

Automorphic pairs of distributions and its application to explicit constructions of Maass forms

千葉工業大学 数学教室 杉山和成 (Kazunari Sugiyama)
(Department of Mathematics, Chiba Institute of Technology)

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1 Automorphic pairs of distributions

Let $\lambda \in \mathbb{C}$, $\varepsilon = 0, 1$. We define the "automorphic factor" $J_{\lambda, \varepsilon}(x)$ on $\mathbb{R}^\times$ by $J_{\lambda, \varepsilon}(x) = \text{sgn}(x)^\varepsilon \cdot |x|^{-2\lambda}$. For $f_0 \in C_0^\infty(\mathbb{R}^\times)$, we put

$$f_\infty(x) = J_{\lambda, \varepsilon}(x) f_0\left(-\frac{1}{x}\right) \quad (x \neq 0). \quad (1)$$

Let $\mathbf{a} = \{a(n)\}_{n \in \mathbb{Z}}$, $\mathbf{b} = \{b(n)\}_{n \in \mathbb{Z}}$ be sequences of complex numbers of polynomial growth, and $N \geq 1$ is a natural number. Consider the mappings $T_0, T_\infty : C_0^\infty(\mathbb{R}^\times) \rightarrow \mathbb{C}$ defined by

$$T_0(\varphi) = \sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}\varphi)(n), \quad T_\infty(\varphi) = \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}\varphi)\left(\frac{n}{N}\right) \quad (\varphi \in C_0^\infty(\mathbb{R}^\times)),$$

where $(\mathcal{F}\varphi)(t)$ denotes the Fourier transform of φ :

$$(\mathcal{F}\varphi)(t) = \int_{\mathbb{R}} \varphi(x) e^{2\pi i x t} dx.$$

If T_0, T_∞ satisfy the condition

$$T_0(f_0) = T_\infty(f_\infty) \quad (2)$$

for all $f_0 \in C_0^\infty(\mathbb{R}^\times)$, then the pair (T_0, T_∞) is called an *automorphic pair of level N with automorphic factor $J_{\lambda, \varepsilon}(x)$* . The relation (2) can be written in a sum formula as

$$\sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}f_0)(n) = \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}f_\infty)\left(\frac{n}{N}\right). \quad (3)$$

Associated Dirichlet series are defined as follows:

$$\begin{aligned} \xi_\pm(\mathbf{a}; s) &= \sum_{n=1}^{\infty} \frac{a(\pm n)}{n^s}, & \xi_\pm(\mathbf{b}; s) &= \sum_{n=1}^{\infty} \frac{b(\pm n)}{n^s}, \\ \Xi_\pm(\mathbf{a}; s) &= (2\pi)^{-s} \Gamma(s) \xi_\pm(\mathbf{a}; s), & \Xi_\pm(\mathbf{b}; s) &= (2\pi)^{-s} \Gamma(s) \xi_\pm(\mathbf{b}; s). \end{aligned} \quad (4)$$

Then we have

Theorem (T. Suzuki [7]). *The L -functions $\xi_{\pm}(\mathbf{a}; s)$ and $\xi_{\pm}(\mathbf{b}; s)$ have analytic continuations to meromorphic functions with a finite number of poles, and satisfy the following functional equations:*

$$\gamma(s) \begin{pmatrix} \Xi_+(\mathbf{a}; s) \\ \Xi_-(\mathbf{a}; s) \end{pmatrix} = N^{2-2\lambda-s} \cdot \Sigma \cdot \gamma(2-2\lambda-s) \begin{pmatrix} \Xi_+(\mathbf{b}; 2-2\lambda-s) \\ \Xi_-(\mathbf{b}; 2-2\lambda-s) \end{pmatrix},$$

where

$$\gamma(s) = \begin{pmatrix} e^{\pi s \sqrt{-1}/2} & e^{-\pi s \sqrt{-1}/2} \\ e^{-\pi s \sqrt{-1}/2} & e^{\pi s \sqrt{-1}/2} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & (-1)^{\varepsilon} \\ 1 & 0 \end{pmatrix}. \quad (5)$$

