

Uniqueness and error bounds for eikonal equations with discontinuities

Klaus Deckelnick & Charles M. Elliott

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. We consider the eikonal equation

$$|\nabla u(x)| = f(x) \quad x \in \Omega \quad (1.1)$$

$$u(x) = \phi(x) \quad x \in \partial\Omega, \quad (1.2)$$

where f and ϕ are given functions. The equation arises for example in geometric optics, computer vision or robotic navigation. In certain situations it is desirable to allow f to be discontinuous, e.g. in geometric optics, when light propagates through a layered medium. The aim of this paper is to study the well-posedness of (1.1), (1.2) for right hand sides f satisfying a one-sided continuity condition (see (2.2) below), that allows certain types of discontinuities. Furthermore, we shall be concerned with an error analysis for a finite difference scheme to approximate the solution of (1.1), (1.2).

The well-posedness of (1.1), (1.2) in the case of continuous f follows from the theory of viscosity solutions for Hamilton–Jacobi equations $H(x, u, \nabla u) = 0$ developed in [4]. The notion of viscosity solution was generalised by Ishii [5] to allow for discontinuous Hamiltonians H . In [11], Tourin proves a comparison result for Hamiltonians, which are allowed to be discontinuous along a smooth surface. Soravia [10] obtains necessary and sufficient conditions for uniqueness of the solution to the boundary value problem. While the work in [11] and [10] is based on Ishii’s notion of solution, several other approaches have been suggested: in [7], Newcomb & Su consider the Dirichlet problem for $H(\nabla u) = f$ and introduce a notion of solution which they call Monge solution. They obtain a comparison result as well as uniqueness for the Dirichlet problem provided that f is lower semicontinuous. Ostrov [8] studies an evolutionary Hamilton–Jacobi equation which occurs in the context of radar satellite tracking and obtains a unique solution as the limit of suitable upper and lower solutions. Recently, Camilli & Siconolfi [3] introduced a new notion of solution for Hamilton–Jacobi equations of the form $H(x, \nabla u) = 0$, which allows measurable dependence of H on x and involves measure-theoretic limits. They prove representation formulae, comparison principles and uniqueness results.

Our work uses Ishii’s definition of solution which we shall recall in §2. For a class of right hand sides f , which satisfy a suitable one-sided continuity condition we obtain well-posedness of the problem (1.1), (1.2). In §3 we discretize the problem with the help of a finite difference scheme on a regular grid. Under a slightly more restrictive condition on f we prove that the error between viscosity solution and discrete approximation is of order $\mathcal{O}(\sqrt{h})$. We have not included all the details of the proofs of existence and of the error analysis. However, a forthcoming paper, which generalises our approach to Hamilton–Jacobi equations of the form $H(\nabla u) = f$, will provide a detailed convergence analysis for a wide class of finite difference schemes as well as numerical tests.

2 Existence and Uniqueness

In order to allow for discontinuous functions f in (1.1) we shall use the following generalisation of the concept of viscosity solution, which was introduced by Ishii in [5].

Definition 2.1. A function $u \in C^0(\bar{\Omega})$ is called a viscosity subsolution (supersolution) of (1.1) if for each $\zeta \in C^\infty(\Omega)$: if $u - \zeta$ has a local maximum (minimum) at a point $x_0 \in \Omega$, then

$$|\nabla \zeta(x_0)| \leq f^*(x_0) \quad (\geq f_*(x_0)).$$

Here,

$$f^*(x) := \limsup_{r \rightarrow 0} \{f(y) \mid y \in B_r(x) \cap \Omega\}, \quad f_*(x) := \liminf_{r \rightarrow 0} \{f(y) \mid y \in B_r(x) \cap \Omega\}.$$

A viscosity solution of (1.1), (1.2) then is a function $u \in C^0(\bar{\Omega})$ which is both a viscosity sub- and supersolution and which satisfies $u(x) = \phi(x)$ for all $x \in \partial\Omega$.

