac{1}{2} \langle \phi, (m^2 - \Delta)\phi \rangle + \frac{i}{\sqrt{N}} \langle \phi^2 - N\beta, \psi \rangle \end{aligned} \quad (6a)$$

$$F(\psi) = \det(1 + i\alpha G\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j] \quad (6b)$$

$\alpha \equiv 2/\sqrt{N}$, c 's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian and $G = (m^2 - \Delta)^{-1}$. Note that $F(\psi)$ is integrable with respect to ψ if and only if $N \geq 3$.

In the same way, the two-point function is given by

$$\langle \phi_0 \phi_x \rangle = \frac{1}{Z} \int \cdots \int (m^2 - \Delta + i\alpha\psi)_{0x}^{-1} F(\psi) \prod \frac{d\psi_j}{2\pi} \quad (7)$$

Set $\nu = 2$ below. Then we can choose m so that $G(0) = \beta$ ($m^2 \sim \exp[-4\pi\beta]$) and

$$F(\psi) = \det_3^{-N/2}(1 + i\alpha G\psi) \exp[-\langle \psi, G^{\circ 2}\psi \rangle], \quad (8)$$

$$\det_3(1 + A) \equiv \det[(1 + A)e^{-A+A^2/2}] \quad (9)$$

where $G^{\circ 2}(x, y) = G(x, y)^2$ so that $\text{Tr}(G\psi)^2 = \langle \psi, G^{\circ 2}\psi \rangle$. Then we expect that the subtracted determinant $\det_3(1 + i\alpha \cdots) \sim 1$ and that exponential decay follows from (7) since $\tilde{Z} = \int F(\psi) \prod d\psi_i/2\pi \sim \int |F(\psi)| \prod d\psi_i/2\pi$.

We justify this argument by renormalization group methods. The cancelation between the first term of the expansion of the determinant and the phase factor $\exp[i\sqrt{N}\beta\psi]$, and the change of effective mass m^2 are carried out recursively.

2. *Proof of the Theorem.* We use the block spin transformation [4] to justify the previous idea. Intuitively speaking, we set

$$\phi(x) = \phi_{<} \left(\left[\frac{x}{L} \right] \right) + \tilde{\phi}(x) \quad (10)$$

$$\psi(x) = \frac{1}{L^2} \psi_{<} \left(\left[\frac{x}{L} \right] \right) + \tilde{\psi}(x) \quad (11)$$

where $\phi(x)$, $\phi_{<}$ and $\tilde{\phi}$ have the momentum $|p_i| \leq \pi$, $|p_i| \leq \pi/L$ and $\pi(1 - 1/L) \leq |p_i| \leq \pi$ ($i = 1, 2$) respectively. The same is true for $\psi(x)$. The point $[x/L] \in Z^2$ means the lattice point nearest to $x/L \in R^2$, then $\phi_{<}(x)$ and $\psi_{<}(x)$ again have the momentum $|p_i| \leq \pi$ and living on the scaled lattice points.

Starting with $\phi_0 = \phi$ and $\psi_0 = \psi$, we recursively define

$$\exp[-W_{n+1}(\phi_{n+1}, \psi_{n+1})] = \int \exp[-W_n(\phi_{n+1} + \tilde{\phi}_n, L^{-2}\psi_{n+1} + \tilde{\psi}_n)] \prod d\tilde{\phi}_n d\tilde{\psi}_n \quad (12)$$

Our theorem follows from the main term of the n 'th action W_n :

$$\begin{aligned} W_n(\phi_n, \psi_n) = & \frac{1}{2} \langle \phi_n, (-\Delta + m_n^2) \phi_n \rangle + \frac{\gamma_n}{2} \sum (\nabla_\mu \phi_n^2(x))^2 \\ & + \langle \psi_n, H_n^{-1} \psi_n \rangle + \frac{i}{\sqrt{N}} \langle (\phi_n^2 - N\beta_n), \psi_n \rangle \end{aligned} \quad (13)$$

where

$$\begin{aligned} m_n^2 &= L^{2n} m_0^2, \quad \gamma_n = \frac{n}{N} \\ \beta_n &= \beta - O(n), \quad H_n^{-1} = O(1) > 0 \end{aligned}$$

