

**STUDIES ON THE APPLICATION OF
LQ OPTIMAL CONTROLLERS**

TORU FUJINAKA

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Contents

1	Introduction	1
2	Outline of optimal control theory	8
2.1	Basic results on optimal control of delay-free continuous-time systems	8
2.2	Results on optimal control of time-delay systems	12
2.3	Basic results on discrete-time optimal control	14
3	Description of unilateral time-delay systems	18
3.1	Continuous-time description	19
3.2	Discretization	22
3.3	Examples	29
4	Continuous-time optimal control of unilateral time-delay systems	37
4.1	Two-station case	38
4.2	Formulation of the problem	41
4.3	Introduction of imaginary delay-free system	42
4.4	Solution for time-delay system	45
4.5	Solution for the initial part	48
4.6	Modification of the controller	53
4.7	Controller with observer	56
4.8	Examples	57
5	Discrete-time optimal control of unilateral time-delay systems	64
5.1	Problem formulation	65
5.1.1	General case	65
5.1.2	Single input case	68
5.2	Introduction of imaginary delay-free system	70
5.3	Solution for time-delay system	73
5.3.1	Single input case	73
5.3.2	Two-station case	78

5.3.3	Multi-station case	82
5.4	Examples	85
6	Discrete-time optimal controller which assigns closed-loop poles in a prescribed region	95
6.1	Problem Formulation	95
6.2	Pole Placement in the Region D_1	97
6.3	Pole Placement in the Region D_2	103
6.4	Examples	107
6.5	Remarks	112
7	Assignable region for the closed-loop poles of discrete-time optimal controller systems	115
7.1	Preliminaries	115
7.2	Assignable region for a real pole	117
7.3	Assignable region for complex poles	119
7.3.1	Characteristic equation of the symplectic matrix . .	120
7.3.2	Region of optimal poles for singular weight	123
7.3.3	Region of optimal poles for general weight	127
7.4	Examples	131
8	Conclusions	135

Chapter 1

Introduction

The linear quadratic optimal control (abbreviated as “LQ control” in the following) scheme has been recognized to be one of the most successful means to design a balanced control law for a complex system, and recently it has come to appear in many applications. In the standard way of applying the LQ control systems, we set up a performance index of quadratic form, find the control input which minimizes the performance index, and implement it in the state feedback or observer feedback form. The feedback gain of the LQ control is given in terms of the positive definite solution of a Riccati equation. Several methods of obtaining the solution of the Riccati equation numerically have been studied by many researchers, and at present fairly reliable and stable method is available. However, it should be noted that the standard LQ control can only be applied to finite dimensional linear systems. Moreover, even in the design of the standard LQ control system, it is not always straightforward to set up an appropriate performance index such that the resultant closed-loop system has desired properties.

The objective of this thesis is twofold. One is to extend the range of application of the LQ control to the control of time-delay systems. The other is to develop a design method of using the LQ control in combination with the pole assignment method. Each issue is discussed in more detail in the following.

Let us discuss the control of time-delay systems using the LQ technique. Time-delays appear in many engineering systems, such as chemical plants, water treatment plants, paper producing processes, etc., and they often cause much difficulty for control systems design.

A wide range of time-delay systems would be described by the difference-differential equation of the following form:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + \sum_{i=1}^j \hat{A}_i x(t - d_i) + \sum_{i=1}^j \hat{E}_i \dot{x}(t - d_i) \\ y(t) &= Cx(t)\end{aligned}\tag{1.1}$$

This class of systems are called neutral type time-delay systems (Hale, 1977), and the control of such systems is extremely difficult due to the complicated structure of the time-delay loops. Time-delay systems described by the following equation are called retarded type time-delay systems (Hale, 1977):

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{i=1}^j \hat{A}_i x(t - d_i) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{1.2}$$

The structure of retarded type time-delay systems is not as much complicated as the neutral type time-delay systems, but it is in general still difficult to control such systems.

When the model of a plant containing material flow is considered, it often happens that the time-delays appear in the unilateral structure as shown in Fig. 1.1. For instance, one can imagine such a plant in which certain material is processed in one station and sent to the next station, consecutively. In this case each station is described by the delay-free subsystem $\mathcal{S}_i$, and the transportation time of the material between the stations corresponds to the time-delay $\mathcal{D}_i$ in the model.

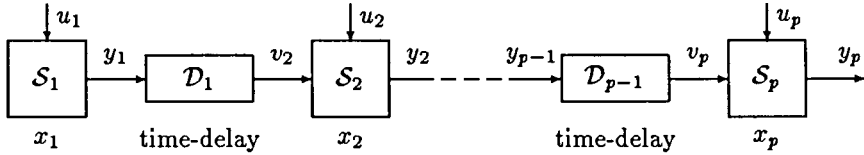


Fig. 1.1 Unilateral time-delay system

The class of systems having the structure of Fig. 1.1 will be called unilateral time-delay systems in the following. Unilateral time-delay systems belong to a subclass of the retarded type time-delay systems. The special allocation of the time-delays, namely the unilateral structure, implies the absence of internal time-delay loops. Thus the difficulties in control system design may be reduced in the design of the controller for unilateral time-delay systems, compared to that for general retarded type time-delay systems. This thesis is concerned with the control of unilateral

time-delay systems, in which the unilateral structure of the time-delays is taken into account for the purpose of reducing the difficulties in designing the controllers. Comparatively simple results are obtained on the design of the controller, and an effective way of reducing the difficulties in implementing the controller is presented.

The solution of the optimal control problem for general retarded type time-delay systems is already given (Fujikawa and Shimemura, 1975). But a partial differential equation must be solved in order to obtain the optimal feedback gain, and an infinite-dimensional state of the time-delay should be measured for implementing the controller. For unilateral time-delay systems, it will be shown that the difficulties in obtaining and implementing the optimal solution can be reduced by taking the advantage of the unilateral structure of the time-delays.

The continuous-time optimal control problem is solved in two parts, namely the steady state and the initial part. The steady state solution is obtained in a comparatively simple form, and a method of implementing the controller based on the steady state solution will be presented.

When a digital computer is used as the controller, it is mandatory that the sampled-data control scheme is implemented. The sampled-data control is known to be an effective means for time-delay systems, and it is applied to the unilateral time-delay systems in this thesis. As a preparation for designing the sampled-data controller, the controlled object, namely the unilateral time-delay system in this thesis, is expressed in discrete-time representation. When a retarded type time-delay system is discretized, it is generally expressed by a difference equation with an infinite number of terms. Thus an approximating assumption on the state variables is often introduced, which results in a difference equation of a finite number of terms. For unilateral time-delay systems, however, no approximating assumption is required, and still the discretized equation consists of a finite number of terms. The formula to derive the difference equation for the unilateral time-delay systems is presented in this thesis.

In the discrete-time description, time-delays are expressed by shift registers, each register corresponding to a state variable. Thus there is no essential difference in the form of state equation between a time-delay system and a delay-free system. But it often happens that the delay time is large, which leads to large dimension of the state equation. This causes difficulties in the design of the controller, and it is important, from

practical point of view, to reduce computational difficulties.

A well-known nonlinear matrix equation of Riccati type is involved in the computation of the optimal feedback gain. Usually a numerical calculation method is employed to obtain the numerical solution of the Riccati equation. This has been enabled by the rapid development of the digital computers, but it is still not easy to solve the Riccati equations of large dimensions. Thus, the computational difficulties can be reduced by reducing the dimension of the Riccati equation to be solved. The reduction is possible by virtue of the unilateral structure, and the formula to obtain the optimal solution with less difficulties is presented in this thesis.

Let us consider the other objective which is to combine the LQ control technique with the pole assignment technique. In this thesis, we will concentrate on the discrete-time systems. In many applications, mere stability of the controlled object is not enough, and it is required that the poles of the closed-loop system should lie in a certain restricted region of stability. Although standard pole placement techniques (Kailath, 1980) can be applied to this problem, the exact specification of many poles at once is very difficult. In this respect, the LQ technique has an advantage in that a stable pole allocation of the closed-loop system is automatically guaranteed. However, the relationship between the weighting matrices of the performance index and the allocation of the resulting closed-loop poles is not simple. Thus, a difficulty arises if the design requirement is to place some of the closed-loop poles at specified locations in the LQ control design.

