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Shuffle-type product formulae of desingularized values of multiple zeta-functions

By

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Abstract

It is known that there are infinitely many singularities of multiple zeta functions and the special values at non-positive integer points are indeterminate. In order to give a suitable rigorous meaning of the special values there, Furusho, Komori, Matsumoto and Tsumura introduced desingularized values by using their desingularization method to resolve all singularities. On the other hand, Ebrahimi-Fard, Manchon and Singer introduced renormalized values by the renormalization method à la Connes and Kreimer and they showed that the values fulfill the shuffle-type product formula. In this paper, we show the shuffle-type product formulae for desingularized values.

§ 0. Introduction

In 1776, Euler ([9]) considered certain power series, the so-called double zeta values, and showed several relations among them. More than 200 years later, the *multiple zeta value* (MZV for short) which is more general series

$$\zeta(k_1,\ldots,k_r) := \sum_{0<m_1<\cdots<m_r} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

converging for $k_1,\ldots,k_r \in \mathbb{N}$ and $k_r > 1$, was discussed by Ecalle ([8]) in 1981. In 1990s, these values also came to be focused by Hoffman ([14]) and Zagier ([21]). MZVs admit an iterated integral expressions, which enable us to regard them as a period of a certain motives ([5], [12] and [20]) and calculate the Kontsevich invariant in knot theory ([17]). MZVs are also related to mathematical physics ([2] and [3]).

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MZVs are regarded as special values at positive integer points of the several variables complex analytic function, the *multiple zeta-function* (MZF for short), which is defined by

$$\zeta(s_1,\dots,s_r):=\sum_{0<m_1<\cdots<m_r}\frac{1}{m_1^{s_1}\cdots m_r^{s_r}}.$$

It converges absolutely in the region

$$\{(s_1,\dots,s_r)\in\mathbb{C}^r \mid \Re(s_{r-k+1}+\cdots+s_r)>k\ (1\le k\le r)\}.$$

In the early 2000s, Zhao ([22]) and Akiyama, Egami and Tanigawa ([1]) independently showed that MZF can be meromorphically continued to $\mathbb{C}^r$. Especially, in [1], the set of all singularities of the function $\zeta(s_1,\dots,s_r)$ is determined as

$$\begin{aligned}&s_r=1,\\ &s_{r-1}+s_r=2,1,0,-2,-4,\dots,\\ &s_{r-k+1}+\cdots+s_r=k-n\quad (3\le k\le r,\ n\in\mathbb{N}_0).\end{aligned}$$

Because almost all of integer points with non-positive arguments are located in the above singularities, the special values of MZF there are indeterminate in all cases except for $\zeta(-k)$ at $k\in\mathbb{N}_0$, and $\zeta(-k_1,-k_2)$ at $k_1,k_2\in\mathbb{N}_0$ with k_1+k_2 odd. Actually, giving a nice definition of "$\zeta(-k_1,\dots,-k_r)$" for $k_1,\dots,k_r\in\mathbb{N}_0$ is one of our most fundamental problems.

In order to resolve all infinitely many singularities of MZF, the desingularization method was introduced by Furusho, Komori, Matsumoto and Tsumura in [10]. By applying this method to $\zeta(s_1,\dots,s_r)$, they constructed the *desingularized MZF* $\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)$ which is entire on the whole space $\mathbb{C}^r$. The functions are represented by finite linear combinations of shifted MZFs (cf. Proposition 2.3.). The *desingularized value*

$$\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)\in\mathbb{C}$$

is defined to be the special value of $\zeta_r^{\mathrm{des}}(s_1,\dots,s_r)$ at $(s_1,\dots,s_r)=(-k_1,\dots,-k_r)$ for $k_1,\dots,k_r\in\mathbb{N}_0$ (see Definition 2.4). In [10], its generating function given by

$$Z_{\text{FKMT}}(t_1,\dots,t_r):=\sum_{k_1,\dots,k_r=0}^{\infty}\frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)\tag{0.1}$$

in $\mathbb{C}[[t_1,\dots,t_r]]$ was calculated and desingularized values were explicitly described in terms of the Bernoulli numbers (see Proposition 2.5.).

In contrast, Connes and Kreimer ([4]) started a Hopf algebraic approach to the renormalization procedure in the perturbative quantum field theory. A fundamental tool in their work is the *algebraic Birkhoff decomposition*. By applying this decomposition to a certain Hopf algebra related to MZVs, Guo and Zhang ([13]) introduced the *renormalized values* which satisfy the harmonic-type product formulae. Later, Manchon and Paycha ([18]) and Ebrahimi-Fard, Manchon and Singer ([7]) introduced the different renormalized values which obey the harmonic-type product formulae by using different Hopf algebras. Ebrahimi-Fard, Manchon and Singer ([6]) also introduced another type of the renormalized values satisfying the shuffle-type product. We denote their values as

$$\zeta_{\mathrm{EMS}}(-k_1,\dots,-k_r)\in\mathbb{C}$$

for $k_1,\dots,k_r\in\mathbb{N}_0$, and their generating function as

$$Z_{\mathrm{EMS}}(t_1,\dots,t_r):=\sum_{k_1,\dots,k_r=0}^{\infty}\frac{(-t_1)^{k_1}\cdots(-t_r)^{k_r}}{k_1!\cdots k_r!}\zeta_{\mathrm{EMS}}(-k_1,\dots,-k_r) \tag{0.2}$$

in $\mathbb{C}[[t_1,\dots,t_r]]$. In the paper [15], the author revealed the following relationship between generating functions (0.1) and (0.2):

$$Z_{\mathrm{EMS}}(t_1,\dots,t_r)=\prod_{i=1}^{r}\frac{1-e^{-t_i-\cdots-t_r}}{t_i+\cdots+t_r}\cdot Z_{\mathrm{FKMT}}(-t_1,\dots,-t_r). \tag{0.3}$$

The following recurrence formulae were essential for the proof of the equation (0.3):

$$Z_{\mathrm{FKMT}}(t_1,\dots,t_r)=Z_{\mathrm{FKMT}}(t_2,\dots,t_r)\cdot Z_{\mathrm{FKMT}}(t_1+\cdots+t_r). \tag{0.4}$$

By these recurrence formulae, we will show our main theorem (Theorem 3.3) that $\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)$ fulfills the following shuffle-type product formula (0.5) which are shown to hold for $\zeta_{\mathrm{EMS}}(-k_1,\dots,-k_r)$ by [6]:

Theorem 3.3 *For $k_1,\dots,k_p,l_1,\dots,l_q\in\mathbb{N}_0$, we have*

$$\begin{aligned}(0.5)\quad &\zeta_p^{\mathrm{des}}(-k_1,\dots,-k_p)\zeta_q^{\mathrm{des}}(-l_1,\dots,-l_q)\\ &=\sum_{\substack{i_1+j_1=l_1\\ \vdots\\ i_q+j_q=l_q}}\prod_{a=1}^{q}(-1)^{i_a}\binom{l_a}{i_a}\zeta_{p+q}^{\mathrm{des}}(-k_1,\dots,-k_{p-1},-k_p-i_1-\cdots-i_q,-j_1,\dots,-j_q).\end{aligned}$$

The above recurrence formula (0.4) also yields

$$\zeta_r^{\mathrm{des}}(-k_1,\dots,-k_r)=\sum_{\substack{i+j=k_r\\ i,j\geq 0}}\binom{k_r}{i}\zeta_{r-1}^{\mathrm{des}}(-k_1,\dots,-k_{r-2},-k_{r-1}-i)\zeta_1^{\mathrm{des}}(-j) \tag{0.6}$$

for $k_1,\dots,k_r\in\mathbb{N}_0$. We will extend the equation (0.6) to the equation (4.15) by replacing $-k_1,\dots,-k_{r-1}\in\mathbb{Z}_{\leq 0}$ with $s_1,\dots,s_{r-1}\in\mathbb{C}$ in Proposition 4.8.