Therefore the integration over ψ_n yields the potential

$$V_n(\phi_n) = \frac{1}{2} \langle \phi_n, (-\Delta + m_n^2) \phi_n \rangle + \frac{\gamma_n}{2} \sum (\nabla_\mu \phi_n^2(x))^2 \frac{1}{N} \sum_x (\phi_n^2(x) - N\beta_n)^2 \quad (14)$$

where $\beta_n \rightarrow 0$ for large n . The term after γ_n is of the form of

$$\sum (\phi_n(x + e_\mu)^2 - \phi_n(x)^2)^2$$

This means that $\phi_n(x) \in R^N$ and $\phi_n(x + e_\mu) \in R^N$ have the same radius and has no effects on the non-existence of phase transition no matter how large γ_n is. Thus the system is the $O(N)$ symmetric Heisenberg model of inverse temperature $N\beta_n = O(N)$ which is in massive phase, see eq. (1).

3. *Block Spin Transformation and Stability Bounds.* To obtain the flow $\{W_n\}$, we use the mathematically controllable block spin transformation introduced by Kupiainen and Gawedzki [12] some decades ago, and integrate $\exp[-W_0]$ recursively from high momentum parts. This is done by decomposing ϕ_n and ψ_n into the next order block spins ϕ_{n+1} and ψ_{n+1} and zero-average fluctuations $Q\xi_n$ and $Q\tilde{\psi}_n$ as

$$\begin{aligned}\phi_n &= A_{n+1}\phi_{n+1} + Q\xi_n \\ \psi_n &= \tilde{A}_{n+1}\psi_{n+1} + Q\tilde{\psi}_n\end{aligned}$$

and by integrating over ξ_n and $\tilde{\psi}_n$ after the substitution. Here A_{n+1} and $\tilde{A}_{n+1}$ are chosen so that

$$\begin{aligned}\langle \phi_n, G_n^{-1}\phi_n \rangle &= \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} \rangle + \langle \xi_n, Q^+ G_n^{-1} Q \xi_n \rangle \\ \langle \psi_n, H_n^{-1}\psi_n \rangle &= \langle \psi_{n+1}, \tilde{H}_{n+1}^{-1}\psi_{n+1} \rangle + \langle \tilde{\psi}_n, Q^+ H_n^{-1} Q \tilde{\psi}_n \rangle\end{aligned}$$

We briefly discuss about matrices A_n , $\tilde{A}_n$ and Q . Let $G_0 = (-\Delta + m_0^2)^{-1}$ and define G_n and $C : R^{\Lambda_n} \rightarrow R^{\Lambda_{n+1}}$ by

$$G_{n+1}(x, y) = (CG_n C^+)(x, y), \quad (Cf)(x) = \frac{1}{L^2} \sum_{z \in \Delta_0} f(Lx + z) \quad (15)$$

where L is a positive integer (e.g. 2, 3, etc.) and Δ_0 is the box of size $L \times L$ centered at the origin. The operator C takes averages of spins over boxes with centers $Lx \in LZ^2$ and scales down the coordinates by L^{-1} . $\Lambda_n = Z^2 \cap L^{-1}\Lambda_{n-1}$ is the lattice space shrunk by L . Let A^+ mean the adjoint of A with respect to the real inner product. The following choice of A_n and Q satisfies our requirement:

$$A_n = G_{n-1} C^+ G_n^{-1} \quad (16)$$

$$Q(x, y) = \begin{cases} 1 & \text{if } x = y \notin L\Lambda_n \\ -1 & \text{if } x \in L\Lambda_n \text{ and } y \in \Delta_x \\ 0 & \text{if otherwise} \end{cases} \quad (17)$$