Several design methods have been reported which utilize the LQ technique to achieve the desired pole allocation. Continuous-time results are found in Solheim (1972) and Kawasaki and Shimemura (1983). Solheim (1972) employs the modal decomposition and successive shifting of a single real pole or a pair of complex conjugate poles. Kawasaki and Shimemura (1983) have derived a method of allocating all the closed-loop poles in a preferable region rather than exact location. However, the continuous-time results cannot be directly extended to discrete-time case. Solheim (1974) has developed a discrete version of Solheim (1972), but the optimality of the closed-loop system is lost due to the difference in the form of continuous and discrete Riccati equations associated with the LQ control problem. More recently, Amin (1984) derived an improved

result in which the optimality of the closed-loop system is assured.

In this thesis, a method of designing discrete-time LQ control systems with its closed-loop poles in a prescribed region is developed. Since the region cannot be specified arbitrary even if it is symmetric about the real axis, two particular regions are considered. One is a disc with its center at the origin of the complex plane, and the other is a disc which contacts the point $1 + j0$ of the complex plane. In each case the radius of the disc can be specified as a design parameter, and the weighting matrices of the performance index is obtained by the proposed design procedure.

This method guarantees that all the closed-loop poles lie inside the specified disc, but the exact location of each pole within the disc cannot be explicitly specified. From this point of view, the design method of employing modal decomposition and shifting one mode at a time has an advantage that the exact location of the closed-loop poles can be specified. However, it should be noted that an arbitrary location cannot be specified as an optimal closed-loop pole, and the region of assignable optimal closed-loop has not been clarified. In fact, the results of Solheim (1972), Solheim (1974) and Amin (1982) utilize only a restricted part of the assignable region in the design. For continuous-time systems, Johnson (1988) clarified the maximal assignable region for single input systems, and Sugimoto et al. (1989) derived a related result for multi input systems, using the solution of the inverse regulator problem. For discrete-time systems, an independent development is required.

In this thesis the maximal region of assignable optimal poles is clarified, which leads to more efficient method of designing discrete-time LQ regulators with specified pole allocation. To this end, the relationship between the weighting matrices and the optimal closed-loop poles is investigated by evaluating the characteristic equation of the symplectic matrix associated with the discrete-time LQ control problem.

This thesis is organized as described in the following.

In Chapter 2, conventional results on the LQ control is briefly reviewed. Basic results on the continuous-time LQ control of delay-free systems are introduced first, followed by the results on the LQ control of general retarded type time-delay systems. Then, basic results on the discrete-time LQ control are introduced. The results of this chapter are used in deriving the results on the LQ control of unilateral time-delay systems in the later chapters.

In Chapter 3, the unilateral time-delay system is described in several forms, and the characteristics of the unilateral structure is clarified. The representation by a set of difference-differential equations is given first, which will be used in deriving the steady-state optimal solution. It is followed by another form of representation, in which a partial differential equation is used for expressing the behavior of the time-delay element. This representation will be used for solving the initial part of the LQ control problem. Then, the unilateral time-delay system is described in the discrete-time form by a difference equation, which is obtained by discretizing the continuous-time equation. The difference equation will be used in the design of sampled data control systems. It will be shown that the transition of the sampled values of vectors can be described by a finite dimensional difference equation without any approximating assumption on the behavior of the state between sampling instants.

In Chapter 4, continuous-time LQ control problem will be studied based on the representations given in Chapter 3. The optimal solution is given in two parts, namely, steady state and initial part. It will be shown that the optimal solution for the steady state can be obtained from the solution for an imaginary delay-free system related to the original time-delay system. As for the initial part, it will be shown that the LQ control problem can be solved in smaller intervals, and that only one time-delay need be considered at a time in deriving the optimal solution for each interval. Then, implementation of the controller based on the steady state solution is considered. The structure of the controller using a finite-interval integrator and time-delay elements is presented. Then, a modification of the controller is proposed, which improves the response to unknown disturbances. The structure of the controller using an observer for the imaginary system is also presented.

In Chapter 5, discrete-time LQ control problem is studied based on the difference equation given in Chapter 3. The purpose of this chapter is to present a method of reducing computational difficulties in the design of the optimal controllers for unilateral time-delay systems. First, an imaginary delay-free system is introduced, as in the continuous-time case. Then a formula is derived, by which the optimal solution of the imaginary system is converted to the solution for the original time-delay system. This method reduces the computational difficulties.

In Chapter 6, the design of discrete-time LQ control systems with its

closed-loop poles in a prescribed region of stability is considered. First, by utilizing the property of Riccati equation with Q being zero matrix, we develop a method for allocating poles in a disc with its center at the origin of the complex plane and with radius less than one. Secondly, we deal with the pole placement in a disc which is in the unit disc and also contacts the point $1 + j0$ of the complex plane. To this end, a bilinear transformation and the results of continuous-time LQ control are employed. In each case, the radius of the disc can be specified as a design parameter, and the weighting matrices of the performance index are obtained to fulfill the desired pole allocations.

In Chapter 7, the design of discrete-time LQ control system by shifting a single real pole or a pair of complex conjugate poles is considered, and the region of assignable optimal closed-loop poles is clarified. The modal decomposition is employed to the controlled object, and the weighting matrices are chosen so that only the specified mode is altered. The assignable region of closed-loop poles is determined by evaluating the characteristic equation of the symplectic matrix associated with the discrete-time LQ control problem.

Chapter 8 is the conclusion of this thesis.

Chapter 2

Outline of optimal control theory

In this chapter, important results about the optimal control are briefly reviewed as preliminary studies. First, the continuous-time optimal control problem of delay-free systems is stated, and the solution is given. In order to calculate the optimal control law, a matrix nonlinear equation of Riccati type (abbreviated as Riccati equation in the following) must be solved. To meet this requirement, numerical methods to solve the Riccati equation are reviewed as well. Second, available results on the optimal control of retarded type time-delay systems are reviewed. Third, the optimal control problem and its solution for the discrete-time systems are reviewed, in a similar manner to the continuous-time case.

2.1 Basic results on optimal control of delay-free continuous-time systems

Consider the linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.1)$$

$$y(t) = Cx(t) \quad (2.2)$$

with the initial state

$$x(0) = x_0 \quad (2.3)$$

given. Here, $x(t)$ denotes the state vector of dimension n , $u(t)$ is the input vector of dimension m , and $y(t)$ is the output vector of dimension r . Accordingly, the size of coefficient matrices A , B , and C are $n \times n$, $n \times m$, and $r \times n$, respectively. The pair (A, B) is assumed to be controllable. Let the performance index J be

$$J = \int_0^T \{x'(t)Qx(t) + u'(t)Ru(t)\}dt + x'(T)P_Tx(t) \quad (2.4)$$

where T is the final time, Q is a symmetric, positive semidefinite matrix of size $n \times n$, R is a symmetric, positive definite matrix of size $m \times m$, and P_T is a symmetric, positive semidefinite matrix of size $n \times n$. The matrices Q , R , and P_T are the weighting coefficients for the state $x(t)$, the input $u(t)$, and the final value of the state $x(T)$, respectively. The symbol $'$ indicates the transposition of the vector, and hereafter it will be used for matrices as well. The pair $(Q^{1/2}, A)$ is assumed to be observable, where $Q^{1/2}$ is defined as the rank $Q \times n$ matrix which satisfies $(Q^{1/2})^T Q^{1/2} = Q$. The optimal regulator problem is to find the input

$$u^*(t), \quad 0 \leq t \leq T$$

which minimizes the performance index J subject to (2.1). The superscript $*$ will be used to denote the optimum function and value hereafter, e.g., the minimum value of the performance index J will be denoted as J^* .