The plan of our paper goes as follows. In §1, we will review the algebraic Birkoff decomposition and the definition of the renormalized values in [6]. In §2, we will recall the definition of the desingularized MZFs and the desingularized values introduced by Furusho, Komori, Matsumoto and Tsumura in [10]. In §3, we will prove the shuffle-type product formulae of desingularized values at non-positive integer points (Theorem 3.3). In §4, we will show the formula (4.14) in Proposition 4.8, which generalizes the equation (0.5) in the case of $q=1$.

§ 1. Algebraic Birkhoff decomposition and renormalized values

In this section, we assume $\mathcal{H}$ is a Hopf algebra over $\mathbb{Q}$, $\mathcal{A} := \mathbb{Q}[[z]][z^{-1}]$ and $\mathcal{L}(\mathcal{H},\mathcal{A}) := \{f : \mathcal{H} \to \mathcal{A} \mid f \text{ is a } \mathbb{Q}\text{-linear map}\}$. For the maps $f, g \in \mathcal{L}(\mathcal{H},\mathcal{A})$, we define the convolution $f * g \in \mathcal{L}(\mathcal{H},\mathcal{A})$ by

$$f * g := m \circ (f \otimes g) \circ \Delta,$$

where m is the product of $\mathcal{A}$ and Δ is the coproduct of $\mathcal{H}$. Then, the subset

$$G(\mathcal{H},\mathcal{A}) := \{\ f \in \mathcal{L}(\mathcal{H},\mathcal{A}) |\ f(1) = 1\}$$

forms a group with the above convolution product $*$.

Theorem 1.1 ([4], [6]: **the algebraic Birkhoff decomposition).**
For $f \in G(\mathcal{H},\mathcal{A})$, there are unique linear maps $f_+ : \mathcal{H} \to \mathbb{Q}[[z]]$ and $f_- : \mathcal{H} \to \mathbb{Q}[z^{-1}]$ with $f_-(1) = 1 \in \mathbb{Q}$ such that

$$f = f_-^{-1} * f_+,$$

where f_-^{-1} is the inverse element of f_- in $G(\mathcal{H},\mathcal{A})$. Moreover the maps f_- and f_+ form algebra homomorphisms if the map f is an algebra homomorphism.

Let $\mathbb{Q}\langle d, y\rangle$ be the $\mathbb{Q}$-vector space generated by the word (including 1) of d and y. We define the $\mathbb{Q}$-algebra $(\mathbb{Q}\langle d, y\rangle, ⧢_0)$ by the new product $⧢_0 : \mathbb{Q}\langle d, y\rangle^{\otimes 2} \to \mathbb{Q}\langle d, y\rangle$ which is a $\mathbb{Q}$-linear map recursively defined by

$$\begin{aligned} 1 ⧢_0 w &:= w ⧢_0 1 := w, \\ yu ⧢_0 v &:= u ⧢_0 yv := y(u ⧢_0 v), \\ du ⧢_0 dv &:= d(u ⧢_0 dv) - u ⧢_0 d^2 v, \end{aligned}$$

for words u,v and w of d and y. This algebra $(\mathbb{Q}\langle d, y\rangle, ⧢_0)$ forms a non-commutative algebra. We consider the following set $\mathcal{S}$:

$$\mathcal{S} := \langle d^k\{d(u ⧢_0 v) - du ⧢_0 v - u ⧢_0 dv\},\ wd \mid u, v, w\text{: words, } k \in \mathbb{N}_0\rangle_{(\mathbb{Q}\langle d,y\rangle, ⧢_0)},$$

that is, to be the two-sided ideal of $(\mathbb{Q}\langle d,y\rangle, ⧢_0)$ algebraically generated by the above elements. Then, the quotient

$$\mathcal{H}_0 := \mathbb{Q}\langle d,y\rangle/\mathcal{S},$$

forms a commutative and cocommutative Hopf algebra (its coproduct is not the deconcatenation coproduct. For detail, see [6],[15]). We define the $\mathbb{Q}$-linear map $\phi : \mathcal{H}_0 \to \mathcal{A}$ by $\phi(1) := 1$ and for $k_1, \dots, k_n \in \mathbb{N}_0$,

$$\phi(d^{k_1}y \cdots d^{k_r}y)(z) := \partial_z^{k_1}(x\partial_z^{k_2}) \cdots (x\partial_z^{k_r})(x(z)),$$

where $x := x(z) := \frac{e^z}{1-e^z} \in \mathbb{Q}[[z]][z^{-1}]$ and ∂_z is the derivative by z.

Proposition 1.2 ([6, §4.2]). *The $\mathbb{Q}$-linear map $\phi : \mathcal{H}_0 \to \mathbb{Q}[[z]][z^{-1}]$ is well-defined and forms algebra homomorphism.*

By applying Theorem 1.1 to this map ϕ, we obtain the algebra homomorphism $\phi_+ : \mathcal{H}_0 \to \mathbb{Q}[[z]]$.

Definition 1.3 ([6, §4.2]). The renormalized values[1] $\zeta_{\text{EMS}}(-k_1, \dots, -k_r)$ is defined by

$$\zeta_{\text{EMS}}(-k_1, \dots, -k_r) := \lim_{z\to 0} \phi_+(d^{k_r}y \cdots d^{k_1}y)(z)$$

for $k_1, \dots, k_r \in \mathbb{N}_0$.

These renormalized values coincide with special values of the meromorphic continuation of MZFs at non-positive integer points which are non-singular, i.e.,

Proposition 1.4 ([6, Theorem 4.3]). *For $k \in \mathbb{N}_0$, we have*

$$\zeta_{\text{EMS}}(-k) = \zeta(-k),$$

and for $k_1, k_2 \in \mathbb{N}_0$ with $k_1 + k_2$ odd, we have

$$\zeta_{\text{EMS}}(-k_1, -k_2) = \zeta(-k_1, -k_2).$$

By Theorem 1.1 and Proposition 1.2, we get the proposition below:

Proposition 1.5 ([6, §4.2]: **shuffle-type product formula**).
For the elements w and w' of $\mathcal{H}_0$, we have

$$\phi_+(w ⧢_0 w') = \phi_+(w)\phi_+(w').$$

Here are examples in lower depth:

[1]If we follow the notations of [6], it should be denoted by $\zeta_+(-k_r, \dots, -k_1)$.