Let us next formulate our assumptions on the data of the problem. We suppose that $f : \Omega \rightarrow \mathbb{R}$ is Borel measurable and that there exist $0 < m \leq M < \infty$ such that

$$m \leq f(x) \leq M \quad \forall x \in \Omega. \quad (2.1)$$

Furthermore, we assume that for every $x \in \Omega$ there exist $\epsilon_x > 0$ and $n_x \in S^{n-1}$ so that for all $y \in \Omega$, $r > 0$ and all $d \in S^{n-1}$ with $|d - n_x| < \epsilon_x$ we have

$$f(y + rd) - f(y) \leq \omega(|y - x| + r), \quad (2.2)$$

where $\omega : [0, \infty) \rightarrow [0, \infty)$ is continuous, nondecreasing and satisfies $\omega(0) = 0$. A similar type of condition was used in [11]; however, in (2.2) it is sufficient to estimate values of f for vectors whose difference is close to a given direction.

Example: Suppose that a surface Γ splits Ω into two subdomains Ω_1 and Ω_2 , that $f|_{\Omega_1} \in C^0(\bar{\Omega}_1)$, $f|_{\Omega_2} \in C^0(\bar{\Omega}_2)$ and that

$$\lim_{y \rightarrow x, y \in \Omega_1} f(y) < \lim_{y \rightarrow x, y \in \Omega_2} f(y) \quad \text{for all } x \in \Gamma.$$

In addition, assume that the following uniform cone property holds: for every $x \in \Gamma$ there exists a neighborhood U_x and a cone C_x (which is congruent to a fixed given cone C_0) such that $y \in U_x \cap \bar{\Omega}_1$ implies that $y + C_x \subset \Omega_1$. Then (2.2) holds with $n = n_x$ given by the direction of the cone C_x .

To see this, observe that the cone condition prevents a situation where $y \in \bar{\Omega}_1$, $y + rd \in \Omega_2$, which would lead to a violation of (2.2) (cf. [11], where Γ is assumed to be smooth).

One can also consider e.g. a two-dimensional domain Ω , where three curves of discontinuity meet at a triple junction.

It is not difficult to verify that (2.2) implies

$$f^*(y + rd) - f_*(y) \leq \omega(|y - x| + r) \quad (2.3)$$

for all $y \in \Omega$, $r > 0$ and $d \in S^{n-1}$, $|d - n_x| < \epsilon_x$.

Finally, we suppose for simplicity that $\phi \equiv 0$.

Lemma 2.2. *There exists a viscosity solution $u \in C^{0,1}(\bar{\Omega})$ of (1.1), (1.2).*

Proof. We only sketch the main ideas. Consider the sup-convolution of f , i.e.

$$f_\epsilon(x) := \sup_{y \in \Omega} \{f(y) - \frac{1}{\epsilon}|x - y|^2\}, \quad x \in \Omega, \epsilon > 0.$$

Clearly, f_ϵ is continuous and $f^*(x) \leq f_\epsilon(x)$ for all $x \in \Omega$. Let

$$L_\epsilon(x, y) := \inf \left\{ \int_0^1 f_\epsilon(\gamma(t)) |\gamma'(t)| dt \mid \gamma \in W^{1,\infty}((0, 1); \bar{\Omega}) \text{ with } \gamma(0) = x, \gamma(1) = y \right\}.$$

It is well-known that $u_\epsilon(x) := \inf_{y \in \partial\Omega} L_\epsilon(x, y)$ is a solution of

$$\begin{aligned} |\nabla u^\epsilon| &= f_\epsilon(x) & x \in \Omega \\ u^\epsilon(x) &= 0 & x \in \partial\Omega \end{aligned}$$

in the viscosity sense. Furthermore, it can be shown that

$$\|u^\epsilon\|_{C^{0,1}(\bar{\Omega})} \leq C(M, \Omega) \quad \text{uniformly in } \epsilon > 0.$$

Thus, there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ with $\epsilon_k \searrow 0, k \rightarrow \infty$ and $u \in C^{0,1}(\bar{\Omega})$ such that $u^{\epsilon_k} \rightarrow u$ uniformly in $\bar{\Omega}$ as $k \rightarrow \infty$. Using well-known arguments from the theory of viscosity solutions one verifies that u is a solution of (1.1), (1.2). $\blacksquare$