The solution of the optimal regulator problem is given as follows (Anderson and Moore, 1971). The optimal input can be given in the form of the linear state feedback control law as

$$u^*(t) = -F(t)x(t) \quad (2.5)$$

The state feedback matrix $F(t)$ which gives the optimal input is called the optimal feedback gain and is given by

$$F(t) = R^{-1}B'P(t) \quad (2.6)$$

where $P(t)$ is the symmetric, positive semidefinite solution of the matrix Riccati differential equation

$$-\dot{P}(t) = P(t)A + A'P(t) - P(t)BR^{-1}B'P(t) + Q \quad (2.7)$$

with the terminal condition

$$P(T) = P_T \quad (2.8)$$

The minimum (i.e., optimal) value of the performance index J is given by

$$J^* = x'(0)P(0)x(0) \quad (2.9)$$

When the final time T is infinity and the performance index J is given as

$$J = \int_0^\infty \{x'(t)Qx(t) + u'(t)Ru(t)\}dt , \quad (2.10)$$

the optimal feedback gain becomes a constant matrix

$$F = R^{-1}B'P \quad (2.11)$$

to result in a time-invariant state feedback law

$$u^*(t) = -Fx(t) \quad (2.12)$$

In this case P is the unique positive-definite solution of the associated matrix Riccati algebraic equation

$$PA + A'P - PBR^{-1}B'P + Q = 0 \quad (2.13)$$

The minimum value of the performance index J is given by

$$J^* = x'(0)Px(0) \quad (2.14)$$

and the closed-loop system

$$\dot{x}(t) = (A - BF)x(t) \quad (2.15)$$

is asymptotically stable.

There are several numerical methods to compute the solution P of the algebraic Riccati equation (2.13). They can be classified as follows:

1. Method based on the Riccati differential equation

Starting with the terminal condition

$$P(0) = 0 \quad (2.16)$$

the Riccati differential equation

$$-\dot{P}(t) = P(t)A + A'P(t) - P(t)BR^{-1}B'P(t) + Q \quad (2.17)$$

is solved in the backward direction. When the steady state is reached, the solution P is obtained as

$$P = P(-\infty) \quad (2.18)$$

An algorithm such as Runge-Kutta method can be employed to solve the differential equation.

2. Method based on the eigenvalues of the Hamilton matrix

The matrix given by

$$H = \begin{bmatrix} -A & BR^{-1}B' \\ Q & A' \end{bmatrix}. \quad (2.19)$$

is called the Hamiltonian matrix. It is known that the eigenvalues of H are allocated symmetrically with respect to the imaginary axis, and that the eigenvalues in the left half plane correspond to the poles of the optimal closed-loop system. Express the eigenvectors of H corresponding to the eigenvalues in the left-half plane as

$$w_i = \begin{bmatrix} w_{1i} \\ w_{2i} \end{bmatrix}, \quad i = 1, \dots, n \quad (2.20)$$

Then the positive definite solution P of the algebraic Riccati equation (2.13) is given by

$$P = [w_{11}, w_{12}, \dots, w_{1n}] [w_{21}, w_{22}, \dots, w_{2n}]^{-1} \quad (2.21)$$

This method can be modified so that the Schur vectors are used instead of eigenvectors. In that case a bilinear transformation is often employed, and the solution is obtained via discrete-time Riccati equation.

3. Method based on successive approximation

First, a matrix F_1 is chosen such that $A - BF_1$ is stable. Next, we calculate the sequence of matrices $P_1, P_2, \dots, P_i, \dots$; and $F_2, F_3, \dots, F_i, \dots$, which satisfy

$$P_i A_i + A_i' P_i + F_i' R F_i + Q = 0 \quad (2.22)$$

$$F_{i+1} = R^{-1} B' P_i \quad (2.23)$$

$$A_i = A - B F_i \quad (2.24)$$

Then, P_i converges to the positive definite solution P of the algebraic Riccati equation (2.13).

2.2 Results on optimal control of time-delay systems

In this section the optimal control of systems which have the structure shown in Fig. 2.1 is considered. Such systems belong to the class of retarded type time-delay systems, but they are not unilateral time-delay systems, as seen from the existence of a loop including the time-delay $\mathcal{D}$.

Let the delay-free part $\mathcal{S}$ be described by

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Ev(t) \quad (2.25)$$

and the time-delay $\mathcal{D}$ be described by

$$v(t) = Lx(t - d). \quad (2.26)$$

Here, v is a vector of dimension r , representing the input from the time-delay, and d is the delay time. The size of coefficient matrix E and L are respectively $n \times r$ and $r \times n$. The state vector $x(t)$, input vector $u(t)$, and coefficient matrices A , B are as defined in Section 2.1. The time-delay $\mathcal{D}$ can also be expressed by the following equations:

$$\frac{\partial}{\partial t}\xi(t, s) + \frac{1}{d}\frac{\partial}{\partial s}\xi(t, s) = 0, \quad 0 < s < 1 \quad (2.27)$$

$$v(t) = \xi(t, 1), \quad \xi(t, 0) = Lx(t) \quad (2.28)$$

Here, $\xi(t, s)$ is a vector of dimension r , representing the state of the time-delay. It is a function of time t and location s , and the initial state

$$\xi(0, s), \quad 0 < s < 1$$

is assumed to be given. The length of the time-delay element is normalized to unity. Thus, d is equal to the reciprocal of the velocity of the signal in the time-delay element. Let the performance index J be

$$J = \int_0^T \{x'(t)Qx(t) + u'(t)Ru(t)\}dt \quad (2.29)$$

The optimal control $u^*(t)$ is given by

$$u^*(t) = -R^{-1}B'P(t)x(t) + \int_0^1 B'\hat{P}(t, s)\xi(t, s)ds \quad (2.30)$$

where $P(t)$ and $\hat{P}(t, s)$ are the solution of the matrix partial differential equations of Riccati type:

$$\frac{d}{dt}P(t) + P(t)A + A'P(t) + Q - P(t)BR^{-1}B'P(t) = 0 \quad (2.31)$$

$$\frac{\partial}{\partial t}\hat{P}(t, s) + \frac{1}{d}\frac{\partial}{\partial s}\hat{P}(t, s) + A\hat{P}(t, s) - P(t)BR^{-1}B'\hat{P}(t, s) = 0 \quad (2.32)$$

with the boundary conditions

$$P(t)E - \frac{1}{d}\hat{P}(t, 1) = 0 \quad (2.33)$$

and terminal conditions

$$P(T) = P_T \quad (2.34)$$

$$2x'(T) \int_0^1 \hat{P}(T, s)\xi(T, s)ds = 0 \quad (2.35)$$

Concerning the above solution of the optimal control law for time-delay systems, there are two difficulties:

1. Difficulty of numerical computation

Partial differential equations (2.31) and (2.32) should be solved to obtain the optimal feedback gain. Generally, it is much more difficult to solve partial differential equations than to solve ordinary differential equations, and the same is true for the present problem. Several researches have been made about numerical methods to solve the partial differential equations of the Riccati type (Fujikawa and Shimemura 1975), but the results are not satisfactory compared with the case of the ordinary differential equations.

2. Difficulty in implementation

The control law is given in the form that the infinite dimensional states of the time-delays are used for feedback. These states must be observed for the implementation of the controller, but such is practically impossible. So we are forced to approximate the optimal control law by some control law that can be implemented. But little are known about which ways of approximation are suitable in guaranteeing stability and keeping the performance index nearly minimum.

2.3 Basic results on discrete-time optimal control

Consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \quad (2.36)$$

$$y(k) = Cx(k) \quad (2.37)$$

with the initial state

$$x(0) = x_0 \quad (2.38)$$

given. Here, $x(k)$ denotes the state vector of dimension n , $u(k)$ is the input vector of dimension m , and $y(k)$ is the output vector of dimension r . Accordingly, the size of coefficient matrices A , B , and C are $n \times n$, $n \times m$, and $r \times n$, respectively. The pair (A, B) is assumed to be controllable. Let the performance index J be

$$J = \sum_{k=0}^{T-1} \{x'(k)Qx(k) + u'(k)Ru(k)\} + x'(T)P_Tx(T) \quad (2.39)$$

where T is the final time, Q is a symmetric, positive semidefinite matrix of size $n \times n$, R is a symmetric, positive definite matrix of size $m \times m$, and P_T is a symmetric, positive semidefinite matrix of size $n \times n$. The matrices Q , R , and P_T are the weighting coefficients for the state $x(k)$, input $u(k)$, and final value of the state $x(T)$, respectively. The pair $(Q^{1/2}, A)$ is assumed to be observable. The optimal regulator problem is to find the optimal control

$$u^*(k), \quad 0 \leq k \leq T$$

which minimizes the performance index J subject to (2.36).