The matrix $Q : R^{\Lambda_n \setminus L\Lambda_{n+1}} \rightarrow R^{\Lambda_n}$ is block-wise diagonal and constructs zero-average fluctuation field $Q\xi$. We then see

$$CA_n = 1, \quad C(Q\xi)(x) = \frac{1}{L^2} \sum_{\zeta \in \Delta_0} (Q\xi)(Lx + \zeta) = 0 \quad (18)$$

The covariance of the fluctuation field $\{\xi_n(x); x \in \Lambda_n \setminus L\Lambda_{n+1}\}$ is given by

$$\Gamma_n = [Q^+ G_n^{-1} Q]^{-1} \quad (19)$$

and we see that $\Gamma_n(x, y)$ decays exponentially fast uniformly in β . Put

$$\mathcal{A}_n = A_1 A_2 \cdots A_n = G_0 (C^+)^n G_n^{-1}, \quad G_0 = \mathcal{G}_0 \quad (20)$$

and define

$$\mathcal{G}_n = \mathcal{A}_n G_n \mathcal{A}_n^+, \quad \mathcal{T}_n = \mathcal{A}_n Q \Gamma_n Q^+ \mathcal{A}_n^+ \quad (21)$$

so that

$$\mathcal{G}_n = \mathcal{G}_{n+1} + \mathcal{T}_n \quad (22)$$

By putting $\phi_0 = A_1 \phi_1 + Q \xi_0$ and integrating over ξ_0 , we obtain the determinants

$$\det^{-N/2} (1 + i\alpha \mathcal{T}_0 \psi)$$

and the Gaussian term of ψ :

$$\exp \left[- \left\langle \psi, \left(\frac{2}{N} (\varphi_1 \varphi_1) \circ \left(Q \frac{1}{P} Q^+ \right) \psi \right) \right\rangle \right] \sim \exp \left[- \left\langle \psi, \left(\frac{2}{N} (\varphi_1 \varphi_1) \circ \mathcal{T}_0 \right) \psi \right\rangle \right] \quad (23)$$

where

$$P(\psi) = \Gamma_0^{-1} + i\alpha Q^+ \psi Q, \quad (24)$$

and $A \circ B$ stands for the Hadamard product of A and B , i.e. $(A \circ B)_{xy} = A_{xy} B_{xy}$, and $A^{\circ 2} = A \circ A$. Remark $\text{Tr}(A\psi)(B\psi) = \langle \psi, (A^t \circ B)\psi \rangle$ for any matrices A and B . We approximate $\varphi_1(x)\varphi_1(y) = N\mathcal{G}_1(x, y) + : \varphi_1(x)\varphi_1(y) :$ by $N\mathcal{G}_1(x, y)$ assuming that the Wick product term is small. There exist configurations which violate this approximation:

$D_w(\varphi_1)$ = minimal paved set such that

$$|\varphi_1(x)\varphi_1(y) - N\mathcal{G}_1(x, y)| < N^{1+\epsilon_1} \exp\left[\frac{c}{10}|x-y|\right], \quad \forall x \in D_w, \forall y \in D_w^c$$

where paved set is a collection of squares $\{\square\}$ each of which consists of squares $\Delta \subset \Lambda$ of size $L \times L$. We call $D_w(\varphi_1)$ domain wall regions. If all spins are in the same direction and their lengths are in $(N\beta_1)^{1/2}(1 \pm N^\epsilon/2\beta_1)$, then $D_w = \emptyset$ by the minimality. Similarly we define the large field region $D(\psi_1)$ of ψ_1 by the paved set such that

$$D(\psi_1) = \{\square \mid |\psi_1(x)| > N^\epsilon, \exists x \in \square\}$$

$\{D_\omega\}$ have small probabilities to exist because of the large energy $\langle \phi_1, (-\Delta)_D \phi_1 \rangle$ of ϕ_1 and the factor $\exp[-i\langle \varphi^2, \psi \rangle / \sqrt{N}]$, where $(-\Delta)_D$ is the restriction of $-\Delta$ on to the region $\{\phi_1(x); x \in D\}$. Similarly $D(\psi_1)$ have small probability to exist because of the determinants. D can be decomposed into connected components $\{D_i\}$. These regions are extracted as $g(D_i, \psi_1, \phi_1)$ from the Gibbs measure as large field regions. (This definition applies for $\beta \gg N$.) These factors satisfy