Uniqueness of the viscosity solution follows from

Theorem 2.3. *Suppose that $u \in C^0(\bar{\Omega})$ is a subsolution of (1.1), $v \in C^0(\bar{\Omega})$ is a supersolution of (1.1) and that at least one of the functions belongs to $C^{0,1}(\bar{\Omega})$. If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in $\bar{\Omega}$.*

Proof. Let us assume that $v \in C^{0,1}(\bar{\Omega})$. We shall use the approach presented in [6] (see also [11]). Fix $\theta \in (0, 1)$ and define $u_\theta(x) := \theta u(x)$. Next, choose $x_0 \in \bar{\Omega}$ such that

$$u_\theta(x_0) - v(x_0) = \max_{x \in \bar{\Omega}} (u_\theta(x) - v(x)) =: \mu, \quad (2.4)$$

and suppose that $\mu > 0$. Upon replacing u, v by $u + k, v + k$, we may assume that $u \geq 0$ in $\bar{\Omega}$, so that $u_\theta \leq u$ in $\bar{\Omega}$. In particular, $u_\theta \leq v$ on $\partial\Omega$, which implies that $x_0 \in \Omega$. Choose $\epsilon = \epsilon_{x_0}$ and $n = n_{x_0} \in S^{m-1}$ according to (2.2) and define for $\lambda > 0, L \geq 1$

$$\Phi(x, y) := u_\theta(x) - v(y) - L\lambda |x - y - \frac{1}{\lambda}n|^2 - |x - x_0|^2, \quad (x, y) \in \bar{\Omega} \times \bar{\Omega}.$$

Choose $(x_\lambda, y_\lambda) \in \bar{\Omega} \times \bar{\Omega}$ such that

$$\Phi(x_\lambda, y_\lambda) = \max_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} \Phi(x, y).$$

Since $x_0 \in \Omega$ we also have $x_0 - \frac{1}{\lambda}n \in \Omega$ for large λ ; using the relation $\Phi(x_\lambda, y_\lambda) \geq \Phi(x_0, x_0 - \frac{1}{\lambda}n)$ together with (2.4) we infer

$$\begin{aligned} L\lambda |x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 &\leq u_\theta(x_\lambda) - v(y_\lambda) - u_\theta(x_0) + v(x_0 - \frac{1}{\lambda}n) \\ &= (u_\theta(x_\lambda) - v(x_\lambda)) - (u_\theta(x_0) - v(x_0)) + v(x_\lambda) - v(y_\lambda) - v(x_0) + v(x_0 - \frac{1}{\lambda}n) \\ &\leq \text{lip}(v) (|x_\lambda - y_\lambda| + \frac{1}{\lambda}) \\ &\leq \text{lip}(v) (|x_\lambda - y_\lambda - \frac{1}{\lambda}n| + \frac{2}{\lambda}). \end{aligned} \quad (2.5)$$

This implies

$$L\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n|^2 + |x_\lambda - x_0|^2 \leq \frac{C}{\lambda},$$

where C depends on $\text{lip}(v)$ and as a consequence,

$$x_\lambda, y_\lambda \rightarrow x_0, \quad \text{as } \lambda \rightarrow \infty \quad (2.6)$$

$$\lambda|x_\lambda - y_\lambda - \frac{1}{\lambda}n| \leq \frac{C}{\sqrt{L}} < \frac{\epsilon}{2+\epsilon} \quad (2.7)$$

provided that L is large enough. Since u is a subsolution, we may deduce from the relation $\Phi(x_\lambda, y_\lambda) \geq \Phi(x, y_\lambda)$ for $x \in \bar{\Omega}$ that

$$|2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda}n) + 2(x_\lambda - x_0)| \leq \theta f^*(x_\lambda)$$

for large λ and similarly,

$$|2L\lambda(x_\lambda - y_\lambda - \frac{1}{\lambda}n)| \geq f_*(y_\lambda).$$

Combining the above inequalities, we infer

$$(1 - \theta)f^*(y_\lambda) \leq 2|x_\lambda - x_0| + \theta(f^*(x_\lambda) - f_*(y_\lambda)). \quad (2.8)$$