The solution of the optimal regulator problem is given as follows (Kwakernaak and Sivan, 1972). The optimal control $u^*(k)$ is given by the linear state feedback control law

$$u^*(k) = -F(k)x(k) \quad (2.40)$$

The optimal feedback gain $F(k)$ is given by

$$F(k) = \{B'P(k)B + R\}^{-1}B'P(k)A \quad (2.41)$$

where $P(k)$ is the symmetric, positive definite solution of the matrix difference equation of the Riccati type:

$$P(k-1) = A'P(k)A + Q - A'P(k)B\{B'P(k)B + R\}^{-1}B'P(k)A \quad (2.42)$$

with the terminal condition

$$P(T) = P_T \quad (2.43)$$

The minimum value of the performance index J is given by

$$J^* = x'(0)P(0)x(0) \quad (2.44)$$

When the performance index J takes the form

$$J = \sum_{k=0}^{\infty} \{x'(k)Qx(k) + u'(k)Ru(k)\} \quad (2.45)$$

the optimal control law becomes a constant gain feedback law

$$u^*(k) = -Fx(k) \quad (2.46)$$

as in the continuous-time case. The optimal feedback gain F is given by

$$F = (B'PB + R)^{-1}B'PA \quad (2.47)$$

where P is determined by the associated matrix Riccati algebraic equation

$$P = A'PA - A'PB(B'PB + R)^{-1}B'PA + Q \quad (2.48)$$

The minimum value of the performance index J is given by

$$J^* = x'(0)Px(0) \quad (2.49)$$

and the closed loop system

$$x(k+1) = (A - BF)x(k) \quad (2.50)$$

is asymptotically stable.

In order to obtain the steady state solution P , several numerical computation methods are available as in the continuous-time case. They can be classified as follows:

1. Method based on the Riccati difference equation

Starting with the terminal condition

$$P(0) = 0 \quad (2.51)$$

the Riccati difference equation

$$P(k-1) = A'P(k)A + Q - A'P(k)B\{B'P(k)B + R\}^{-1}B'P(k)A \quad (2.52)$$

is solved recursively in the backward direction. When the steady state is reached, the solution P is obtained as

$$P = P(-\infty) \quad (2.53)$$

In order to reduce truncation errors in the computation, the iteration formula can be modified as

$$P(k-1) = A'P(k)\{I + BR^{-1}B'P(k)\}^{-1}A + Q \quad (2.54)$$

in which no subtraction appears.

2. Method based on the eigenvalues of a symplectic matrix

The matrix given by

$$H = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B' \\ QA^{-1} & A' + QA^{-1}BR^{-1}B' \end{bmatrix}. \quad (2.55)$$

is called the symplectic matrix. It is known that the eigenvalues of H are such that the reciprocal of every eigenvalue is also an eigenvalue. The eigenvectors of H corresponding to the eigenvalues outside the unit circle are calculated and expressed as

$$w_i = \begin{bmatrix} w_{1i} \\ w_{2i} \end{bmatrix}, \quad i = 1, \dots, n \quad (2.56)$$

Then the solution P of the Riccati equation is given by

$$P = [w_{11}, w_{12}, \dots, w_{1n}] [w_{21}, w_{22}, \dots, w_{2n}]^{-1} \quad (2.57)$$

In order to reduce computational difficulties in calculating the eigenvectors of the symplectic matrix H , a similar method is known in which Schur vectors are employed instead of eigenvectors (Laub, 1979).

3. Method based on successive approximation

First, a matrix F_1 is chosen such that $A - BF_1$ is stable in discrete-time sense. Next, we calculate the sequence of matrices $P_1, P_2, \dots, P_i, \dots$; and $F_2, F_3, \dots, F_i, \dots$, which satisfy

$$P_i = A_i' P_i A_i + F_i' R F_i + Q \quad (2.58)$$

$$F_{i+1} = (B' P_i B + R)^{-1} B' P_i A \quad (2.59)$$

$$A_i = A - B F_i \quad (2.60)$$

Then, P_i converges to the solution P of the Riccati equation.

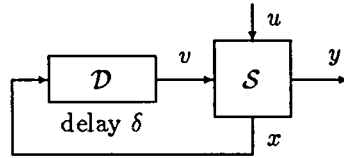


Fig. 2.1 Retarded type time-delay system

Chapter 3

Description of unilateral time-delay systems

In this chapter, the description of the unilateral time-delay systems are given, and its characteristics are considered. Continuous-time representation is given first, which is in the form of difference-differential equations. This representation is used in solving the steady state part of the continuous-time optimal control problem. It is not adequate for solving the initial part of the optimal control problem, however, since the state of the time-delay does not appear explicitly in this form. Thus the representation as a distributed parameter system is also introduced, in which a partial differential equation is used for expressing the behavior of the time-delay element.

Then the discrete-time representation of the unilateral time-delay systems is considered. Namely, the continuous-time representation given in the form of difference-differential equation is discretized, which results in the form of difference equation. This representation is used in solving the discrete-time optimal control problem.

It should be noted that the discrete-time equation can be derived without any approximating assumptions in the behavior of the state between the sampling instants, and that the resultant equation still consists of finite terms. This property is due to the unilateral structure of the system, and it does not apply to general retarded type time-delay systems. It should be also noted that the discrete-time unilateral time-delay systems are described by the difference equation, and this is essentially the same form as delay-free systems.

In the description of the discretization procedure, both continuous-time quantities and discrete-time quantities appear in one expression. Thus, the subscript c is attached to the former throughout this chapter in order to avoid confusion.

3.1 Continuous-time description

Consider the time-delay system $\mathcal{S}$ shown in Fig. 3.1. It is a unilateral time-delay system which consists of delay-free subsystems $\mathcal{S}_{ci}$ connected in series by time-delays $\mathcal{D}_{ci}$. Here, each $\mathcal{S}_{ci}$ ($i = 1, \dots, p$) represents a time-invariant linear system described by

$$\dot{x}_{ci}(t) = A_{ci}x_{ci}(t) + B_{ci}u_{ci}(t) \quad (3.1)$$

$$y_{ci}(t) = C_{ci}x_{ci}(t) \quad (3.2)$$

where x_{ci} is the n_{ci} dimensional state vector, u_{ci} is the m_{ci} dimensional manipulating vector, v_{ci} is the r_{ci-1} dimensional output of the preceding block, and y_{ci} is the r_{ci} dimensional output vector. The block $\mathcal{D}_{ci}$ ($i = 1, \dots, p-1$) is a delay line described by

$$v_{ci+1}(t) = y_{ci}(t - \tau_{ci}) \quad (3.3)$$

By substituting (3.1)–(3.3) into (3.8), $\mathcal{S}_c$ can be expressed by the following difference-differential equation:

$$\dot{x}_c(t) = A_c x_c(t) + \sum_{i=1}^{p-1} \hat{A}_{ci} x_c(t - \hat{\tau}_i) + B_c u_c(t) \quad (3.4)$$