$$|g(D_i, \varphi_1, \psi_1)| \leq \exp[-\text{const. } N^{1+\varepsilon} |D|]$$

In other regions, the fields are small and smooth, we can extract a Gaussian factor:

$$\begin{aligned} & \det^{-N/2} (1 + i\alpha \mathcal{T}_0 \psi) \\ &= \det_3^{-N/2} (1 + i\alpha \mathcal{T}_0 \psi) \times \exp[-i\sqrt{N} \langle \mathcal{T}_0, \psi \rangle - \langle \psi, \mathcal{T}_0^2 \psi \rangle] \end{aligned} \quad (25)$$

This and the previous factor yield a new Gaussian term of ψ :

$$\begin{aligned} & \exp \left[-\frac{i}{\sqrt{N}} \langle (\varphi_1^2 - N\beta_1), \psi \rangle - \langle \psi, \tilde{H}_1^{-1} \psi \rangle \right] \\ & \tilde{H}_1^{-1} = \mathcal{T}_0^2 + 2\mathcal{G}_1 \circ \mathcal{T}_0 \end{aligned}$$

Here $\beta_1 = \beta_0 - \mathcal{T}_0(x, x)$, ($\beta_0 = \beta$). Since $\beta_0 \gg 1$, $\tilde{H}_1^{-1} \sim 2\beta_1 \mathcal{T}_0$ is again a Laplacian with small mass term. But we see that $\tilde{H}_n^{-1}$ becomes soon massive.

We need another block spin transformation of the auxiliary field ψ to decompose the bilinear form of ψ . Since the field ψ has the dimension $(\text{length})^{-2}$, we define the block spin operator $C' = L^2 C$ of ψ by

$$(C'\psi)(x) = L^2(C\psi)(x) = \sum_{\zeta \in \Delta_0} \psi(Lx + \zeta) \quad (26)$$

Since $\mathcal{T}_0(x, y)$ decreases exponentially fast in $|x - y|$, and $\mathcal{G}_1(x, y)$ is a slowly decreasing function such that $\mathcal{G}_1(x, y) \sim \beta_1$ for $|x - y| < O(1)$, $\mathcal{T}_0^2 + 2\mathcal{G}_1 \circ \mathcal{T}_0$ has two types of eigenvectors. The first one is (almost) a block-wise constant vector corresponding to the eigenvalue $O(1)$ and the second ones are the zero-average eigenvectors corresponding to the eigenvalues of order $O(\beta_1)$. Put

$$\psi(x) = \tilde{A}_1 \psi_1 + Q \tilde{\psi}_0 \quad (27)$$

$$\psi_1(x) = (C'\psi)(x) = \sum_{\zeta \in \Delta_0} \psi(Lx + \zeta) \quad (28)$$

so that

$$\langle \psi, \tilde{H}_1^{-1} \psi \rangle = \langle \psi_1, H_1^{-1} \psi_1 \rangle + \langle \tilde{\psi}_0, Q^+ \tilde{H}_1^{-1} Q \tilde{\psi}_0 \rangle \quad (29)$$

$$\langle : \varphi_1^2 : , \psi \rangle = \langle : \varphi_1^2 : , \tilde{A}_1 \psi_1 \rangle + \langle : \varphi_1^2 : , Q \tilde{\psi}_0 \rangle \quad (30)$$

where $: \varphi_1^2 := \varphi_1^2 - N\beta_1$ and $H_1^{-1} = \tilde{A}_1^+ \tilde{H}_1^{-1} \tilde{A}_1$. Contrary to $CA_n = 1$, it holds that $C\tilde{A}_n = L^{-2}C'\tilde{A}_n = L^{-2}$. Thus the Gaussian integration over $\tilde{\psi}_0$ yields