In order to apply (2.2) we write $x_\lambda = y_\lambda + r_\lambda d_\lambda$, where

$$d_\lambda = \frac{n + w_\lambda}{|n + w_\lambda|}, \quad r_\lambda = \frac{1}{\lambda} |n + w_\lambda|, \quad w_\lambda = \lambda \left(x_\lambda - y_\lambda - \frac{1}{\lambda}n \right). \quad (2.9)$$

Recalling (2.7) we deduce

$$|d_\lambda - n| \leq \frac{2|w_\lambda|}{1 - |w_\lambda|} \leq \frac{\frac{2\epsilon}{2+\epsilon}}{1 - \frac{\epsilon}{2+\epsilon}} = \epsilon$$

and (2.3) therefore yields

$$f^*(x_\lambda) - f_*(y_\lambda) = f^*(y_\lambda + r_\lambda d_\lambda) - f_*(y_\lambda) \leq \omega(|y_\lambda - x_0| + r_\lambda). \quad (2.10)$$

If we send $\lambda \rightarrow \infty$ in (2.8) we finally obtain from (2.1), (2.10) and (2.6) that $(1 - \theta)m \leq 0$, a contradiction. Thus, $u_\theta \leq v$ in $\bar{\Omega}$ and sending $\theta \nearrow 1$ gives the desired result. $\blacksquare$

3 Numerical scheme and error analysis

Let us assume that $\Omega = \Pi_{i=1}^n(0, b_i)$ and that the grid size $h > 0$ is chosen in such a way that $b_i = N_i h$ for some $N_i \in \mathbb{N}$, $i = 1, \dots, n$. We then define

$$\Omega_h := \mathbf{Z}_h^n \cap \Omega, \quad \partial\Omega_h := \mathbf{Z}_h^n \cap \partial\Omega, \quad \bar{\Omega}_h := \Omega_h \cup \partial\Omega_h,$$

where $\mathbf{Z}_h^n = \{x_\alpha = (h\alpha_1, \dots, h\alpha_n) \mid \alpha_i \in \mathbf{Z}, i = 1, \dots, n\}$. Our aim is to approximate the viscosity solution u of (1.1), (1.2) by a grid function $U : \bar{\Omega}_h \rightarrow \mathbb{R}$ and to prove an estimate for $\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)|$. Let us abbreviate $U_\alpha = U(x_\alpha)$ and recall the usual backward and forward difference quotients,

$$D_k^- U_\alpha := \frac{U_\alpha - U_{\alpha - e_k}}{h}, \quad D_k^+ U_\alpha := \frac{U_{\alpha + e_k} - U_\alpha}{h}, \quad x_\alpha \in \Omega_h, \quad k = 1, \dots, n.$$

In order to define the numerical method we introduce the function $G : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ as

$$G(p_1, q_1, \dots, p_n, q_n) := \left(\sum_{k=1}^n \max(p_k^+, -q_k^-)^2 \right)^{\frac{1}{2}},$$

where $x^+ = \max(x, 0)$, $x^- = \min(x, 0)$. The discrete problem now reads: find $U : \bar{\Omega}_h \rightarrow \mathbb{R}$ such that

$$G(D_1^- U_\alpha, D_1^+ U_\alpha, \dots, D_n^- U_\alpha, D_n^+ U_\alpha) = f(x_\alpha) \quad x_\alpha \in \Omega_h \quad (3.1)$$

$$U_\alpha = 0 \quad x_\alpha \in \partial\Omega_h. \quad (3.2)$$

The above scheme was examined for continuous f in [9] in the context of shape-from-shading and convergence to the viscosity solution was obtained as a consequence of a result of Barles and Souganidis [2]. In the case of a constant right hand side $f \equiv 1$, Zhao [12] recently obtained an $\mathcal{O}(h)$ error bound. The scheme can be derived by interpreting the viscosity solution u as the value function of an optimal control problem. For further information and a corresponding list of references we refer to Appendix A (written by M. Falcone) in [1].