Example. Let $\operatorname{Re} \lambda > 1/2$, $\varepsilon = 0$, $N = 1$. We have

$$\zeta(2\lambda-1) \cdot (\mathcal{F}f_0)(0) + \sum_{n \neq 0} \sigma_{1-2\lambda}(|n|) (\mathcal{F}f_0)(n) = \zeta(2\lambda-1) \cdot (\mathcal{F}f_{\infty})(0) + \sum_{n \neq 0} \sigma_{1-2\lambda}(|n|) (\mathcal{F}f_{\infty})(n),$$

where $\sigma_a(n) := \sum_{0 < d|n} d^a$. This equality is proved by using the Fourier expansion of the distribution E_{λ} defined by

$$E_{\lambda}(f_0) = \frac{1}{2} \sum_{m, n \neq 0} |m|^{-2\lambda} f_0\left(\frac{n}{m}\right). \quad (\text{"Eisenstein distribution"}) \quad (6)$$

2 Principal series representations of $G = SL_2(\mathbb{R})$.

We introduce the following function space:

$$\mathcal{V}_{\lambda, \varepsilon}^{\infty} = \{f_0 \in C^{\infty}(\mathbb{R}) \mid f_{\infty}(x), \text{ defined by (1), can be extended to an element of } C^{\infty}(\mathbb{R})\}.$$

The action of $G = SL_2(\mathbb{R})$ on $\mathcal{V}_{\lambda, \varepsilon}^{\infty}$ is defined by

$$(\pi_{\lambda, \varepsilon}(g)f_0)(x) = \begin{cases} J_{\lambda, \varepsilon}(-cx+a)f_0\left(\frac{dx-b}{-cx+a}\right) & (\text{if } -cx+a \neq 0) \\ J_{\lambda, \varepsilon}(-dx+b)f_{\infty}\left(\frac{-cx+a}{-dx+b}\right) & (\text{if } -dx+b \neq 0) \end{cases} \quad (7)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL_2(\mathbb{R})$ and $f_0 \in \mathcal{V}_{\lambda, \varepsilon}^{\infty}$. To be precise, elements of $\mathcal{V}_{\lambda, \varepsilon}^{\infty}$ should be regarded as sections of a line bundle over $\mathbb{P}^1(\mathbb{R}) \cong G/P$. We set $f_0(\infty) := f_{\infty}(0)$. It is known that $\mathcal{V}_{\lambda, \varepsilon}^{\infty}$ is one of the realizations of the (non-unitary) principal series representations of G .

We define a topology through seminorms on $\mathcal{V}_{\lambda, \varepsilon}^{\infty}$ given by

$$f_0 \mapsto \sup_{x \in K} \left| \frac{d^N f_0}{dx^N}(x) \right|, \quad f_0 \mapsto \sup_{x \in K} \left| \frac{d^N f_{\infty}}{dx^N}(x) \right|,$$

where K is any compact subset of $\mathbb{R}$ and $N \in \mathbb{Z}_{\geq 0}$. We call a continuous linear mapping $T : \mathcal{V}_{\lambda, \varepsilon}^{\infty} \rightarrow \mathbb{C}$ a distribution on $\mathcal{V}_{\lambda, \varepsilon}^{\infty}$, and denote by $\mathcal{V}_{\lambda, \varepsilon}^{-\infty}$ the space of distributions. For $g \in G$ and $T \in \mathcal{V}_{\lambda, \varepsilon}^{-\infty}$, we define $(\pi_{-\lambda, \varepsilon}(g)T)(f_0) = T(\pi_{\lambda, \varepsilon}(g^{-1})f_0)$.

For a subgroup Γ of $SL_2(\mathbb{Z})$ of finite index, we define

$$(\mathcal{V}_{\lambda, \varepsilon}^{-\infty})^{\Gamma} = \left\{ T \in \mathcal{V}_{\lambda, \varepsilon}^{-\infty} \mid \pi_{-\lambda, \varepsilon}(\gamma)T = T \text{ for all } \gamma \in \Gamma \right\}.$$