$$\mathcal{F}_1 = \frac{1}{4N} \langle : \varphi_1^2 : , Q f_1 Q^+ : \varphi_1^2 : \rangle$$

where $f_1 = [Q^+ \tilde{H}_1^{-1} Q]^{-1}$. Then we have

$$\begin{aligned} \mathcal{F}_1 &= \frac{1}{32N\beta_1} \sum_{\mu} \sum_x (\nabla_{\mu} \varphi^2(x))^2 \\ &\sim \frac{1}{8N\beta_1} \langle \nabla_{\mu} \varphi_1, (\varphi_1 \otimes \varphi_1) \nabla_{\mu} \varphi_1 \rangle \end{aligned}$$

This is a reminiscence of $(\varphi_1^2 - N\beta_1)^2$ which shows that the fluctuation field of φ_1 is perpendicular with φ_1 itself. The RG flow of this term is different from that of $(\varphi_n^2 - N\beta_n)^2$ since $\mathcal{F}$ is made at each step and the latter term keeps its form with a slight change of β_n .

The origin of this term is found in the hierarchical approximation of Dyson-Wilson type [7, 8] and rediscovered in [11]. This is a part of the probability that two spins $\phi_{\pm} \equiv \phi \pm \xi$ form the block spin ϕ such that $\phi^2 = x$. In fact put $\phi = (\varphi, 0) \in R_+ \times R^{N-1}$ and $\xi = (s, u) \in R \times R^{N-1}$. Then

$$\begin{aligned} &\int f((\phi + \xi)^2) f((\phi - \xi)^2) ds d^{N-1}u \\ &= \int f((\varphi + s)^2 + u^2) f((\varphi - s)^2 + u^2) ds d^{N-1}u \\ &= \frac{1}{\sqrt{x}} \int_0^{N\beta} \int_0^{N\beta} f(p) f(q) \left(\frac{p+q}{2} - x - \frac{(p-q)^2}{16x} \right)^{(N-3)/2} dp dq \end{aligned}$$

where

$$\frac{(p-q)^2}{16x} = \frac{(\phi_+^2 - \phi_-^2)^2}{16\phi^2} = \frac{\langle \phi, \xi \rangle^2}{\phi^2} \quad (31)$$

corresponds to $\mathcal{F}_1$. In the hierarchical model, this is restricted to each block, and does not enter the next step. In the real systems, however, this enters the next step since $(\phi_+^2 - \phi_-^2)^2$ is replaced by $\sum_{\mu} (\nabla_{\mu} \phi^2)^2$, and this term increases slowly in n .

Let us see the role of ψ integration. We observe

$$\int \frac{e^{-i:\varphi^2:\psi}}{(1+i\psi)^{N/2}} d\psi = \text{const.} e^{:\varphi^2:} \times \begin{cases} (-: \varphi^2 :)^{N/2-1} & \text{if } : \varphi^2 : < 0 \\ 0 & \text{otherwise} \end{cases}$$

Applying this discussion to the block Δ in which $|\varphi^2| = |\varphi^2 - N\beta|$ is large, one finds a constant $\beta_1 = \beta - O(1)$ such that for $\varphi^2 < N\beta$

$$\begin{aligned} & \exp[L^2(\varphi^2 - N\beta) + \frac{L^2}{2}(N-2)\log(N\beta - \varphi^2)] \\ & \sim \exp\left[-\frac{L^2}{N}(\varphi^2 - N\beta_1)^2\right] \end{aligned} \quad (32)$$

This is the probability density that the arithmetic average of L^2 balls takes its value at φ . This is a rediscovery of the facts found in the hierarchical model approximation which goes back some decades [7, 8]. This means that the fluctuation of φ^2 is considerably small.