Example 1.6. For $a, b, c \in \mathbb{N}_0$, we have

$$\zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b) = \sum_{k=0}^{a} (-1)^k \binom{a}{k} \zeta_{\text{EMS}}(-b-k, -a+k),$$

$$\zeta_{\text{EMS}}(-a) \cdot \zeta_{\text{EMS}}(-b, -c) = \sum_{\substack{i_1+j_1=b \\ i_2+j_2=c}} (-1)^{i_1+i_2} \binom{b}{i_1} \binom{c}{i_2} \zeta_{\text{EMS}}(-a-i_1-i_2, -j_1, -j_2).$$

In the paper [15], the author showed the explicit formula of $\zeta_{\text{EMS}}(-k_1, \ldots, -k_n)$:

Proposition 1.7 ([15]). *For $r \in \mathbb{N}$, we have*

$$Z_{\text{EMS}}(t_1, \ldots, t_r) = \prod_{i=1}^{r} \frac{(t_i + \cdots + t_r) - (e^{t_i + \cdots + t_r} - 1)}{(t_i + \cdots + t_r)(e^{t_i + \cdots + t_r} - 1)},$$

where $Z_{\text{EMS}}(t_1, \ldots, t_r)$ is the generating function (0.2) *of $\zeta_{\text{EMS}}(-k_1, \ldots, -k_r)$.*

§ 2. Desingularization of multiple zeta-functions

In this section, we review desingularized values introduced by Furusho, Komori, Matsumoto and Tsumura in [10]. In §1.1, we recall the definition of the desingularized MZF, and explain some remarkable properties of this function. In §1.2, we review desingularized values and their generating function.

§ 2.1. Desingularized MZFs

In this subsection, we review the definition of desingularized MZF and the two properties, i.e., desingularized MZF can be analytically continued to $\mathbb{C}^r$ as an entire function (Proposition 2.2) and can be represented by a finite "linear" combination of MZFs (Proposition 2.3). We consider the generating function[2] $\tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c) \in \mathbb{C}[[t_1, \ldots, t_r]]$ (cf. [10, Definition 1.9]):

$$\begin{aligned}
\tilde{\mathfrak{H}}_r(t_1, \ldots, t_r; c) &:= \prod_{j=1}^{r} \left(\frac{1}{\exp\left(\sum_{k=j}^{r} t_k\right) - 1} - \frac{c}{\exp\left(c \sum_{k=j}^{r} t_k\right) - 1} \right) \\
&= \prod_{j=1}^{r} \left(\sum_{m=1}^{\infty} (1 - c^m) B_m \frac{\left(\sum_{k=j}^{r} t_k\right)^{m-1}}{m!} \right)
\end{aligned}$$

for $c \in \mathbb{R}$. Here B_m $(m \geq 0)$ is the Bernoulli number which is defined by

$$\frac{x}{e^x - 1} := \sum_{m \geq 0} \frac{B_m}{m!} x^m. \tag{2.1}$$

We note that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$.

[2] It is denoted by $\tilde{\mathfrak{H}}_n((t_j); (1); c)$ in [10].

Definition 2.1 ([10, Definition 3.1]). For non-integral complex numbers $s_1, \dots, s_r$, *desingularized MZF* $\zeta_r^{\mathrm{des}}(s_1, \dots, s_r)$ is defined by

$$
\begin{aligned}
(2.2) \qquad & \zeta_r^{\mathrm{des}}(s_1, \dots, s_r) \\
& := \lim_{\substack{c \to 1 \\ c \in \mathbb{R} \setminus \{1\}}} \frac{1}{(1-c)^r} \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)} \int_{\mathcal{C}^r} \tilde{\mathfrak{H}}_r(t_1, \dots, t_r; c) \prod_{k=1}^{r} t_k^{s_k - 1} dt_k.
\end{aligned}
$$

Here $\mathcal{C}$ is the path consisting of the positive real axis (top side), a circle around the origin of radius ε (sufficiently small), and the positive real axis (bottom side).

One of the remarkable properties of desingularized MZF is that it is an entire function, i.e., the equation (2.2) is well-defined as an analytic function by the following proposition.

Proposition 2.2 ([10, Theorem 3.4]). *The equation $\zeta_r^{\mathrm{des}}(s_1, \dots, s_r)$ can be analytically continued to $\mathbb{C}^r$ as an entire function in $(s_1, \dots, s_r) \in \mathbb{C}^r$ by the following integral expression:*

$$
\begin{aligned}
\zeta_r^{\mathrm{des}}(s_1, \dots, s_r) &= \prod_{k=1}^{r} \frac{1}{(e^{2\pi i s_k} - 1)\Gamma(s_k)} \\
& \cdot \int_{\mathcal{C}^n} \prod_{j=1}^{r} \lim_{\substack{c \to 1 \\ c \in \mathbb{R} \setminus \{1\}}} \frac{1}{1-c} \left(\frac{1}{\exp\left(\sum_{k=j}^{r} t_k\right) - 1} - \frac{c}{\exp\left(c \sum_{k=j}^{r} t_k\right) - 1} \right) \prod_{k=1}^{r} t_k^{s_k - 1} dt_k.
\end{aligned}
$$

For indeterminates u_j and v_j $(1 \le j \le r)$, we set

$$
(2.3) \qquad \mathcal{G}_r(u_1, \dots, u_r; v_1, \dots, v_r) := \prod_{j=1}^{r} \{1 - (u_j v_j + \cdots + u_r v_r)(v_j^{-1} - v_{j-1}^{-1})\}
$$

with the convention $v_0^{-1} := 0$, and we define the set of integers $\{a_{\boldsymbol{l}, \boldsymbol{m}}^r\}$ by

$$
(2.4) \qquad \mathcal{G}_r(u_1, \dots, u_r; v_1, \dots, v_r) = \sum_{\substack{\boldsymbol{l} = (l_j) \in \mathbb{N}_0^r \\ \boldsymbol{m} = (m_j) \in \mathbb{Z}^r \\ |\boldsymbol{m}| = 0}} a_{\boldsymbol{l}, \boldsymbol{m}}^r \prod_{j=1}^{r} u_j^{l_j} v_j^{m_j}.
$$

Here, $|\boldsymbol{m}| := m_1 + \cdots + m_r$.
Another remarkable properties of desingularized MZF is that the function is given by a finite "linear" combination of shifted MZFs, i.e.,

Proposition 2.3 ([10, Theorem 3.8]). *For $s_1, \dots, s_r \in \mathbb{C}$, we have the following equality between meromorphic functions of the complex variables $(s_1, \dots, s_r)$:*

$$
(2.5) \qquad \zeta_r^{\mathrm{des}}(s_1, \dots, s_r) = \sum_{\substack{\boldsymbol{l} = (l_j) \in \mathbb{N}_0^r \\ \boldsymbol{m} = (m_j) \in \mathbb{Z}^r \\ |\boldsymbol{m}| = 0}} a_{\boldsymbol{l}, \boldsymbol{m}}^r \left(\prod_{j=1}^{r} (s_j)_{l_j} \right) \zeta(s_1 + m_1, \dots, s_r + m_r).
$$

Here, $(s)_k$ *is the Pochhammer symbol, that is, for* $k \in \mathbb{N}$ *and* $s \in \mathbb{C}$ $(s)_0 := 1$ *and* $(s)_k := s(s+1)\cdots(s+k-1)$.

§ 2.2. Desingularized values

We review the definition of desingularized values and their explicit formula (Proposition 2.5), and then we give a recurrence formula of desingularized values (Corollary 2.6).

Definition 2.4. For $k_1, \ldots, k_r \in \mathbb{N}_0$, *desingularized value* $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r) \in \mathbb{C}$ is defined to be the special value of desingularized MZF $\zeta_r^{\mathrm{des}}(s_1, \ldots, s_r)$ at $(s_1, \ldots, s_r) = (-k_1, \ldots, -k_r)$.

The generating function $Z_{\mathrm{FKMT}}(t_1, \ldots, t_r)$ of $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ in the equation (0.1) is explicitly calculated as follows.