Here, x_c and u_c are defined as

$$x_c(t) = [x'_{cp}(t), \dots, x'_{c1}(t)]' \quad (3.5)$$

$$u_c(t) = [u'_{cp}(t), \dots, u'_{c1}(t)]' \quad (3.6)$$

and the coefficient matrices turn out to be

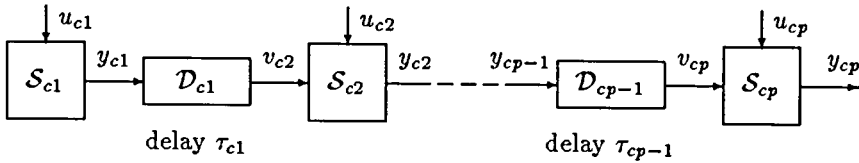


Fig. 3.1 Continuous-time unilateral time-delay system $\mathcal{S}_c$

$$\begin{aligned}
A_c &= \begin{bmatrix} A_{cp} & & 0 \\ & \ddots & \\ 0 & & A_{c1} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{cp} & & 0 \\ & \ddots & \\ 0 & & B_{c1} \end{bmatrix}, \\
\hat{A}_{cp} &= \begin{bmatrix} 0 & E_{cp}C_{cp-1} & 0 & \cdots & \cdots & 0 \\ & 0 & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & 0 & 0 \\ & & & & 0 & 0 \\ 0 & & & & & 0 \end{bmatrix}, \dots, \\
\hat{A}_{c1} &= \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ & 0 & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & 0 & 0 \\ & & & & 0 & E_{c2}C_{c1} \\ 0 & & & & & 0 \end{bmatrix} \quad (3.7)
\end{aligned}$$

From the above description it is clear that unilateral time-delay systems belongs to the class of retarded type time-delay systems. As seen from (3.7), the coefficient matrices $\hat{A}_{ci}$ are sparse, such that there is only one nonzero submatrix $E_{ci+1}C_{ci}$ in each $\hat{A}_{ci}$. This leads to the reduction of difficulties in the control of unilateral time-delay systems, compared to that of general retarded type time-delay systems.

In (3.3) time-delay $\mathcal{D}_{ci}$ is described by a difference equation. When the state of the time-delay is required in the design of the optimal controller, namely, when the initial state of the time-delay $\mathcal{D}_{ci}$ is taken into consideration, the following representation should be used:

$$\frac{\partial}{\partial t}\xi_{ci}(t, s) + \frac{1}{\tau_{ci-1}}\frac{\partial}{\partial s}\xi_{ci}(t, s) = 0, \quad 0 < s < 1 \quad (3.8)$$

$$\xi_{ci}(t, 0) = y_{ci}(t) \quad (3.9)$$

$$v_{ci+1}(t) = \xi_{ci}(t, 1) \quad (3.10)$$

Thus, the unilateral time-delay system $\mathcal{S}_c$ can be described as a distributed parameter system, in which $\xi_{ci}(t, s)$ represents the state of the time-delay $\mathcal{D}_{ci}$.

Now, recall that in Section 2.1.2 the optimal solution for retarded type time-delay systems described by (2.25) and (2.26) has been given by (2.30)–(2.35). The system described by (2.25)–(2.26) is not restricted to unilateral time-delay systems, but the time-delay included in (2.25)–(2.26) is only that of length d . On the other hand, the unilateral time-delay system $\mathcal{S}_c$ in this section has time-delays of different lengths. Thus, the result of Section 2.1.2 cannot be directly applied to system $\mathcal{S}_c$.

However, this difficulty can be avoided by taking the advantage of the structure of the unilateral time-delay systems. Consider the system $\hat{\mathcal{S}}_c$ shown in Fig. 3.2. This system can be described as:

$$\frac{d}{dt}x_c(t) = A_c x_c(t) + B_c u_c(t) + E_c \xi_c(t, 1) \quad (3.11)$$

$$\frac{\partial}{\partial t} \xi_c(t, s) + \frac{1}{\tau_c} \frac{\partial}{\partial s} \xi_c(t, s) = 0, \quad 0 < s < 1 \quad (3.12)$$

$$\xi_c(t, 0) = 0 \quad (3.13)$$

System $\hat{\mathcal{S}}_c$ is certainly a unilateral time-delay system which contains only one delay line of length τ_c . It has a restricted form of the system $\mathcal{S}_c$ shown in Fig. 3.1, and at the same time it belongs to the class of systems described by (2.25)–(2.26) in Section 2.1.2. When the continuous-time optimal control problem for the initial part is considered, the problem is solved via the solution for the system $\hat{\mathcal{S}}_c$. Detailed discussion of this problem will be described in Chapter 4.

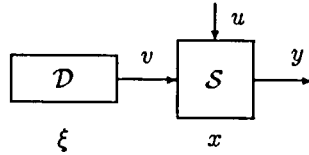


Fig. 3.2 Time-delay system $\hat{\mathcal{S}}_c$

3.2 Discretization

It is assumed that sampled data controllers with the zero-order-hold circuits are implemented at each input u_{ci} . Namely, each $u_{ci}(t)$ is assumed to be constant between the sampling instants:

$$u_{ci}(t) = u_i(k), \quad kT \leq t < (k+1)T \quad (3.14)$$

Let μ_i and σ_i be the integer and the real number satisfying

$$(\mu_i - 1)T < \tau_{ci} \leq \mu_i T, \quad i = 1, \dots, p-1 \quad (3.15)$$

$$\sigma_i = \mu_i T - \tau_{ci}, \quad i = 1, \dots, p-1. \quad (3.16)$$

By the standard formulae of linear systems, the following equations are obtained for the state $x_{ci}(t)$ between kT and $(k+1)T$.

$$\begin{aligned} x_{ci}(t) = & e^{A_{ci}(t-kT)} x_{ci}(kT) + \int_0^{t-kT} e^{A_{ci}(t-kT-\sigma)} B_{ci} u_i(k) d\sigma \\ & + \int_0^{t-kT} e^{A_{ci}(t-kT-\sigma)} E_{ci} v_{ci}(kT + \sigma) d\sigma \end{aligned} \quad (3.17)$$

By using (3.8) and (3.3) recurrently, a set of difference equations are obtained, which describes the transition from time kT to time $(k+1)T$:

$$\begin{aligned} x_{ci}((k+1)T) = & \sum_{j=1}^i \sum_{h=0}^{i-j} \{ A_{ijh} x_{cj}((k - \mu_{ijh})T) \\ & + B_{ijh} u_{cj}((k - \mu_{ijh})T) \} \end{aligned} \quad (3.18)$$

Here, integers μ_{ijh} are defined for $i = 1, \dots, p$; $j = 1, \dots, i$; $h = 0, \dots, i-j$ as

$$\mu_{ijh} = \begin{cases} 0, & \text{if } j = i \\ \mu_{i-1} + \mu_{i-2} + \dots + \mu_j - h, & \text{if } j < i \end{cases} \quad (3.19)$$

The coefficient matrices A_{ijh} , B_{ijh} are calculated by the formulae given in the following. The subscripts i and j are used to indicate stations S_i and S_j , respectively. The state variable of station S_i is influenced by the past value of the state variable and the manipulating variable of the preceding stations S_j , and the coefficient matrices A_{ijh} , B_{ijh} show the magnitude of the influence from station S_j to station S_i . It should be noted that

the influence from station S_j to station S_i is described by multiple terms. The number of terms depends on the value $i - j$, namely the number of stations between station S_j to station S_i , and the suffix h is used to identify the terms within the same pair of i and j .