For very large ψ ($|\psi| \geq O(N^{1/2})$) where $1/P(\psi)$ is small and the Gaussian factor is small, we have the stability of the determinant which comes from the determinant inequality

$$|\det^2(1 + i\alpha ABA^*)| \geq \det(1 + k_0^2 \alpha^2 B^2) \quad (33)$$

where $k_0 = \inf \text{spec} AA^*$ and $B = B^*$. (We put $A = (\Gamma_0)^{1/2}$, $B = Q^+ \psi Q$). This is the reason why we need $N \geq 3$.

4. Renormalization Group Flow. We combine two types of block transformations to $W_n(\phi_n, \psi_n)$. One is the block spin transformation of the N component boson model of mass m_n^2 , and the other is the block spin transformation of the auxiliary field ψ_n which has the dimension $(\text{length})^{-2}$.

The induction assumption is that the main part of $W_n(\phi_n, \psi_n)$ is given by (13), and we have to prove that the change of W_n is absorbed by the parameters m_n^2 in G_n^{-1} , γ_n and $u_n = N\beta_n$. Moreover $H_0^{-1} = 0$, $\gamma_0 = 0$, $\beta_0 = \beta$ and we discarded irrelevant terms. Compared with W_0 , the most strange term is

$$\gamma_n \sum (\nabla_\mu \phi_n^2(x))^2 \sim 4\gamma_n \langle \nabla_\mu \phi_n, (\phi_n \otimes \phi_n) \nabla_\mu \phi_n \rangle$$

which means that the fluctuation field $\xi_n \sim \nabla_\mu \phi_n$ is almost orthogonal to the block spin field ϕ_n since $\gamma_n \geq 0$ increases as $n \rightarrow \infty$. This term is a reminiscence of $\langle (\phi_k^2 - u_k), \psi_k \rangle$, $k \leq n$ and they sum up to yield γ_n .

Let $\Lambda_n = L^{-n}\Lambda \cap Z^2$ and let ϕ_n be the n th block spin ($\phi_{n+1} = C\phi_n$):

1. Set $\phi_n = A_{n+1}\phi_{n+1} + Q\xi_n$ so that

$$\langle \phi_n, G_n^{-1}\phi_n \rangle = \langle \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} \rangle + \langle \xi_n, \Gamma_n^{-1}\xi_n \rangle$$

where $G_{n+1}^{-1} = A_{n+1}^+ G_n^{-1} A_{n+1}$ and $Q^+ G_n^{-1} Q = \Gamma_n^{-1}$.

2. The Gaussian part of ξ also comes from γ_n and we have

$$\begin{aligned} \gamma_n \langle \nabla_\mu \phi_n, (\phi_n \otimes \phi_n) \nabla_\mu \phi_n \rangle &= \gamma_n \langle \nabla_\mu \phi_{n+1}, (\phi_{n+1} \otimes \phi_{n+1}) \nabla_\mu \phi_{n+1} \rangle \\ &+ \gamma_n \langle \nabla_\mu Q \xi_n, (\phi_{n+1} \otimes \phi_{n+1}) \nabla_\mu Q \xi_n \rangle \end{aligned}$$

where we assume $\phi_n(x)$ changes slowly in x (i.e. outside of the domain wall region).

Moreover we have

$$\frac{i}{\sqrt{N}} \langle \phi_n^2, \psi_n \rangle = \frac{i}{\sqrt{N}} \langle \phi_{n+1}^2 + 2\phi_{n+1}(Q\xi_n) + (Q\xi_n)^2, \psi_n \rangle$$

3. The ξ_n integration is strongly affected by the block spin ϕ_{n+1} .

$$d\mu(\xi_n) = \left[-\frac{1}{2} \langle \xi_n, P_n \xi_n \rangle \right] \prod_x d\xi_n(x) \quad (34)$$

$$P_n = 1_N \otimes [\Gamma_n^{-1} + i\alpha Q^+ \Psi_n Q] + \gamma_n [\phi_{n+1} \otimes \phi_{n+1}] \otimes_x \Gamma_n^{-1} \quad (35)$$

where $([\phi \otimes \phi] \otimes_x \Gamma_n^{-1})(x, y) = \phi(x) \otimes \phi(x) \Gamma_n^{-1}(x, y)$ is an $N \times N$ matrix.