We call T an *automorphic distribution* after Miller and Schmid [4]. Now we take $T \in (\mathcal{V}_{\lambda, \varepsilon}^{-\infty})^{\Gamma_0(N)}$, where $\Gamma_0(N)$ is the congruence subgroup of level N . Let $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma_0(N)$. The invariance of T under γ_1 implies that T has a Fourier expansion as

$$T(f_0) = a(\infty)f_0(\infty) + \sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}f_0)(n),$$

and since

$$(\pi_{\lambda, \varepsilon}(g)f_{\infty})(x) = J_{\lambda, \varepsilon}(bx + d)f_{\infty}\left(\frac{ax + c}{bx + d}\right),$$

the invariance under γ_2 implies that

$$T(f_{\infty}) = b(\infty)f_{\infty}(\infty) + \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}f_{\infty})\left(\frac{n}{N}\right).$$

Hence one can construct an automorphic pair of distributions of level N from $T \in (\mathcal{V}_{\lambda, \varepsilon}^{-\infty})^{\Gamma_0(N)}$.

3 Poisson transforms

Note that $G/P \cong \mathbb{P}^1(\mathbb{R})$ is the boundary of $G/K \cong \mathcal{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. Roughly speaking, we construct automorphic forms on G/K for Γ from Γ -invariant distributions on G/P .

Now we define the Poisson transform after Lewis and Zagier [3], Unterberger [9]. For $z \in \mathcal{H}$, $\lambda \in \mathbb{C}$, $l \in \mathbb{Z}$, we define the Poisson kernel $f_{\lambda, l}(t, z)$ by

$$f_{\lambda, l}(t, z) = \frac{y^{\lambda}}{|z - t|^{2\lambda}} \cdot \left(\frac{z - t}{|z - t|}\right)^{-l} = \frac{y^{\lambda}}{|z - t|^{2\lambda - l}(z - t)^l}.$$

When we fix $z \in \mathcal{H}$ (resp. $t \in \mathbb{P}^1(\mathbb{R})$) and regard $f_{\lambda, l}(t, z)$ as a function of t (resp. z), we write $f_{\lambda, l, z}(t)$ (resp. $f_{\lambda, l, t}(z)$).

Lemma. (1) $f_{\lambda, l, z}$ is an element of $\mathcal{V}_{\lambda, \varepsilon(l)}^{\infty}$, where $\varepsilon(l) = 0(l \equiv 0 \pmod{2}), = 1(l \equiv 1 \pmod{2})$.

(2) For $g \in SL_2(\mathbb{R})$, we have $(\pi_{\lambda, \varepsilon(l)}(g)f_{\lambda, l, z})(t) = (f_{\lambda, l, t}|_l g)(z)$, where $|_l$ is the slash operator defined by

$$(F|_l g)(z) = \left(\frac{cz + d}{|cz + d|}\right)^{-l} F\left(\frac{az + b}{cz + d}\right) \quad (z \in \mathcal{H}).$$

(3) $\Delta_l f_{\lambda, l, t}(z) = \lambda(1 - \lambda)f_{\lambda, l, t}(z)$, where Δ_l is the Laplace-Beltrami operator defined by

$$\Delta_l = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i l y \frac{\partial}{\partial x}.$$

Definition. For $T \in \mathcal{V}_{\lambda, \varepsilon}^{-\infty}$ and $l \in \mathbb{Z}$ with $\varepsilon(l) = \varepsilon$, we define the *Poisson transform* $\mathcal{P}_{\lambda, l}$ by

$$\mathcal{P}_{\lambda, l}(T)(z) = T(f_{\lambda, l, z}).$$

Definition. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index, and χ a character of Γ . A function $F : \mathcal{H} \rightarrow \mathbb{C}$ is said to be a *Maass form for Γ of weight $l \in \mathbb{Z}$ with eigenvalue $\lambda \in \mathbb{C}$ and character χ* if the following conditions are satisfied:

- (1) $(F|_l \gamma)(z) = \chi(\gamma) \cdot F(z)$ for $z \in \mathcal{H}$ and $\gamma \in \Gamma$.
- (2) $\Delta_l F = \lambda(1 - \lambda)F$
- (3) F is slowly increasing at every cusp of Γ .

We denote by $\mathcal{M}_l(\Gamma, \lambda; \chi)$ the space of the Maass forms.