Proposition 2.5 ([10, Theorem 3.7]). *We have*

$$Z_{\mathrm{FKMT}}(t_1, \ldots, t_r) = \prod_{i=1}^{r} \frac{(1 - t_i - \cdots - t_r)e^{t_i + \cdots + t_r} - 1}{(e^{t_i + \cdots + t_r} - 1)^2}.$$

In terms of $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ *for* $k_1, \ldots, k_r \in \mathbb{N}_0$, *the above equation is reformulated to*

$$\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r) = (-1)^{k_1 + \cdots + k_r} \sum_{\substack{\nu_{1i} + \cdots + \nu_{ii} = k_i \\ 1 \le i \le r}} \prod_{i=1}^{r} \frac{k_i!}{\prod_{j=i}^{r} \nu_{ij}!} B_{\nu_{ii} + \cdots + \nu_{ir} + 1}.$$

By the above proposition we have the following recurrence formula:

Corollary 2.6.

$$Z_{\mathrm{FKMT}}(t_1, \ldots, t_r) = Z_{\mathrm{FKMT}}(t_2, \ldots, t_r) \cdot Z_{\mathrm{FKMT}}(t_1 + \cdots + t_r) \quad (r \in \mathbb{N}). \tag{2.6}$$

In terms of $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$, *the equation* (2.6) *is reformulated to*

$$\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r) = \sum_{\substack{i_2 + j_2 = k_2 \\ \vdots \\ i_r + j_r = k_r}} \prod_{a=2}^{r} \binom{k_a}{i_a} \zeta_{r-1}^{\mathrm{des}}(-i_2, \ldots, -i_r) \zeta_1^{\mathrm{des}}(-k_1 - j_2 - \cdots - j_r) \tag{2.7}$$

for $k_1, \ldots, k_r \in \mathbb{N}_0$.

In the paper [15], the author showed that the desingularized values $\zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_r)$ and renormalized values $\zeta_{\mathrm{EMS}}(-k_1, \ldots, -k_r)$ in [6] are equivalent (the equation (0.3)), i.e.

Theorem 2.7. *For $r \in \mathbb{N}$, we have*

$$Z_{\mathrm{EMS}}(t_1, \dots, t_r) = \prod_{i=1}^{r} \frac{1 - e^{-t_i - \cdots - t_r}}{t_i + \cdots + t_r} \cdot Z_{\mathrm{FKMT}}(-t_1, \dots, -t_r).$$

The following is an example of our equivalence:

Example 2.8. For $k \in \mathbb{N}_0$, we have

$$\zeta_{\mathrm{EMS}}(-k) = \sum_{i+j=k} \binom{k}{i} \frac{(-1)^j}{i+1} \zeta_{\mathrm{FKMT}}(-j),$$
$$\zeta_{\mathrm{FKMT}}(-k) = (-1)^k \sum_{i+j=k} \binom{k}{i} B_i \zeta_{\mathrm{EMS}}(-j).$$

§ 3. The product formulae at non-positive integer points

In this section, we prove the shuffle-type product formulae of desingularized values at non-positive integer points (Theorem 3.3).

Lemma 3.1. *For $r \in \mathbb{N}$, we have*

$$Z_{\mathrm{FKMT}}(u_1) \cdots Z_{\mathrm{FKMT}}(u_r) = Z_{\mathrm{FKMT}}(u_1 - u_2, u_2 - u_3, \dots, u_{r-1} - u_r, u_r). \tag{3.1}$$

Proof. Let $r \in \mathbb{N}$. Using the equation (2.6) repeatedly, we get

$$Z_{\mathrm{FKMT}}(t_1, \dots, t_r) = \prod_{i=1}^{r} Z_{\mathrm{FKMT}}(t_i + \cdots + t_r).$$

We replace $t_i + \cdots + t_r$ to u_i for $i = 1, \dots, r$ in this formula. Then, we obtain the equation (3.1). □

Calculating simply, we obtain the following lemma.

Lemma 3.2. *For $r \in \mathbb{N}$, $a_1, \dots, a_r \in \mathbb{C}$ and $f : \mathbb{N}_0 \to \mathbb{C}$, we have*

$$\sum_{k=0}^{\infty} \frac{(a_1 + \cdots + a_r)^k}{k!} f(k) = \sum_{k=0}^{\infty} \frac{f(k)}{k!} \sum_{i_1 + \cdots + i_r = k} \frac{k!}{i_1! \cdots i_r!} a_1^{i_1} \cdots a_r^{i_r}$$
$$= \sum_{i_1, \dots, i_r = 0}^{\infty} \frac{a_1^{i_1} \cdots a_r^{i_r}}{i_1! \cdots i_r!} f(i_1 + \cdots + i_r).$$

Using the above two lemmas, we have the following theorem.

Theorem 3.3. *For $p, q \in \mathbb{N}$ and $k_1, \ldots, k_p, l_1, \ldots, l_q \in \mathbb{N}_0$, we have*

$$\begin{aligned}(3.2)\quad &\zeta_p^{\mathrm{des}}(-k_1, \ldots, -k_p)\zeta_q^{\mathrm{des}}(-l_1, \ldots, -l_q) \\ &= \sum_{\substack{i_1+j_1=l_1 \\ \vdots \\ i_q+j_q=l_q}} \prod_{a=1}^{q} (-1)^{i_a}\binom{l_a}{i_a} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_{p-1}, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q).\end{aligned}$$

Proof. Using the equation (2.6) repeatedly, we get

$$Z_{\mathrm{FKMT}}(s_1, \ldots, s_p)Z_{\mathrm{FKMT}}(t_1, \ldots, t_q) = Z_{\mathrm{FKMT}}(s_1+\cdots+s_p) \cdots Z_{\mathrm{FKMT}}(s_p)Z_{\mathrm{FKMT}}(t_1+\cdots+t_q) \cdots Z_{\mathrm{FKMT}}(t_q).$$

By putting $u_i = \begin{cases} s_i + \cdots + s_p & (1 \leq i \leq p), \\ t_{i-p} + \cdots + t_q & (p+1 \leq i \leq p+q), \end{cases}$ and applying the equation (3.1) to the above equation, we have

$$\begin{aligned} &Z_{\mathrm{FKMT}}(s_1, \ldots, s_p)Z_{\mathrm{FKMT}}(t_1, \ldots, t_q) \\ &= Z_{\mathrm{FKMT}}(s_1, \ldots, s_{p-1}, s_p - t_1 - \cdots - t_q, t_1, \ldots, t_q) \\ &= \sum_{k_1, \ldots, k_p \geq 0} \frac{(-s_1)^{k_1} \cdots (-s_{p-1})^{k_{p-1}}(-s_p + t_1 + \cdots + t_q)^{k_p}}{k_1! \cdots k_{p-1}! k_p!} \\ &\qquad \cdot \sum_{j_1, \ldots, j_q \geq 0} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p, -j_1, \ldots, -j_q) \\ &= \sum_{\substack{k_1, \ldots, k_{p-1} \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{(-s_1)^{k_1} \cdots (-s_{p-1})^{k_{p-1}}}{k_1! \cdots k_{p-1}!} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!} \\ &\qquad \cdot \sum_{k_p \geq 0} \frac{(-s_p + t_1 + \cdots + t_q)^{k_p}}{k_p!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p, -j_1, \ldots, -j_q). \end{aligned}$$