The function G has the following crucial properties:

a) *Consistency*:

$$G(p_1, p_1, \dots, p_n, p_n) = |p| \quad \text{for all } p = (p_1, \dots, p_n) \in \mathbb{R}^n. \quad (3.3)$$

b) *Monotonicity*:

let $a = (a_1, a_2, \dots, a_{2n-1}, a_{2n})$, $b = (b_1, b_2, \dots, b_{2n-1}, b_{2n}) \in \mathbb{R}^{2n}$ with $a_k \geq b_k$ for $k = 1, \dots, 2n$. Then

$$G(t - a_1, a_2 - t, \dots, t - a_{2n-1}, a_{2n} - t) \leq G(t - b_1, b_2 - t, \dots, t - b_{2n-1}, b_{2n} - t) \quad \forall t \in \mathbb{R}. \quad (3.4)$$

Note that the above properties imply in particular that the solution of (3.1), (3.2) cannot have a local minimum in Ω_h and therefore $U_\alpha \geq 0$ in $\bar{\Omega}_h$. In order to carry out our error analysis we need to strengthen (2.2) in that we assume that there exist $\epsilon > 0$, $K \geq 0$ such that for all $x \in \Omega$ there is a direction $n = n_x \in S^{n-1}$ with

$$f(y + rd) - f(y) \leq Kr \quad \forall y \in \Omega, |y - x| < \epsilon \quad \forall d \in S^{n-1}, |d - n| < \epsilon \quad \forall r > 0. \quad (3.5)$$

Theorem 3.1. *Let u be the viscosity solution of (1.1), (1.2) and U a solution of (3.1), (3.2). Then there exists a constant C , which is independent of h such that*

$$\max_{x_\alpha \in \bar{\Omega}_h} |u(x_\alpha) - U(x_\alpha)| \leq C\sqrt{h}.$$

Proof. We again only sketch the main ideas. As it seems difficult to use the argument from the uniqueness proof in order to control the maximum of $u - U$, we shall resort to the Kruřkov transform. Thus, let $\tilde{u} : \bar{\Omega} \rightarrow \mathbb{R}$, $\tilde{U} : \bar{\Omega}_h \rightarrow \mathbb{R}$ be defined by

$$\tilde{u}(x) := -e^{-u(x)}, \quad x \in \bar{\Omega}, \quad \tilde{U}_\alpha := -e^{-U_\alpha}, \quad x_\alpha \in \bar{\Omega}_h.$$

One verifies (cf. [4]) that $\tilde{u}$ is a viscosity solution of

$$f(x)\tilde{u} + |\nabla\tilde{u}| = 0 \quad x \in \Omega \quad (3.6)$$

$$\tilde{u}(x) = -1 \quad x \in \partial\Omega, \quad (3.7)$$

and that $\tilde{U}$ satisfies

$$f(x_\alpha)\tilde{U}_\alpha + G(D_1^-\tilde{U}_\alpha, D_1^+\tilde{U}_\alpha, \dots, D_n^-\tilde{U}_\alpha, D_n^+\tilde{U}_\alpha) = F_\alpha^h \quad x_\alpha \in \Omega_h \quad (3.8)$$

$$\tilde{U}_\alpha = -1 \quad x_\alpha \in \partial\Omega_h, \quad (3.9)$$

where

$$\max_{x_\alpha \in \Omega_h} |F_\alpha^h| \leq Ch. \quad (3.10)$$

Next, choose $x_\beta \in \bar{\Omega}_h$ such that

$$|\tilde{u}(x_\beta) - \tilde{U}_\beta| = \max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha|$$

and assume that $\tilde{u}(x_\beta) \geq \tilde{U}_\beta$. The opposite case can be treated similarly. If $\text{dist}(x_\beta, \partial\Omega) \leq \sqrt{h}$, it follows from (3.7), (3.9) and the Lipschitz continuity of $\tilde{u}$ that

$$\max_{x_\alpha \in \bar{\Omega}_h} |\tilde{u}(x_\alpha) - \tilde{U}_\alpha| = \tilde{u}(x_\beta) - \tilde{U}_\beta \leq C\sqrt{h}.$$