Theorem. The Poisson transform $\mathcal{P}_{\lambda, l}$ defines a map from $(\mathcal{V}_{\lambda, \varepsilon}^{-\infty})^{\Gamma, \chi}$ to $\mathcal{M}_l(\Gamma, \lambda; \chi^{-1})$, where

$$(\mathcal{V}_{\lambda, \varepsilon}^{-\infty})^{\Gamma, \chi} = \{T \in \mathcal{V}_{\lambda, \varepsilon}^{-\infty} \mid \pi_{-\lambda, \varepsilon}(\gamma)T = \chi(\gamma)T \text{ for all } \gamma \in \Gamma\}.$$

Example. Let $l \in 2\mathbb{Z}$ and $\text{Re } \lambda > 1/2$. We define the "genuine" Eisenstein distribution by

$$E_\lambda^\circ(f_0) = \frac{1}{2} \sum_{(m, n) \neq (0, 0)} |m|^{-2\lambda} f_0\left(\frac{n}{m}\right).$$

Here, for $m = 0$, we put $|m|^{-2\lambda} f_0(\frac{n}{m}) := f_0(\infty) \cdot |n|^{-2\lambda}$. This distribution differs from E_λ defined in (6) by the constant terms. The Poisson transform $\mathcal{P}_{\lambda, l}(E_\lambda^\circ)$ is nothing but the so-called real analytic Eisenstein series

$$E_l(\lambda, z) = \frac{1}{2} \sum_{(m, n) \neq (0, 0)} \frac{y^\lambda}{|mz + n|^{2\lambda - l} \cdot (mz + n)^l}.$$

4 A converse theorem for automorphic distributions

Let N be a positive integer, λ a complex number with $\text{Re}(\lambda) > 1/2$ and $2 - 2\lambda \notin \mathbb{Z}_{\leq 0}$. Let $\varepsilon = 0, 1$. Further, let χ be a Dirichlet character of mod N such that $\chi(-1) = (-1)^\varepsilon$. For complex sequences $\mathbf{a} = \{a(n)\}_{n \in \mathbb{Z} \setminus \{0\}}$, $\mathbf{b} = \{b(n)\}_{n \in \mathbb{Z} \setminus \{0\}}$ of polynomial growth, we define the Dirichlet series $\xi_\pm(\mathbf{a}; s)$, $\xi_\pm(\mathbf{b}; s)$ and the completed zeta functions $\Xi_\pm(\mathbf{a}; s)$, $\Xi_\pm(\mathbf{b}; s)$ by (4).

Let r be an odd prime with $(N, r) = 1$. We take an arbitrary Dirichlet character ψ of mod r and define the twisted zeta functions $\xi_\pm(\mathbf{a}, \psi; s)$, $\Xi_\pm(\mathbf{a}, \psi; s)$, $\xi_\pm(\mathbf{b}, \psi; s)$, $\Xi_\pm(\mathbf{b}, \psi; s)$ by

$$\begin{aligned} \xi_\pm(\mathbf{a}, \psi; s) &= \sum_{n=1}^{\infty} \frac{a(\pm n) \tau_\psi(\pm n)}{n^s}, & \Xi_\pm(\mathbf{a}, \psi; s) &= (2\pi)^{-s} \Gamma(s) \xi_\pm(\mathbf{a}, \psi; s), \\ \xi_\pm(\mathbf{b}, \psi; s) &= \sum_{n=1}^{\infty} \frac{b(\pm n) \tau_\psi(\pm n)}{n^s}, & \Xi_\pm(\mathbf{b}, \psi; s) &= (2\pi)^{-s} \Gamma(s) \xi_\pm(\mathbf{b}, \psi; s), \end{aligned}$$

where $\tau_\psi(n)$ is the Gauss sum defined by

$$\tau_\psi(n) = \sum_{\substack{(m, r)=1 \\ \text{mod } r}} \psi(m) e^{2\pi \sqrt{-1} mn/r}.$$

These twisted zeta functions were first considered by Razar [6]. We assume

[A1] $\xi_\pm(\mathbf{a}; s)$, $\xi_\pm(\mathbf{b}; s)$ converges absolutely for $\text{Re } s > 1$ and have analytic continuations to meromorphic functions of s to $\mathbb{C}$.