The formulae to calculate the coefficient matrices A_{ijh} , B_{ijh} are presented. Define matrices $A_{ijh}(t)$, $B_{ijh}(t)$ ($i = 1, \dots, p$; $j = 1, \dots, i$; $h = 0, \dots, i-j$) for $0 < t \leq T$ as follows:

(i) if $j = i$ and $h = 0$

$$A_{ijh}(t) = A_{ii0}(t) = e^{A_{ci}t}, \quad 0 < t \leq T \quad (3.20)$$

$$B_{ijh}(t) = B_{ii0}(t) = \int_0^t e^{A_{ci}(t-\tau)} B_{ci} d\tau, \quad 0 < t \leq T \quad (3.21)$$

(iia) if $j = i-1$ and $h = 0$

$$A_{ijh}(t) = A_{ii-10}(t) = \begin{cases} \int_0^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} A_{i-1i-10}(\sigma_{i-1} + \tau) d\tau, & 0 < t \leq T - \sigma_{i-1} \\ \int_0^{T-\sigma_{i-1}} e^{A_{ci}(T-\sigma_{i-1}-\tau)} E_{ci} C_{ci-1} A_{i-1i-10}(\sigma_{i-1} + \tau) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.22)$$

$$B_{ijh}(t) = B_{ii-10}(t) = \begin{cases} \int_0^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} B_{i-1i-10}(\sigma_{i-1} + \tau) d\tau, & 0 < t \leq T - \sigma_{i-1} \\ \int_0^{T-\sigma_{i-1}} e^{A_{ci}(T-\sigma_{i-1}-\tau)} E_{ci} C_{ci-1} B_{i-1i-10}(\sigma_{i-1} + \tau) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.23)$$

(iib) if $j = i-1$ and $h = i-j = 1$

$$A_{ijh}(t) = A_{ii-11}(t)$$

$$= \begin{cases} 0, & 0 < t \leq T - \sigma_{i-1} \\ \int_{T-\sigma_{i-1}}^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} A_{i-1i-10}(\sigma_{i-1} + \tau - T) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.24)$$

$$B_{ijh}(t) = B_{i-11}(t)$$

$$= \begin{cases} 0, & 0 < t \leq T - \sigma_{i-1} \\ \int_{T-\sigma_{i-1}}^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} B_{i-1i-10}(\sigma_{i-1} + \tau - T) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.25)$$

(iiia) if $j < i-1$ and $h = 0$

$$A_{ijh}(t) = A_{ij0}(t)$$

$$= \begin{cases} \int_0^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} A_{i-1j0}(\sigma_{i-1} + \tau) d\tau, & 0 < t \leq T - \sigma_{i-1} \\ \int_0^{T-\sigma_{i-1}} e^{A_{ci}(T-\sigma_{i-1}-\tau)} E_{ci} C_{ci-1} A_{i-1j0}(\sigma_{i-1} + \tau) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.26)$$

$$B_{ijh}(t) = B_{ij0}(t)$$

$$= \begin{cases} \int_0^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} B_{i-1j0}(\sigma_{i-1} + \tau) d\tau, & 0 < t \leq T - \sigma_{i-1} \\ \int_0^{T-\sigma_{i-1}} e^{A_{ci}(T-\sigma_{i-1}-\tau)} E_{ci} C_{ci-1} \cdot B_{i-1j0}(\sigma_{i-1} + \tau) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.27)$$

(iiib) if $j < i-1$ and $0 < h < i-j$

$$A_{ijh}(t)$$

$$= \begin{cases} \int_0^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} A_{i-1jh}(\sigma_{i-1} + \tau) d\tau, & 0 < t \leq T - \sigma_{i-1} \\ \int_0^{T-\sigma_{i-1}} e^{A_{ci}(T-\sigma_{i-1}-\tau)} E_{ci} C_{ci-1} A_{i-1jh}(\sigma_{i-1} + \tau) d\tau, \\ + \int_{T-\sigma_{i-1}}^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} A_{i-1jh}(\sigma_{i-1} + \tau - T) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.28)$$

$$B_{ijh}(t)$$

$$= \begin{cases} \int_0^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} B_{i-1jh}(\sigma_{i-1} + \tau) d\tau, & 0 < t \leq T - \sigma_{i-1} \\ \int_0^{T-\sigma_{i-1}} e^{A_{ci}(T-\sigma_{i-1}-\tau)} E_{ci} C_{ci-1} B_{i-1jh}(\sigma_{i-1} + \tau) d\tau, \\ + \int_{T-\sigma_{i-1}}^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} B_{i-1jh}(\sigma_{i-1} + \tau - T) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.29)$$

(iiic) if $j < i-1$ and $h = i-j$

$$A_{ijh}(t) = A_{ij\ i-j}(t)$$

$$= \begin{cases} 0, & 0 < t \leq T - \sigma_{i-1} \\ \int_{T-\sigma_{i-1}}^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} A_{i-1\ j\ i-j-1}(\sigma_{i-1} + \tau - T) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.30)$$

$$B_{ijh}(t) = B_{ij\ i-j}(t)$$

$$= \begin{cases} 0, & 0 < t \leq T - \sigma_{i-1} \\ \int_{T-\sigma_{i-1}}^t e^{A_{ci}(t-\tau)} E_{ci} C_{ci-1} B_{i-1\ j\ i-j-1}(\sigma_{i-1} + \tau - T) d\tau, & T - \sigma_{i-1} < t \leq T \end{cases} \quad (3.31)$$

Then the coefficient matrices A_{ijh} , B_{ijh} of equation (3.18) are given by

$$A_{ijh} = A_{ijh}(T) , \quad B_{ijh} = B_{ijh}(T) \quad (3.32)$$

$$i = 1, \dots, p ; \quad j = 1, \dots, i ; \quad h = 0, \dots, i-j$$

The dimension of matrices A_{ijh} and B_{ijh} are $n_{ci} \times n_{ci}$ and $n_{ci} \times m_{ci}$, respectively. If the states between sampling instants are assumed to be constant, the coefficient matrices A_{ijh} and B_{ijh} with $i-j \geq 2$ vanish. Thus, the difference $i-j$ of the subscripts approximately indicates the order of the magnitude of the terms. In practical application, the terms which correspond to large $i-j$ may be neglected.

It should be noted that no approximation is made in deriving equation (3.18), and still its right-hand side consists of finite terms. This property is the consequence resulted from the unilateral structure, i.e., the structure that the signal flows through the delays in one direction, of the system and does not apply to general time-delay systems. For general time-delay systems, the right-hand side of (3.18) turns out to be an infinite series as shown in Koepcke (1965). Now, put

$$d_i = \mu_i - 1 , \quad i = 1, \dots, p-1 \quad (3.33)$$

and define the discrete-time vectors $x_i(k)$, $y_i(k)$, $z_i(k)$, $v_i(k)$ by

$$v_i(k) = v_{ci}(kT) , \quad i = 1, \dots, p \quad (3.34)$$

$$x_1(k) = x_{c1}(kT) ,$$

$$x_i(k) = [x'_{ci}(kT), v'_i(k-1), \dots, v'_i(k-i+1)]' , \quad i = 2, \dots, p \quad (3.35)$$

$$y_i(k) = [x'_{ci}(kT), v'_i(k-i+1), u'_{ci}(kT)]' , \quad i = 1, \dots, p-1$$

$$y_p(k) = y_{cp}(kT) \quad (3.36)$$

$$z_i(k) = [y'_i(k-d_i), \dots, y'_i(k-1)]' , \quad i = 1, \dots, p-1 \quad (3.37)$$

$$v_{i+1}(k) = y_i(k-d_i) , \quad i = 1, \dots, p-1 \quad (3.38)$$

Then the following discrete-time state equations are obtained:

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + E_i v_i(k) , \\ y_i(k) &= C_i x_i(k) + D_i u_i(k) \end{aligned} \quad (3.39)$$

$$i = 1, \dots, p$$

$$\begin{aligned}
z_i(k+1) &= F_i z_i(k) + G_i y_i(k), \\
v_i(k) &= H_i z_i(k)
\end{aligned} \tag{3.40}$$

$i = 1, \dots, p$

The coefficient matrices are given as follows:

$$A_1 = A_{110}, \quad B_1 = B_{110}, \quad C_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{3.41}$$

$$A_i = \begin{bmatrix} A_{ii0} & \tilde{E}_{i,i-2} & \cdots & \cdots & \tilde{E}_{i0} \\ 0 & 0 & & & \\ \vdots & I & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & I & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{ii0} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix},$$

$$C_i = \begin{bmatrix} I & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & I \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix},$$

$$E_i = \begin{bmatrix} \tilde{E}_{i,i-1} \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = 2, \dots, p \tag{3.42}$$

$$F_i = \begin{bmatrix} 0 & I & & & \\ & \ddots & \ddots & & \\ & & \ddots & I & \\ & & & & 0 \end{bmatrix}, \quad G_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix},$$

$$H_i = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}, \quad i = 1, \dots, p \tag{3.43}$$