Now suppose that $\text{dist}(x_\beta, \partial\Omega) > \sqrt{h}$ and define

$$\Phi(x, x_\alpha) := \tilde{u}(x) - \tilde{U}_\alpha - \frac{L_1}{\sqrt{h}} |x - x_\alpha - \sqrt{h}n|^2 - L_2\sqrt{h} |x_\alpha - x_\beta|^2, \quad (x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h.$$

Here, $n = n_{x_\beta}$ and $L_1, L_2 \geq 0$ are constants that do not depend on h and which will be chosen later. There exists $(x_h, x_{\alpha_h}) \in \bar{\Omega} \times \bar{\Omega}_h$ such that

$$\Phi(x_h, x_{\alpha_h}) = \max_{(x, x_\alpha) \in \bar{\Omega} \times \bar{\Omega}_h} \Phi(x, x_\alpha).$$

Since $\text{dist}(x_\beta, \partial\Omega) > \sqrt{h}$, we have $x_\beta + \sqrt{h}n \in \bar{\Omega}$ and therefore

$$\Phi(x_h, x_{\alpha_h}) \geq \Phi(x_\beta + \sqrt{h}n, x_\beta).$$

From this we infer in a similar way as in (2.5) that

$$|x_{\alpha_h} - x_\beta| < \epsilon, \quad (3.11)$$

$$\frac{1}{\sqrt{h}} |x_h - x_{\alpha_h} - \sqrt{h}n| < \frac{\epsilon}{2 + \epsilon} \quad (3.12)$$

provided that $L_i = L_i(\text{lip}(\tilde{u}), \epsilon)$, $i = 1, 2$ are sufficiently large (ϵ from (3.5)).

Suppose first that $(x_h, x_{\alpha_h}) \in \Omega \times \Omega_h$. Since $\tilde{u}$ is a subsolution of (3.6) we infer

$$f^*(x_h)\tilde{u}(x_h) + \left| \frac{2L_1}{\sqrt{h}} (x_h - x_{\alpha_h} - \sqrt{h}n) \right| \leq 0. \quad (3.13)$$

Keeping the first component of Φ fixed we obtain on the other hand for all $x_\alpha \in \bar{\Omega}_h$

$$\begin{aligned} \tilde{U}_\alpha &\geq \tilde{U}_{\alpha_h} + \frac{L_1}{\sqrt{h}} \left(|x_h - x_{\alpha_h} - \sqrt{h}n|^2 - |x_h - x_\alpha - \sqrt{h}n|^2 \right) \\ &\quad + L_2\sqrt{h} (|x_{\alpha_h} - x_\beta|^2 - |x_\alpha - x_\beta|^2) \\ &=: \tilde{V}_\alpha. \end{aligned}$$

Since $\tilde{U}_{\alpha_h} = \tilde{V}_{\alpha_h}$, (3.4) and (3.3) imply

$$\begin{aligned} G(D_1^- \tilde{U}_{\alpha_h}, D_1^+ \tilde{U}_{\alpha_h}, \dots, D_n^- \tilde{U}_{\alpha_h}, D_n^+ \tilde{U}_{\alpha_h}) &\leq G(D_1^- \tilde{V}_{\alpha_h}, D_1^+ \tilde{V}_{\alpha_h}, \dots, D_n^- \tilde{V}_{\alpha_h}, D_n^+ \tilde{V}_{\alpha_h}) \\ &\leq \left| \frac{2L_1}{\sqrt{h}} (x_h - x_{\alpha_h} - \sqrt{h}n) - 2L_2\sqrt{h}(x_{\alpha_h} - x_\beta) \right| + C\sqrt{h}. \end{aligned}$$

Combining this inequality with (3.8) and (3.10) then yields

$$f(x_{\alpha_h})\tilde{U}_{\alpha_h} + \left| \frac{2L_1}{\sqrt{h}} (x_h - x_{\alpha_h} - \sqrt{h}n) \right| \geq -|F_{\alpha_h}^h| - C\sqrt{h} \geq -C\sqrt{h}. \quad (3.14)$$