[A2] (1) $\Xi_{\pm}(\mathbf{a}; s), \Xi_{\pm}(\mathbf{b}; s)$ satisfy the functional equation

$$\gamma(s) \begin{pmatrix} \Xi_{+}(\mathbf{a}; s) \\ \Xi_{-}(\mathbf{a}; s) \end{pmatrix} = N^{2-2\lambda-s} \cdot \Sigma \cdot \gamma(2-2\lambda-s) \begin{pmatrix} \Xi_{+}(\mathbf{b}; 2-2\lambda-s) \\ \Xi_{-}(\mathbf{b}; 2-2\lambda-s) \end{pmatrix},$$

where $\gamma(s)$ and Σ are defined by (5).

(2) $\Xi_{\pm}(\mathbf{a}, \psi; s), \Xi_{\pm}(\mathbf{b}, \psi; s)$ satisfy the functional equation

$$\gamma(s) \begin{pmatrix} \Xi_{+}(\mathbf{a}, \psi; s) \\ \Xi_{-}(\mathbf{a}, \psi; s) \end{pmatrix} = \overline{\chi(r)} \cdot \overline{\psi(-N)} \cdot r^{2\lambda-2} \cdot (Nr^2)^{2-2\lambda-s} \\ \cdot \Sigma \cdot \gamma(2-2\lambda-s) \begin{pmatrix} \Xi_{+}(\mathbf{b}, \overline{\psi}; 2-2\lambda-s) \\ \Xi_{-}(\mathbf{b}, \overline{\psi}; 2-2\lambda-s) \end{pmatrix}.$$

[A3] $\xi_{\pm}(\mathbf{a}; s), \xi_{\pm}(\mathbf{b}; s), \xi_{\pm}(\mathbf{a}, \psi; s), \xi_{\pm}(\mathbf{b}, \overline{\psi}; s)$ have poles only at $s = 1, 2-2\lambda$ of order at most 1, and the residues satisfy the following relations:

$$\begin{aligned} \operatorname{Res}_{s=1} \xi_{\pm}(\mathbf{a}, \psi; s) &= \overline{\chi(r)} \cdot \overline{\psi(-N)} \cdot r^{-2\lambda} \cdot \tau_{\overline{\psi}}(0) \cdot \operatorname{Res}_{s=1} \xi_{\pm}(\mathbf{a}; s), \\ \operatorname{Res}_{s=2-2\lambda} \xi_{\pm}(\mathbf{a}, \psi; s) &= \overline{\chi(r)} \cdot \overline{\psi(-N)} \cdot r^{2\lambda-2} \cdot \tau_{\overline{\psi}}(0) \cdot \operatorname{Res}_{s=2-2\lambda} \xi_{\pm}(\mathbf{a}; s), \\ \operatorname{Res}_{s=1} \left(\overline{\chi(r)} \cdot \overline{\psi(-N)} \cdot r^{2\lambda} \xi_{\pm}(\mathbf{b}, \overline{\psi}; s) \right) &= \tau_{\psi}(0) \operatorname{Res}_{s=1} \xi_{\pm}(\mathbf{b}; s), \\ \operatorname{Res}_{s=2-2\lambda} \left(\overline{\chi(r)} \cdot \overline{\psi(-N)} \cdot r^{2-2\lambda} \xi_{\pm}(\mathbf{b}, \overline{\psi}; s) \right) &= \tau_{\psi}(0) \operatorname{Res}_{s=2-2\lambda} \xi_{\pm}(\mathbf{b}; s). \end{aligned}$$

[A4] $\xi_{\pm}(\mathbf{a}; s), \xi_{\pm}(\mathbf{a}, \psi; s), \xi_{\pm}(\mathbf{b}; s), \xi_{\pm}(\mathbf{b}, \psi; s)$ have finite order in lacunary vertical strips, i.e., For any $\alpha_1 < \alpha_2 (\alpha_1, \alpha_2 \in \mathbb{R})$, there exists some $\tau_0, K, \rho > 0$ such that

$$\begin{aligned} |\xi_{\pm}(\mathbf{a}; \alpha + \sqrt{-1}\tau)|, |\xi_{\pm}(\mathbf{a}, \psi; \alpha + \sqrt{-1}\tau)| &< K \cdot e^{|\tau|^{\rho}} \\ |\xi_{\pm}(\mathbf{b}; \alpha + \sqrt{-1}\tau)|, |\xi_{\pm}(\mathbf{b}, \psi; \alpha + \sqrt{-1}\tau)| &< K \cdot e^{|\tau|^{\rho}} \end{aligned}$$

for any $\alpha \in [\alpha_1, \alpha_2]$ and τ with $|\tau| > \tau_0$.