Using Lemma 3.2, we get

$$\begin{aligned} &Z_{\mathrm{FKMT}}(s_1, \ldots, s_p)Z_{\mathrm{FKMT}}(t_1, \ldots, t_q) \\ &= \sum_{\substack{k_1, \ldots, k_{p-1} \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{(-s_1)^{k_1} \cdots (-s_{p-1})^{k_{p-1}}}{k_1! \cdots k_{p-1}!} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!} \\ &\qquad \cdot \sum_{k_p, i_1, \ldots, i_q \geq 0} \frac{(-s_p)^{k_p} t_1^{i_1} \cdots t_q^{i_q}}{k_p! i_1! \cdots i_q!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q) \\ &= \sum_{k_1, \ldots, k_p \geq 0} \frac{(-s_1)^{k_1} \cdots (-s_p)^{k_p}}{k_1! \cdots k_p!} \\ &\qquad \cdot \sum_{\substack{i_1, \ldots, i_q \geq 0 \\ j_1, \ldots, j_q \geq 0}} \frac{t_1^{i_1} \cdots t_q^{i_q}}{i_1! \cdots i_q!} \frac{(-t_1)^{j_1} \cdots (-t_q)^{j_q}}{j_1! \cdots j_q!} \zeta_{p+q}^{\mathrm{des}}(-k_1, \ldots, -k_p - i_1 - \cdots - i_q, -j_1, \ldots, -j_q) \end{aligned}$$

$$
\begin{aligned}
&Z_{\text{FKMT}}(s_1,\dots,s_p)Z_{\text{FKMT}}(t_1,\dots,t_q)\\
&=\sum_{k_1,\dots,k_p\ge 0}\frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\\
&\quad\cdot\sum_{\substack{i_1,\dots,i_q\ge 0\\ j_1,\dots,j_q\ge 0}}\frac{(-t_1)^{i_1+j_1}\cdots(-t_q)^{i_q+j_q}}{i_1!\cdots i_q!j_1!\cdots j_q!}(-1)^{i_1+\cdots+i_q}\zeta_{p+q}^{\text{des}}(-k_1,\dots,-k_p-i_1-\cdots-i_q,-j_1,\dots,-j_q)\\
&=\sum_{\substack{k_1,\dots,k_p\ge 0\\ l_1,\dots,l_q\ge 0}}\frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\frac{(-t_1)^{l_1}\cdots(-t_q)^{l_q}}{l_1!\cdots l_q!}\\
&\quad\cdot\sum_{\substack{i_1+j_1=l_1\\ \vdots\\ i_q+j_q=l_q}}\prod_{a=1}^{q}\binom{l_a}{i_a}(-1)^{i_a}\zeta_{p+q}^{\text{des}}(-k_1,\dots,-k_p-i_1-\cdots-i_q,-j_1,\dots,-j_q).
\end{aligned}
$$

On the other hand, by the definition of $Z_{\text{FKMT}}(t_1,\dots,t_q)$, we have

$$
\begin{aligned}
&Z_{\text{FKMT}}(s_1,\dots,s_p)Z_{\text{FKMT}}(t_1,\dots,t_q)\\
&\quad=\left\{\sum_{k_1,\dots,k_p\ge 0}\frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\zeta_p^{\text{des}}(-k_1,\dots,-k_p)\right\}\\
&\qquad\qquad\cdot\left\{\sum_{l_1,\dots,l_q\ge 0}\frac{(-t_1)^{l_1}\cdots(-t_q)^{l_q}}{l_1!\cdots l_q!}\zeta_q^{\text{des}}(-l_1,\dots,-l_q)\right\}\\
&\quad=\sum_{\substack{k_1,\dots,k_p\ge 0\\ l_1,\dots,l_q\ge 0}}\frac{(-s_1)^{k_1}\cdots(-s_p)^{k_p}}{k_1!\cdots k_p!}\frac{(-t_1)^{l_1}\cdots(-t_q)^{l_q}}{l_1!\cdots l_q!}\zeta_p^{\text{des}}(-k_1,\dots,-k_p)\zeta_q^{\text{des}}(-l_1,\dots,-l_q).
\end{aligned}
$$

Therefore, we obtain the equation (3.2). $\square$

Here are examples for $(p,q)=(1,1),\ (1,2)$.

Example 3.4. For $a,b,c\in\mathbb{N}_0$, we have

$$
\begin{aligned}
\zeta_1^{\text{des}}(-a)\zeta_1^{\text{des}}(-b)&=\sum_{i_1+j_1=b}(-1)^{i_1}\binom{b}{i_1}\zeta_2^{\text{des}}(-a-i_1,-j_1),\\
\zeta_1^{\text{des}}(-a)\zeta_2^{\text{des}}(-b,-c)&=\sum_{\substack{i_1+j_1=b\\ i_2+j_2=c}}(-1)^{i_1+i_2}\binom{b}{i_1}\binom{c}{i_2}\zeta_3^{\text{des}}(-a-i_1-i_2,-j_1,-j_2).
\end{aligned}
$$

Remark. In order to prove Theorem 3.3, we essentially used only the property (2.6) of $Z_{\text{FKMT}}(t_1,\dots,t_r)$, which also holds for $Z_{\text{EMS}}(t_1,\dots,t_r)$, so $\zeta_r^{\text{des}}(-k_1,\dots,-k_r)$ satisfies the same shuffle-type product formula to $\zeta_{\text{EMS}}(-k_1,\dots,-k_r)$ introduced in [6].

§ 4. More general product formulae

In this section, we prove a generalization of the equation (0.6) in Proposition 4.8 and *general* "shuffle product" between $\zeta_r^{\mathrm{des}}(s_1,\ldots,s_{r-1})$ and $\zeta_1^{\mathrm{des}}(-l)$ in Proposition 4.9. We assume $r\in\mathbb{N}_{\geq 2}$ in this section. We start with the following lemma on the property of the Pochhammer symbol.

Lemma 4.1. *For $a,b\in\mathbb{C}$ and $n\in\mathbb{N}_0$, we have*

$$(a+b)_n=\sum_{i+j=n}\binom{n}{i}(a)_i(b)_j.$$

Proof. By considering the Taylor expansion of $(1-t)^{-a}$, we get

$$(1-t)^{-a}=\sum_{n\geq 0}\frac{(a)_n}{n!}t^n.$$

Because we have $(1-t)^{-a-b}=(1-t)^{-a}(1-t)^{-b}$, by comparing the coefficient of this equation, we obtain the claim. □

The above lemma is used in the proof of Proposition 4.7.
Next, we prove a property of $\mathcal{G}_r((u_j);(v_j))$ defined by the equation (2.3).