As a result of (3.13), (3.14)

$$\begin{aligned} f(x_{\alpha_h})(\tilde{u}(x_h) - \tilde{U}_{\alpha_h}) &\leq C\sqrt{h} + e^{-u(x_h)}(f^*(x_h) - f(x_{\alpha_h})) \\ &= C\sqrt{h} + e^{-u(x_h)}(f^*(x_{\alpha_h} + r_h d_h) - f(x_{\alpha_h})) \end{aligned} \quad (3.15)$$

where similar to (2.9), $d_h = \frac{n + w_h}{|n + w_h|}$, $r_h = \sqrt{h}|n + w_h|$, $w_h = \frac{1}{\sqrt{h}}(x_h - x_{\alpha_h} - \sqrt{h}n)$. Since

$$\begin{aligned} \tilde{u}(x_h) - \tilde{U}_{\alpha_h} &= \Phi(x_h, x_{\alpha_h}) + \frac{L_1}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}n|^2 + L_2\sqrt{h}|x_{\alpha_h} - x_\beta|^2 \\ &\geq \Phi(x_\beta, x_\beta) = \tilde{U}_\beta - \tilde{u}(x_\beta) - L_1\sqrt{h}, \end{aligned}$$

we finally deduce from (2.1), (3.15) and (3.5) that

$$m(\tilde{u}(x_\beta) - \tilde{U}_\beta) \leq C\sqrt{h} + Kr_h \leq C\sqrt{h}.$$

The cases $x_h \in \partial\Omega$ or $x_{\alpha_h} \in \partial\Omega_h$ can be examined with the help of the boundary conditions (3.7), (3.9). Transforming back to u and U implies the desired error bound. ■

References

- [1] M. Bardi, I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman Equations*, Birkhäuser Boston 1997.
- [2] G. Barles, P.E. Souganidis, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic Anal. **4**, 271–283 (1991).
- [3] F. Camilli, A. Siconolfi, *Hamilton–Jacobi equations with measurable dependence on the state variable*, Preprint, (2002).
- [4] M.G. Crandall, P.L. Lions, *Viscosity solutions of Hamilton–Jacobi equations*, Trans. Amer. Math. Soc. **277**, 1–42 (1983).
- [5] H. Ishii, *Hamilton–Jacobi equations with discontinuous Hamiltonians on arbitrary open sets*, Bull. Fac. Sci. Engrg. Chuo. Univ. **28**, 33–77 (1985).
- [6] H. Ishii, *A simple, direct proof of uniqueness for solutions of the Hamilton–Jacobi equations of eikonal type*, Proc. Amer. Math. Soc. **100**, Number 2, 247–251 (1987).
- [7] R.T. Newcomb II, J. Su, *Eikonal equations with discontinuities*, Differential Integral Equations **8**, 1947–1960 (1995).

- [8] D. Ostrov, *Viscosity solutions and convergence of monotone schemes for synthetic aperture radar shape-from-shading equations with discontinuous intensities*, SIAM J. Appl. Math. **59**, 2060–2085 (1999).
- [9] E. Rouy, A. Tourin, *A viscosity solutions approach to shape-from-shading*, SIAM J. Numer. Anal. **29**, 867–884 (1992).
- [10] P. Soravia, *Boundary value problems for Hamilton–Jacobi equations with discontinuous Lagrangian*, Indiana Univ. Math. J. **51**, 451–477 (2002).
- [11] A. Tourin, *A comparison theorem for a piecewise Lipschitz continuous Hamiltonian and applications to shape-from-shading*, Numer. Math. **62**, 75–85 (1992).
- [12] H. Zhao, *Fast sweeping method for eikonal equations I: distance function*, Preprint.

Klaus Deckelnick
 Institut für Analysis und Numerik
 Otto-von-Guericke-Universität Magdeburg
 Universitätsplatz 2
 39106 Magdeburg
 Germany

Charles M. Elliott
 Centre for Mathematical Analysis and Its Applications
 School of Mathematical Sciences
 University of Sussex
 Falmer Brighton BN1 9QH
 United Kingdom