Theorem. We assume that [A1]–[A4] hold for every (not necessarily primitive) Dirichlet character ψ of mod r . We put

$$\begin{aligned} a(0) &= \left(\frac{2\pi}{N} \right)^{2\lambda-2} \Gamma(2-2\lambda) \left\{ e^{\frac{\pi\sqrt{-1}}{2}(2-2\lambda)} \operatorname{Res}_{s=2-2\lambda} \xi_{+}(\mathbf{b}; s) + e^{-\frac{\pi\sqrt{-1}}{2}(2-2\lambda)} \operatorname{Res}_{s=2-2\lambda} \xi_{-}(\mathbf{b}; s) \right\}, \\ a(\infty) &= \frac{N}{2} \left(\operatorname{Res}_{s=1} \xi_{+}(\mathbf{b}; s) + \operatorname{Res}_{s=1} \xi_{-}(\mathbf{b}; s) \right), \\ b(0) &= (-1)^{\varepsilon} (2\pi)^{2\lambda-2} \Gamma(2-2\lambda) \left\{ e^{\frac{\pi\sqrt{-1}}{2}(2-2\lambda)} \operatorname{Res}_{s=2-2\lambda} \xi_{+}(\mathbf{a}; s) + e^{-\frac{\pi\sqrt{-1}}{2}(2-2\lambda)} \operatorname{Res}_{s=2-2\lambda} \xi_{-}(\mathbf{a}; s) \right\}, \\ b(\infty) &= \frac{(-1)^{\varepsilon}}{2} \left(\operatorname{Res}_{s=1} \xi_{+}(\mathbf{a}; s) + \operatorname{Res}_{s=1} \xi_{-}(\mathbf{a}; s) \right), \end{aligned}$$

and define the linear functionals T_0, T_{∞} on $\mathcal{V}_{\lambda, \varepsilon}^{\infty}$ by

$$\begin{aligned} T_0(\varphi) &= a(\infty)\varphi(\infty) + \sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}\varphi)(n), \\ T_{\infty}(\varphi) &= b(\infty)\varphi(\infty) + \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}\varphi)\left(\frac{n}{N}\right) \end{aligned}$$

for $\varphi \in \mathcal{V}_{\lambda, \varepsilon}^{\infty}$. Then, $T_0(f_0) = T_{\infty}(f_{\infty})$ and T_0 is an automorphic distribution for $\Gamma_0(N)$ with character χ .

Corollary. *The Poisson transform $(\mathcal{P}_{\lambda, l}T_0)(z)$ is a Maass form for $\Gamma_0(N)$ of weight l , with eigenvalue λ and character χ^{-1} . Moreover, $(\mathcal{P}_{\lambda, l}T_0)(z)$ has the following Fourier expansion:*

$$\begin{aligned} (\mathcal{P}_{\lambda, l}T_0)(z) &= a(\infty)y^{\lambda} + a(0) \cdot (-1)^{\frac{l}{2}} \frac{(2\pi)^{2^{1-2\lambda}} \Gamma(2\lambda - 1)}{\Gamma(\lambda + \frac{l}{2}) \Gamma(\lambda - \frac{l}{2})} y^{1-\lambda} \\ &\quad + (-1)^{\frac{l}{2}} \pi^{\lambda} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |n|^{\lambda-1} a(n) \frac{W_{\frac{\text{sgn}(n)l}{2}, \lambda - \frac{1}{2}}(4\pi|n|y)}{\Gamma\left(\lambda + \frac{\text{sgn}(n)l}{2}\right)} e^{2\pi i n x}. \end{aligned}$$

Remark. (1) “It is an open question whether or not Weil’s argument applies to Maass forms. A key point for Weil is that radially symmetric holomorphic functions are necessarily constant; this is not true in the non-holomorphic case.” (quoted from Gelbart and Miller [2]).