Proposition 4.2. *We have*

$$\begin{aligned}\mathcal{G}_r&\left(u_1,\ldots,u_r;v_1,\ldots,v_{r-1},\frac{u_r+z}{u_r}v_{r-1}\right)\\&=(z+1)\mathcal{G}_{r-1}(u_1,\ldots,u_{r-2},u_{r-1}+u_r+z;v_1,\ldots,v_{r-1}).\end{aligned}\tag{4.1}$$

Proof. By the definition of $\mathcal{G}_r((u_j);(v_j))$, we have

$$\begin{aligned}&\mathcal{G}_r\left(u_1,\ldots,u_r;v_1,\ldots,v_{r-1},\frac{u_r+z}{u_r}v_{r-1}\right)\\&=\prod_{j=1}^{r-1}\left\{1-\left(u_jv_j+\cdots+u_{r-1}v_{r-1}+u_r\frac{u_r+z}{u_r}v_{r-1}\right)(v_j^{-1}-v_{j-1}^{-1})\right\}\\&\quad\cdot\left\{1-u_r\frac{u_r+z}{u_r}v_{r-1}\left(\left(\frac{u_r+z}{u_r}v_{r-1}\right)^{-1}-v_{r-1}^{-1}\right)\right\}\\&=\prod_{j=1}^{r-1}\left\{1-(u_jv_j+\cdots+u_{r-1}v_{r-1}+(u_r+z)v_{r-1})(v_j^{-1}-v_{j-1}^{-1})\right\}\\&\quad\cdot\left\{1-(u_r+z)v_{r-1}\left(\frac{u_r}{u_r+z}-1\right)v_{r-1}^{-1}\right\}\end{aligned}$$

$$=\prod_{j=1}^{r-1}\left\{1-\left(u_jv_j+\cdots+u_{r-2}v_{r-2}+(u_{r-1}+u_r+z)v_{r-1}\right)\left(v_j^{-1}-v_{j-1}^{-1}\right)\right\}$$
$$\cdot\{1-(u_r-(u_r+z))\}$$
$$=(z+1)\mathcal{G}_{r-1}(u_1,\dots,u_{r-2},u_{r-1}+u_r+z;v_1,\dots,v_{r-1}).$$

□

It is easy to prove the following lemma by comparing coefficients $a^r_{\boldsymbol{l},\boldsymbol{m}}$ of the equations (2.3) and (2.4).

Lemma 4.3. *Let $\boldsymbol{l}:=(l_j)\in\mathbb{N}_0^r$ and $\boldsymbol{m}:=(m_j)\in\mathbb{Z}^r$. If $m_r\neq l_r-1,l_r$ or $m_r<0$, then we have*

$$a^r_{\boldsymbol{l},\boldsymbol{m}}=0. \tag{4.2}$$

For our simplicities, we employ the following symbols:

Notation 4.4. Let $s_1,\dots,s_r$ and z be indeterminates. For r-tuple symbol $\boldsymbol{s}:=(s_1,\dots,s_r)$, the symbols $\boldsymbol{s}'$ and $\boldsymbol{s}^-$ are defined by

$$\boldsymbol{s}':=(s_1,\dots,s_{r-2},s_{r-1}+s_r),$$
$$\boldsymbol{s}^-:=(s_1,\dots,s_{r-1}),$$
$$|\boldsymbol{s}|:=s_1+\cdots+s_r,$$

and we define $\boldsymbol{z}:=(\underbrace{0,\dots,0}_{r-1},z)$.

Lemma 4.5. *For the functions $f:\mathbb{Z}^r\to\mathbb{C}$ and $g:\mathbb{N}_0^{r+1}\to\mathbb{C}$ with*

$$\#\{\boldsymbol{n}\in\mathbb{Z}^r\mid f(\boldsymbol{n})\neq 0\}<\infty \text{ and } \#\{\boldsymbol{a}\in\mathbb{N}_0^{r+1}\mid g(\boldsymbol{a})\neq 0\}<\infty,$$

we have

$$\sum_{\substack{\boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\|\boldsymbol{n}|=0}} f(\boldsymbol{n})=\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\ \sum_{\substack{p+q=m_{r-1}\\p,q\in\mathbb{Z}}} f(\boldsymbol{m}^-,p,q), \tag{4.3}$$

$$\sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r} g(\boldsymbol{l}',l_{r-1},l_r)=\sum_{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1}}\ \sum_{\substack{p+q=k_{r-1}\\p,q\in\mathbb{N}_0}} g(\boldsymbol{k},p,q). \tag{4.4}$$

Proof. We only prove the equation (4.3), because the proof of the equation (4.4) can be done in the same way to that of the equation (4.3). We have

$$\sum_{\substack{\boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\|\boldsymbol{n}|=0}} f(\boldsymbol{n})=\sum_{n_1,\dots,n_{r-2},n_{r-1}\in\mathbb{Z}} f(n_1,\dots,n_{r-2},n_{r-1},-n_1-\cdots-n_{r-2}-n_{r-1})$$
$$=\sum_{m_1,\dots,m_{r-2}\in\mathbb{Z}}\ \sum_{n_{r-1}\in\mathbb{Z}} f(m_1,\dots,m_{r-2},n_{r-1},-m_1-\cdots-m_{r-2}-n_{r-1}).$$

When we put $m_{r-1} := -m_1 - \cdots - m_{r-2}$, then m_{r-1} can run over all integers. So we get

$$= \sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{n_{r-1}\in\mathbb{Z}} f(m_1,\ldots,m_{r-2},n_{r-1},m_{r-1}-n_{r-1}).$$

When we put $p := n_{r-1}$ and $q := m_{r-1} - n_{r-1}$, then p and q run over all integers with $p+q = m_{r-1}$. So we obtain

$$= \sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1}\\ p,q\in\mathbb{Z}}} f(\boldsymbol{m}^-,p,q).$$

□

Using Lemma 4.3 and Lemma 4.5, we get the following corollary.

Corollary 4.6. *For $\boldsymbol{l} := (l_1,\ldots,l_r) \in \mathbb{N}_0^r$ and $\boldsymbol{m} := (m_1,\ldots,m_{r-1}) \in \mathbb{Z}^{r-1}$ with $|\boldsymbol{m}| = 0$, we have*

$$a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)} + a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)} = \binom{l_{r-1}+l_r}{l_{r-1}} a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}} = -a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}. \tag{4.5}$$

Proof. Let $r \in \mathbb{N}$. By the equation (2.4) (the definition of the coefficient $a^r_{\boldsymbol{l},\boldsymbol{m}}$ of the function $\mathcal{G}_r$), we have

$$\mathcal{G}_r\left(u_1,\ldots,u_r;v_1,\ldots,v_{r-1},\frac{u_r+z}{u_r}v_{r-1}\right) = \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^r u_j^{l_j}\right)\left(\prod_{j=1}^{r-1} v_j^{n_j}\right)\left(\frac{u_r+z}{u_r}v_{r-1}\right)^{n_r}. \tag{4.6}$$

By using the equation (4.3) of Lemma 4.5, we have

$$\begin{aligned}
&\mathcal{G}_r\left(u_1,\ldots,u_r;v_1,\ldots,v_{r-1},\frac{u_r+z}{u_r}v_{r-1}\right)\\
&= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1}\\ p,q\in\mathbb{Z}}} a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} \left(\prod_{j=1}^r u_j^{l_j}\right)\left(\prod_{j=1}^{r-2} v_j^{m_j}\right) v_{r-1}^p \left(\frac{u_r+z}{u_r}v_{r-1}\right)^q\\
&= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=m_{r-1}\\ p,q\in\mathbb{Z}}} a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)} \left(\prod_{j=1}^r u_j^{l_j}\right)\left(\frac{u_r+z}{u_r}\right)^q \left(\prod_{j=1}^{r-1} v_j^{m_j}\right).
\end{aligned}$$

By Lemma 4.3, we get $a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)}=0$ for $q\neq l_r-1,l_r$. So we have

$$\begin{aligned}
&\mathcal{G}_r\left(u_1,\dots,u_r;v_1,\dots,v_{r-1},\frac{u_r+z}{u_r}v_{r-1}\right)\\
&=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}}\left\{a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)u_r(u_r+z)^{l_r-1}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\right.\\
&\qquad\left.+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\right\}\\
&=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}}\left\{a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(-z)(u_r+z)^{l_r-1}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\right.\\
&\qquad+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\\
&\qquad\left.+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\right\}.
\end{aligned}$$