Originally we have $[(\phi \otimes \phi) \circ \Gamma_n^{-1}]_{(i,x),(j,y)} \equiv \phi_i(x) \phi_j(y) \Gamma_n^{-1}(x, y)$. This is approximated as above when $\Gamma_n^{-1}(x, y)$ decays fast in $|x - y|$.

4. The determinant $\det^{-1/2}(P_n)$ and P_n^{-1} depends on an approximate projection operator $\varphi_n \otimes \varphi_n$ and the fluctuations parallel with ϕ_n are very much depressed and the fluctuations perpendicular with ϕ_n are not affected.

$$\begin{aligned} &\int \exp[-i\alpha \langle \xi_n, Q^+(\phi_{n+1} \psi_n) \rangle] d\mu(\xi_n) \\ &= \det^{-1/2}(P_n) \exp \left[-\frac{1}{N} \langle \psi_n, \phi_{n+1} Q \frac{1}{P_n} Q^+ \phi_{n+1} \psi_n \rangle \right] \end{aligned}$$

5. If γ_n is small, then

$$\phi_{n+1} Q P_n^{-1} Q^+ \phi_{n+1} \sim N G_{n+1} \circ T_n \sim N \beta_{n+1} T_n$$

where $T_n = Q \Gamma_n Q^+$. This is very large. If γ_n is large then

$$\phi_{n+1} Q P_n^{-1} Q^+ \phi_{n+1} = \frac{1}{\gamma_n} T_n \sim 0$$

In the same way, for small γ_n ,

$$\det^{-1/2}(P_n) = \det^{-N/2}(1 + i\alpha T_n \psi_n)$$

and for large γ_n

$$\det^{-1/2}(P_n) = \det^{-(N-1)/2}(1 + i\alpha T_n \psi_n)$$

6. Thus we obtain the Gaussian term of ψ_n expanding the determinant up to the second order. The first term is used to decrease β_n by $T_n = O(1)$ and the second term is used to make the Hamiltonian of ψ_n . Remark that $\langle \varphi_n, \Psi_n \rangle \sim \langle \phi_n, \psi_n \rangle$, $\langle \varphi_n^2, \Psi_n \rangle \sim \langle \phi_n^2, \psi_n \rangle$ etc., thanks to the properties of $\mathcal{A}_n$ and $\tilde{\mathcal{A}}_n$.

Then $H(\psi_n) = \langle \Psi_n (\mathcal{T}_n^{\circ 2} + 2\mathcal{T}_n \circ \mathcal{G}_{n+1}) \Psi_n \rangle$ for small γ_n and $H(\psi_n) = \langle \Psi_n, \mathcal{T}_n^{\circ 2} \Psi_n \rangle$ for large γ_n . Put $\psi_n = \tilde{A}_{n+1} \psi_{n+1} + Q \tilde{\psi}_n$. The integral of $\langle \phi_{n+1}^2, Q \tilde{\psi}_n \rangle$ by $\tilde{\psi}_n$ yields a new factor of order $O(N^{-1})$ of form γ_n since Q^+ acts as a differential operator on ϕ_{n+1}^2 .

7. As a conclusion, the term proportional to γ_n does not have strong effects on the flow. The flow of u_n is not affected by γ_n . The curvature of the potential at $\phi_n^2 = u_n = N(\beta_0 - O(n))$ is N^{-1} uniformly in n . This is what happens in the hierarchical model approximation of Dyson-Wilson type of the sigma model with large N .
8. Thus our iteration continues except for the regions of the large fields and domain walls which have a small probability to exist. Thus this transformation iterates well and we reach at the high-temperature region [16].

As we discussed, our conclusion follows from the form of W_n of large n such that $\beta_n = O(1)$. We are not sure if the idea used here can be applied to the study of the non-abelian lattice gauge theory.

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