- (2) Recently, Diamantis and Goldfeld [1] proved the converse theorem for double Dirichlet series associated with metaplectic Eisenstein series. Their twists of L -functions involve the Gauss sum $\tau_{\psi}(n)$, not the value $\psi(n)$ of the character ψ . Moreover, it is necessary to include non-primitive Dirichlet characters. We have followed Diamantis-Goldfeld’s method.
- (3) Diamantis-Goldfeld’s result is a converse theorem for vector-valued Dirichlet series, where the dimension of the vector (=the number of Dirichlet series) is equal to the number of cusps of $\Gamma_0(N)$. On the contrary, our argument is rather irrelevant to the discrete subgroup in question.

5 Application to zeta functions associated with quadratic forms

We recall the zeta functions studied by Peter [5], Ueno [8]. Put $V = \mathbb{C}^{m+2}$ and let $Q(x)$ be a non-degenerate integral quadratic form on V of the form

$$Q(x) = x_0 x_{m+1} + \sum_{1 \leq i, j \leq m} a_{ij} x_i x_j,$$

where $a_{ij} = a_{ji} \in \frac{1}{2}\mathbb{Z}$ ($i \neq j$) and $a_{ii} \in \mathbb{Z}$. The matrix of Q is given by

$$\begin{pmatrix} 0 & 0 & 1/2 \\ 0 & A & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

with $A = (a_{ij})$. We consider the maximal subgroup of $SO(Q)$ of the form

$$P = \left\{ \begin{pmatrix} a & -2a^t u A h & -a A[u] \\ 0 & h & u \\ 0 & 0 & a^{-1} \end{pmatrix} \middle| \begin{array}{l} a \in \mathbb{C}^{\times} \\ h \in SO(A) \\ u \in \mathbb{C}^m \end{array} \right\}.$$

Then the triplet $(P \times GL_1(\mathbb{C}), V)$ is a prehomogeneous vector space.

Let $D = \det(2A)$. For positive integers l, n , we put

$$\begin{aligned} r(l, n) &= \#\{v \in \mathbb{Z}^m / (l\mathbb{Z})^m \mid A[v] \equiv n \pmod{l}\}, \\ r^*(l, n) &= \#\{v^* \in \mathbb{Z}^m / 2lA\mathbb{Z}^m \mid 4^{-1} \cdot |D| A^{-1}[v^*] \equiv n \pmod{|D|l}\}. \end{aligned}$$

and define the Dirichlet series $Z(n, w), Z^*(n, w)$ ($n \in \mathbb{Z}$) by

$$Z(n, w) = \sum_{l=1}^{\infty} r(l, n) l^{-w}, \quad Z^*(n, w) = \sum_{l=1}^{\infty} r^*(l, n) l^{-w}$$

The prehomogeneous zeta functions associated with $(P \times GL_1(\mathbb{C}), V)$ coincide with

$$\zeta_{\epsilon}(w, s) = \sum_{n=1}^{\infty} Z(\epsilon n, w) n^{-s}, \quad \zeta_{\eta}^*(w, s) = |D|^s \sum_{n=1}^{\infty} Z^*(\eta n, w) n^{-s} \quad (\epsilon, \eta = \pm).$$

By using the theory of prehomogeneous vector spaces, Ueno proved that $\zeta_{\epsilon}(w, s)$ and $\zeta_{\eta}^*(w, s)$ have analytic continuations to meromorphic functions on $\mathbb{C}^2$ and satisfy functional equations.

Theorem. Assume that m is even and let $D = \det(2A)$. Then, under a suitable adjustment ($w = 2\lambda - 1 + \frac{m}{2}$, etc.), ζ_{ϵ} and ζ_{η}^* satisfy the assumption of our converse theorem, and we can construct Maass forms for $\Gamma_0(|D|)$ or $\Gamma_0(4|D|)$ with explicit Fourier coefficients defined by $Z(n, w)$ and $Z^*(n, w)$.

Remark. When m is odd, it is expected that ζ_{ϵ} and ζ_{η}^* correspond to Maass forms of half-integral weight. To include the Maass forms of general weight in our framework, we need to consider the principal series representation of the universal covering group $\tilde{G}$ of $G = SL_2(\mathbb{R})$.

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Chiba Institute of Technology
email: sugiyama.kazunari@p.chibakoudai.jp