By Lemma 4.3, we get $a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}+1,-1)}=0$ (i.e. the case of $l_r=0$). By replacing l_r-1 with l_r, we have

$$\begin{aligned}
&\mathcal{G}_r\left(u_1,\dots,u_r;v_1,\dots,v_{r-1},\frac{u_r+z}{u_r}v_{r-1}\right)\\
&=-z\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}}\left\{a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\right\}\\
&\qquad+\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}}\left\{\left(a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\right)\right.\\
&\qquad\qquad\left.\cdot\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\right\}.
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&(z+1)\mathcal{G}_{r-1}(u_1,\ldots,u_{r-2},u_{r-1}+u_r+z;v_1,\ldots,v_{r-1})\\
&=(z+1)\sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1}\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}}\left(\prod_{j=1}^{r-2}u_j^{k_j}\right)(u_{r-1}+u_r+z)^{k_{r-1}}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right)\\
&=(z+1)\sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1}\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}}\left(\prod_{j=1}^{r-2}u_j^{k_j}\right)\sum_{\substack{p+q=k_{r-1}\\ p,q\in\mathbb{N}_0}}\binom{k_{r-1}}{p}u_{r-1}^p(u_r+z)^q\left(\prod_{j=1}^{r-1}v_j^{m_j}\right).
\end{aligned}
$$

By using the equation (4.4) of Lemma 4.5, we have

$$
\begin{aligned}
&(z+1)\mathcal{G}_{r-1}(u_1,\ldots,u_{r-2},u_{r-1}+u_r+z;v_1,\ldots,v_{r-1})\\
&\quad=(z+1)\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}\\ \boldsymbol{m}=(n_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}}\binom{l_{r-1}+l_r}{l_{r-1}}a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}}\left(\prod_{j=1}^{r-1}u_j^{l_j}\right)(u_r+z)^{l_r}\left(\prod_{j=1}^{r-1}v_j^{m_j}\right).
\end{aligned}
\tag{4.7}
$$

By comparing the coefficients of (4.6) and (4.7), we obtain (4.5). □

By tracing the proof of Corollary 4.6 inversely, we get the following proposition.

Proposition 4.7. *For $s_1,\ldots,s_r,z\in\mathbb{C}$,*

$$
\begin{aligned}
&\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r}\\ |\boldsymbol{n}|=0}} a^{r}_{\boldsymbol{l},\boldsymbol{n}}\left(\prod_{j=1}^{r}(s_j)_{l_j}\right)\frac{\Gamma(s_r+n_r+z)\Gamma(-z)}{\Gamma(s_r+n_r)}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}')\\
&\qquad\qquad=(1+z)\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta^{\mathrm{des}}_{r-1}(\boldsymbol{s}'+\boldsymbol{z}'),
\end{aligned}
\tag{4.8}
$$

holds except for singularities.

Proof. Let $s_1,\ldots,s_r,z\in\mathbb{C}$. Using Corollary 4.6, we have

(4.9)

$$
\begin{aligned}
&(z+1)\sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}}\binom{l_{r-1}+l_r}{l_{r-1}}a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\\
&=-z\sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}}\left\{a^{r}_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\right\}\\
&\quad+\sum_{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}}\left\{\left(a^{r}_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}+a^{r}_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\right)\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\right\}.
\end{aligned}
$$

By multiplying the function $\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')$ and taking summation over $\boldsymbol{m}\in\mathbb{Z}^{r-1}$ with $|\boldsymbol{m}|=0$, we have

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}(\text{R.H.S. of (4.9)})\cdot\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\\
&=-z\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\left\{a^r_{(\boldsymbol{l}^-,l_r+1),(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\right\}\\
&\quad+\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\left\{\left(a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\right)\right.\\
&\qquad\qquad\left.\cdot\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\right\}\\
&=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\left\{a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(-z)(s_r+z)_{l_r-1}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\right.\\
&\qquad+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\\
&\qquad\left.+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\right\}\\
&=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\left\{a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r+1,l_r-1)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+l_r-1)(s_r+z)_{l_r-1}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\right.\\
&\qquad\left.+a^r_{\boldsymbol{l},(\boldsymbol{m}^-,m_{r-1}-l_r,l_r)}\left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right)(s_r+z)_{l_r}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\right\}.
\end{aligned}$$

By Lemma 4.3, we get $a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)}=0$ for $q\neq l_r-1,l_r$. So we have

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}(\text{R.H.S. of (4.9)})\cdot\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}')\\
&=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}}\sum_{\substack{p+q=m_{r-1}\\p,q\in\mathbb{Z}}}a^r_{\boldsymbol{l},(\boldsymbol{m}^-,p,q)}\left(\prod_{j=1}^{r}(s_j)_{l_j}\right)\frac{(s_r+z)_q}{(s_r)_q}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').
\end{aligned}$$

By using the equation (4.3) of Lemma 4.5, we have

$$
\begin{aligned}
(4.10) \qquad & \sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} (\text{R.H.S. of (4.9)}) \cdot \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \\
&= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r}(s_j)_{l_j}\right) \frac{(s_r+z)_{n_r}}{(s_r)_{n_r}} \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}').
\end{aligned}
$$

We have $\Gamma(s+n)=(s)_n\Gamma(s)$ for $s\in\mathbb{C}$ and $n\in\mathbb{N}_0$, by the relation $\Gamma(s+1)=s\Gamma(s)$. By multiplying the equation (4.10) with $\Gamma(s_r+z)\Gamma(-z)/\Gamma(s_r)$, we obtain

$$
\begin{aligned}
(4.11) \qquad & \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \cdot (\text{L.H.S. of (4.10)}) \\
&= \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^r\\ |\boldsymbol{n}|=0}} a^r_{\boldsymbol{l},\boldsymbol{n}} \left(\prod_{j=1}^{r}(s_j)_{l_j}\right) \frac{\Gamma(s_r+n_r+z)\Gamma(-z)}{\Gamma(s_r+n_r)} \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}').
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& \sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} (\text{L.H.S. of (4.9)}) \cdot \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \\
&= (z+1) \sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^r\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \binom{l_{r-1}+l_r}{l_{r-1}} a^{r-1}_{\boldsymbol{l}',\boldsymbol{m}} \left(\prod_{j=1}^{r-1}(s_j)_{l_j}\right) (s_r+z)_{l_r} \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').
\end{aligned}
$$

By using the equation (4.4) of Lemma 4.5, we have

$$
\begin{aligned}
&= (z+1) \sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1}\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\ |\boldsymbol{m}|=0}} \sum_{\substack{p+q=k_{r-1}\\ p,q\in\mathbb{N}_0}} \binom{k_{r-1}}{p} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}} \\
&\qquad \cdot \left(\prod_{j=1}^{r-2}(s_j)_{k_j}\right) (s_{r-1})_p (s_r+z)_q \zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').
\end{aligned}
$$

Using Lemma 4.1, we have

$$\sum_{\substack{\boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}} (\text{L.H.S. of (4.9)})\cdot\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}') \tag{4.12}$$
$$=(z+1)\sum_{\substack{\boldsymbol{k}=(k_j)\in\mathbb{N}_0^{r-1}\\ \boldsymbol{m}=(m_j)\in\mathbb{Z}^{r-1}\\|\boldsymbol{m}|=0}} a^{r-1}_{\boldsymbol{k},\boldsymbol{m}}\left(\prod_{j=1}^{r-2}(s_j)_{k_j}\right)(s_{r-1}+s_r+z)_{k_r-1}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{m}+\boldsymbol{z}').$$

By multiplying the equation (4.12) with $\Gamma(s_r+z)\Gamma(-z)/\Gamma(s_r)$ and by the equation (2.5) of the desingularized function $\zeta_r^{\mathrm{des}}(\boldsymbol{s})$, we obtain

$$\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\cdot(\text{R.H.S. of (4.12)})=(1+z)\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}'). \tag{4.13}$$

So we obtain the equation (4.8), by combining the equations (4.11) and (4.13) because we have (4.10) = (4.12). □

Proposition 4.8. *For $s_1,\dots,s_{r-1}\in\mathbb{C}$ and $k\in\mathbb{N}_0$, we have*

$$\zeta_r^{\mathrm{des}}(s_1,\dots,s_{r-1},-k)=\sum_{i+j=k}\binom{k}{i}\zeta_{r-1}^{\mathrm{des}}(s_1,\dots,s_{r-2},s_{r-1}-i)\zeta_1^{\mathrm{des}}(-j). \tag{4.14}$$

Proof. Let $\boldsymbol{s}:=(s_1,\dots,s_r)\in\mathbb{C}^r$. By Mellin-Barnes integral formula, we obtain the following formula ([19, the equation (3.7)]);

$$\zeta_r(\boldsymbol{s})=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{z}')\zeta(-z)dz,$$

for $\Re(s_j)>1$ $(1\le j\le r)$, $-\Re(s_r)<c<0$ and the path of integration is the vertical line $\Re(z)=c$. By this formula and the definition of $\zeta_r^{\mathrm{des}}(\boldsymbol{s})$, we have

$$\zeta_r^{\mathrm{des}}(\boldsymbol{s})=\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r}\\|\boldsymbol{n}|=0}} a^{r}_{\boldsymbol{l},\boldsymbol{n}}\left(\prod_{j=1}^{r}(s_j)_{l_j}\right)\zeta(\boldsymbol{s}+\boldsymbol{n}).$$
$$=\frac{1}{2\pi i}\int_{(c)}\sum_{\substack{\boldsymbol{l}=(l_j)\in\mathbb{N}_0^{r}\\ \boldsymbol{n}=(n_j)\in\mathbb{Z}^{r}\\|\boldsymbol{n}|=0}} a^{r}_{\boldsymbol{l},\boldsymbol{n}}\left(\prod_{j=1}^{r}(s_j)_{l_j}\right)\frac{\Gamma(s_r+n_r+z)\Gamma(-z)}{\Gamma(s_r+n_r)}$$
$$\cdot\,\zeta_{r-1}(\boldsymbol{s}'+\boldsymbol{n}'+\boldsymbol{z}')\zeta(-z)dz.$$

Using Proposition 4.7, we get

$$=\frac{1}{2\pi i}\int_{(c)}(1+z)\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta(-z)dz.$$

By Proposition 2.3, we have the formula $\zeta_1^{\mathrm{des}}(s) = (1-s)\zeta(s)$, so we obtain

$$=\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z)dz.$$

For $M \in \mathbb{N}$ and sufficiently small $\varepsilon > 0$, we set $\mathcal{D} := \{z \in \mathbb{C} \mid c < \Re(z) < M - \varepsilon\}$. For $z \in \mathcal{D}$, we have $\Re(s_r + z) > 0$ by $-\Re(s_r) < c < 0$. So singularities of the above integrand, which lie on $\mathcal{D}$, are only $z = 0, 1, 2, \ldots, M-1$. By using the residue theorem, we get

$$\begin{aligned}
=&-\sum_{j=0}^{M-1} \operatorname{Res}\left[\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z), z=j\right] \\
&+\frac{1}{2\pi i}\int_{(M-\varepsilon)} \frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)} \zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z)dz.
\end{aligned}$$

By the same arguments to those of [19], the above second term converge. By using the fact that the residue of gamma function $\Gamma(s)$ at $s = -j$ is $\frac{(-1)^j}{j!}$, we get

$$\begin{aligned}
=&\sum_{j=0}^{M-1} \binom{-s_r}{j} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1}+s_r+j)\zeta_1^{\mathrm{des}}(-j) \\
&+\frac{1}{2\pi i\Gamma(s_r)}\int_{(M-\varepsilon)} \Gamma(s_r+z)\Gamma(-z)\zeta_{r-1}^{\mathrm{des}}(\boldsymbol{s}'+\boldsymbol{z}')\zeta_1^{\mathrm{des}}(-z)dz.
\end{aligned}$$

Setting $s_r = -k$ and $M = k+1$ for $k \in \mathbb{N}_0$, we obtain

$$\zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-1}, -k) = \sum_{j=0}^{k} \binom{k}{j} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1}-k+j)\zeta_1^{\mathrm{des}}(-j),$$

because $1/\Gamma(-k) = 0$ for $k \in \mathbb{N}_0$. □

Remark. In case of $r = 2$, the above theorem recovers the equation

$$\zeta_2^{\mathrm{des}}(s, -N) = \sum_{i+j=N} \binom{N}{i} \zeta_1^{\mathrm{des}}(s-i)\zeta_1^{\mathrm{des}}(-j)$$

shown in [11, Proposition 4.3].

By Proposition 4.8, we obtain the following corollary.

Proposition 4.9. *For $s_1, \ldots, s_{r-1} \in \mathbb{C}$ and $l \in \mathbb{N}_0$, we have*

$$\zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-1})\zeta_1^{\mathrm{des}}(-l) = \sum_{i+j=l} (-1)^i \binom{l}{i} \zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1}-i, -j). \tag{4.15}$$

Proof. We prove this claim by induction on l. It is clear that the case of $l = 0$ follows from the case of $k = 0$ of Proposition 4.8. By putting $k = l_0$ (≥ 1) in the equation (4.14), we get

$$\begin{aligned}&\zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-1})\zeta_1^{\mathrm{des}}(-l_0)\\&\qquad= \zeta_r^{\mathrm{des}}(s_1, \ldots, s_{r-1}, -l_0) - \sum_{j=0}^{l_0-1} \binom{l_0}{j} \zeta_{r-1}^{\mathrm{des}}(s_1, \ldots, s_{r-2}, s_{r-1} - l_0 + j)\zeta_1^{\mathrm{des}}(-j).\end{aligned}$$

In the second term of the right hand side of this equation, by using our induction hypothesis (i.e. the equation (4.15) in the case of $0 \leq l \leq l_0 - 1$), we obtain the equation (4.15) of $l = l_0$. □

Putting $p = r - 1$ and $q = 1$ in Theorem 3.3, we obtain

$$\zeta_{r-1}^{\mathrm{des}}(-k_1, \ldots, -k_{r-1})\zeta_1^{\mathrm{des}}(-l) = \sum_{i+j=l} (-1)^i \binom{l}{i} \zeta_r^{\mathrm{des}}(-k_1, \ldots, -k_{r-2}, -k_{r-1} - i, -j)$$

for $k_1, \ldots, k_{r-1}, l \in \mathbb{N}_0$. Therefore the equation (4.15) can be regarded as a generalization of this equation.

Remark. In our forthcoming paper [16], we will show a more general formula which extends both Theorem 3.3 and Proposition 4.9.

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