The partitions of the matrices correspond to the partitions of the vectors $x_i(k)$, $y_i(k)$, $z_i(k)$, and $v_i(k)$ given in (3.34)–(3.38). Submatrices $\tilde{E}_{ih}$ in A_i and E_i are composite matrices defined as

$$\tilde{E}_{ih} = [A_{i,i-1h} \cdots A_{i1h} \ B_{i1h} \cdots B_{i,i-1h}],$$

$$i = 2, \dots, p; h = 0, \dots, i-1 \quad (3.44)$$

Here, A_{ijh} , B_{ijh} ($j = 1, \dots, i-1$) are as defined above for $j \leq i-h$, and $A_{ijh} = 0$, $B_{ijh} = 0$ for $j > i-h$. Note that $\tilde{E}_{i,i-1}$ in E_i is given by putting $h = i-1$, which turns out to be

$$\tilde{E}_{i,i-1} = [0 \ \dots \ 0 \ A_{i,1,i-1} \ B_{i,1,i-1} \ 0 \ \dots \ 0] \quad (3.45)$$

As seen from the form of F_i , the system $\mathcal{S}_i$ is the discrete-time delay line whose delay time is $d_i T$. Therefore discrete-time system has a parallel structure to the original continuous-time system as illustrated in Fig. 3.3. It should be noted that the dimension n_i of the discrete-time system $\mathcal{S}_i$ becomes much larger than the dimension n_{ci} of the continuous-time system $\mathcal{S}_{ci}$. From practical viewpoint this is not desirable, and if $\mathcal{S}_{ci}$ are stable it can be avoided by neglecting those terms of (3.18) which have small coefficients.

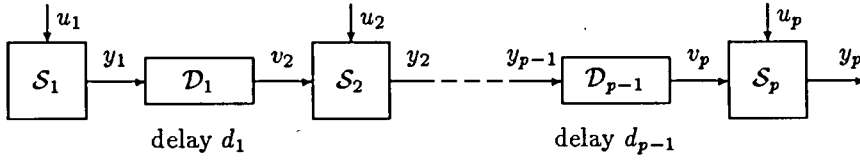


Fig. 3.3 Discrete-time unilateral time-delay system $\mathcal{S}$

In the following, the dimension of the vectors x_i , u_i , and y_i will be denoted by n_i , m_i and r_i , respectively. Note that the dimension of v_i is equal to r_i . By putting

$$x(k) = [x'_p(k), z'_{p-1}(k), x'_{p-1}(k), \dots, z'_1(k), x'_1(k)]' \quad (3.46)$$

$$u(k) = [u'_p(k), u'_{p-1}(k), \dots, u'_1(k)]' \quad (3.47)$$

the state equation can be written as

$$x(k+1) = Ax(k) + Bu(k) \quad (3.48)$$

where

$$A = \begin{bmatrix} A_p & E_p H_{p-1} & 0 & \cdots & 0 & 0 \\ 0 & F_{p-1} & G_{p-1} C_{p-1} & \cdots & 0 & 0 \\ 0 & 0 & A_{p-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & F_1 & G_1 C_1 \\ 0 & 0 & 0 & \cdots & 0 & A_1 \end{bmatrix} \quad (3.49)$$

$$B = \begin{bmatrix} B_p & 0 & \cdots & 0 \\ 0 & G_{p-1} D_{p-1} & \cdots & 0 \\ 0 & B_{p-1} & \ddots & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \ddots & G_1 D_1 \\ 0 & 0 & \cdots & & B_1 \end{bmatrix} \quad (3.50)$$

The dimension of the vectors x and u are n and m , respectively, where

$$n = \sum_{i=1}^p n_i + \sum_{i=1}^{p-1} d_i r_i, \quad m = \sum_{i=1}^p m_i \quad (3.51)$$

3.3 Examples

Example 1

A system with $p = 2$ is considered as an example, with the other parameters

$$\begin{aligned} \mathcal{S}_{c1}: \quad A_{c1} &= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & -\frac{2}{5} \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{5} \end{bmatrix}, \\ C_{c1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{c1} = 0 \end{aligned} \quad (3.52)$$

$$D_{c1}: \quad \tau_{c1} = 4 \quad (3.53)$$

$$\begin{aligned}
S_{c2}: \quad A_{c2} &= \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{2}{9} \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{9} \end{bmatrix}, \\
E_{c2} &= \begin{bmatrix} \frac{3}{20} & \frac{1}{10} \\ \frac{3}{27} & \frac{2}{27} \end{bmatrix}, \quad C_{c1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
D_{c2} &= 0
\end{aligned} \tag{3.54}$$

The block diagram of the system is given in Fig. 3.4. Let the sampling period be $T = 1$, considering the time constants of the system. In this case μ_1 and σ_1 are given by (3.15) and (3.16) as

$$\mu_1 = 4, \quad \sigma_1 = 0 \tag{3.55}$$

Since $p = 2$ in this example, the integers μ_{ijh} are defined as

$$\begin{aligned}
\mu_{110} &= 0, \\
\mu_{210} &= \mu_1 - 0 = 4, \\
\mu_{211} &= \mu_1 - 1 = 3, \\
\mu_{220} &= 0
\end{aligned} \tag{3.56}$$

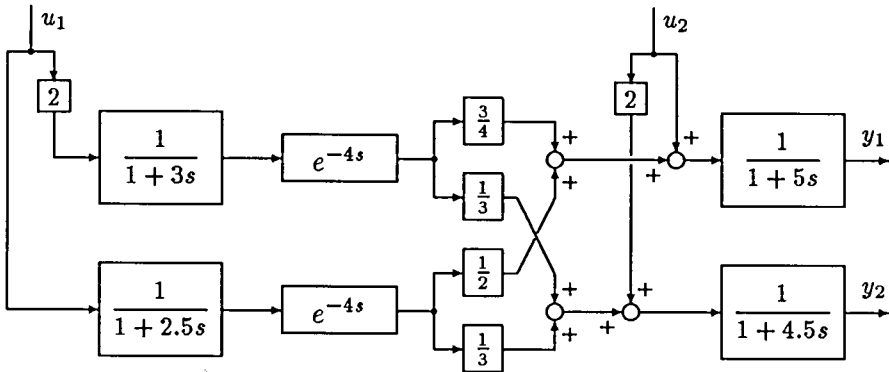


Fig. 3.4 Example of unilateral time-delay system

Thus the state transition from time kT to time $(k+1)T$ is described in the following form:

$$x_{c1}((k+1)T) = A_{110}x_{c1}((k-\mu_{110})T) + B_{110}u_{c1}((k-\mu_{110})T) \quad (3.57)$$

$$\begin{aligned} x_{c2}((k+1)T) &= A_{210}x_{c1}((k-\mu_{210})T) + B_{210}u_{c1}((k-\mu_{210})T) \\ &\quad + A_{211}x_{c1}((k-\mu_{211})T) + B_{211}u_{c1}((k-\mu_{211})T) \\ &\quad + A_{220}x_{c2}((k-\mu_{220})T) + B_{220}u_{c2}((k-\mu_{220})T) \end{aligned} \quad (3.58)$$

Using the equations (3.20)–(3.31) the matrices $A_{ijh}(t)$ and $B_{ijh}(t)$ are given by

$$A_{110}(t) = e^{A_{c1}t} \quad (3.59)$$

$$B_{110}(t) = \int_0^t e^{A_{c1}(t-\tau)} B_{c1} d\tau \quad (3.60)$$

$$A_{210}(t) = \int_0^t e^{A_{c2}(t-\tau)} E_{c2} C_{c1} A_{110}(\tau) d\tau \quad (3.61)$$

$$B_{210}(t) = \int_0^t e^{A_{c2}(t-\tau)} E_{c2} C_{c1} B_{110}(\tau) d\tau \quad (3.62)$$

$$A_{211}(t) = 0 \quad (3.63)$$

$$B_{211}(t) = 0 \quad (3.64)$$

$$A_{220}(t) = e^{A_{c2}t} \quad (3.65)$$

$$B_{220}(t) = \int_0^t e^{A_{c2}(t-\tau)} B_{c2} d\tau \quad (3.66)$$

The coefficient matrices A_{ijh} and B_{ijh} are calculated as follows:

$$A_{110} = A_{110}(1) = e^{A_{c1}} \quad (3.67)$$

$$B_{110} = B_{110}(1) = \int_0^1 e^{A_{c1}(1-\tau)} B_{c1} d\tau = A_{c1}^{-1}(e^{A_{c1}} - I)B_{c1} \quad (3.68)$$

$$\begin{aligned} A_{210} &= A_{210}(1) = \int_0^1 e^{A_{c2}(1-\tau)} E_{c2} C_{c1} A_{110}(\tau) d\tau \\ &= \int_0^1 e^{A_{c2}(1-\tau)} E_{c2} C_{c1} e^{A_{c1}\tau} d\tau \end{aligned} \quad (3.69)$$

$$\begin{aligned}
B_{210} &= B_{210}(1) = \int_0^1 e^{A_{c2}(1-\tau)} E_{c2} C_{c1} B_{110}(\tau) d\tau \\
&= \int_0^1 e^{A_{c2}(1-\tau)} E_{c2} C_{c1} A_{c1}^{-1} (e^{A_{c1}} - I) B_{c1} d\tau \quad (3.70)
\end{aligned}$$

$$A_{211} = A_{211}(1) = 0 \quad (3.71)$$

$$B_{211} = B_{211}(1) = 0 \quad (3.72)$$

$$A_{220} = A_{220}(1) = e^{A_{c2}} \quad (3.73)$$

$$B_{220} = B_{220}(1) = \int_0^1 e^{A_{c2}(\tau)} B_{c2} d\tau = A_{c2}^{-1} (e^{A_{c2}} - I) B_{c2} \quad (3.74)$$

Matrices $\tilde{E}_{ih}$ are defined by (3-50) as

$$\tilde{E}_{20} = [A_{210} \ B_{210}] \quad (3.75)$$

$$\tilde{E}_{21} = [A_{211} \ B_{211}] = 0 \quad (3.76)$$

Then the coefficient matrices of the discrete-time equations turn out to be

$$\begin{aligned}
\mathcal{S}_1 : \quad A_1 &= A_{110} = \begin{bmatrix} e^{-\frac{1}{3}} & 0 \\ 0 & e^{-\frac{2}{5}} \end{bmatrix}, \\
B_1 &= B_{110} = \begin{bmatrix} -2(e^{-\frac{1}{3}} - 1) \\ -(e^{-\frac{2}{5}} - 1) \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (3.77)
\end{aligned}$$

$$\mathcal{D}_1 : \quad F_1 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \quad (9 \times 9), \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \quad (9 \times 3),$$

$$H_1 = [I \ 0 \ 0] \quad (3 \times 9),$$

$$d_1 = \mu_1 - 1 = 3 \quad (3.78)$$

$$\begin{aligned}
\mathcal{S}_2: A_2 &= \begin{bmatrix} A_{220} & \tilde{E}_{20} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{220} & A_{210} & B_{210} \\ 0 & 0 & 0 \end{bmatrix} \quad (5 \times 5), \\
B_2 &= \begin{bmatrix} B_{220} \\ 0 \end{bmatrix} \quad (3 \times 1), \\
C_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} \tilde{E}_{21} \\ I \end{bmatrix} \quad (5 \times 3) \tag{3.79}
\end{aligned}$$

$$\begin{aligned}
A_{220} &= \begin{bmatrix} e^{-\frac{1}{5}} & 0 \\ 0 & e^{-\frac{2}{9}} \end{bmatrix}, \\
A_{210} &= \begin{bmatrix} \frac{9}{8}(e^{-\frac{1}{5}} - e^{-\frac{1}{3}}) & \frac{1}{2}(e^{-\frac{1}{5}} - e^{-\frac{2}{5}}) \\ \frac{2}{3}(e^{-\frac{2}{9}} - e^{-\frac{1}{3}}) & \frac{5}{12}(e^{-\frac{2}{9}} - e^{-\frac{2}{5}}) \end{bmatrix}, \\
B_{210} &= \begin{bmatrix} \frac{1}{2}e^{-\frac{2}{5}} - \frac{19}{4}e^{-\frac{1}{5}} + \frac{9}{4}e^{-\frac{1}{3}} + 2 \\ -\frac{4}{11}e^{-\frac{2}{9}} + \frac{3}{4}e^{-\frac{1}{3}} + \frac{5}{12}e^{-\frac{2}{5}} + 1 \end{bmatrix}, \\
B_{220} &= \begin{bmatrix} -(e^{-\frac{1}{5}} - 1) \\ -2(e^{-\frac{2}{9}} - 1) \end{bmatrix}, \\
\tilde{E}_{21} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.80}
\end{aligned}$$

In this example the sampling period T was chosen such that the delay time τ_{c1} is a multiple of T . Namely, $\sigma_1 = 0$ in this case. This leads to a relatively simple form of the coefficient matrices in the discrete-time equation. Especially, the coefficient matrices A_{221} and B_{221} vanish. On

the other hand, the discrete-time equation is of the form that a time-delay of one sampling period is included in station S_2 . Accordingly, the length of delay line $\mathcal{D}_1$ is $d_1 = 3$. Thus the discrete-time equation may be redefined such that no time-delay is included in station S_2 , while the length of delay line $\mathcal{D}_1$ is $d_1 = 4$.

Example 2

Another example of discretization is described here, in which the length of the time-delay is different from a multiple of the sampling period. Let the parameters of the continuous-time system be the same as the preceding example, except that the length of the delay line $\mathcal{D}_{c1}$ is

$$\tau_{c1} = 3.6 \quad (3.81)$$

First, consider the case where the sampling period is adjusted as

$$T = 0.9 \quad (3.82)$$

In this case it follows that

$$\mu_1 = 4, \quad \sigma_1 = 0 \quad (3.83)$$

and the discrete-time equation is quite similar to the result of the preceding example. In fact, the only difference in the discretized equation is that $T = 0.9$ instead of $T = 1$. Thus the value of the coefficient matrices can be calculated as

$$A_{110} = A_{110}(0.9) = e^{0.9A_{c1}} \quad (3.84)$$

and so on.

Now the case where

$$T = 1 \quad (3.85)$$

is considered. Note that the delay time τ_{c1} is not a multiple of the sampling period T . In this case μ_1 and σ_1 are given as

$$\mu_1 = 4, \quad \sigma_1 = 0.4 \quad (3.86)$$

Thus the difference from the preceding example is that $\sigma_1 = 0.4$ instead of $\sigma_1 = 0$. The coefficient matrices A_{110} , B_{110} , A_{220} , and B_{220} are identical to those of the preceding example, since the parameter σ_1 does not appear in the definition of these matrices. The coefficient matrices A_{210} , B_{210} , A_{211} , and B_{211} , which depend on σ_1 , are given as follows:

$$A_{210}(t) = \begin{cases} \int_0^t e^{A_{c2}(t-\tau)} E_{c2} C_{c1} A_{110}(0.4 + \tau) d\tau, & 0 < t \leq 0.6 \\ \int_0^{0.6} e^{A_{c2}(0.6-\tau)} E_{c2} C_{c1} A_{110}(0.4 + \tau) d\tau, & 0.6 < t \leq 1 \end{cases} \quad (3.87)$$

$$B_{210}(t) = \begin{cases} \int_0^t e^{A_{c2}(t-\tau)} E_{c2} C_{c1} B_{110}(0.4 + \tau) d\tau, & 0 < t \leq 0.6 \\ \int_0^{0.6} e^{A_{c2}(0.6-\tau)} E_{c2} C_{c1} B_{110}(0.4 + \tau) d\tau, & 0.6 < t \leq 1 \end{cases} \quad (3.88)$$

$$A_{211}(t) = \begin{cases} 0, & 0 < t \leq 0.6 \\ \int_{0.6}^t e^{A_{c2}(t-\tau)} E_{c2} C_{c1} A_{110}(-0.6 + \tau) d\tau, & 0.6 < t \leq 1 \end{cases} \quad (3.89)$$

$$B_{211}(t) = \begin{cases} 0, & 0 < t \leq 0.6 \\ \int_{0.6}^t e^{A_{c2}(t-\tau)} E_{c2} C_{c1} B_{110}(-0.6 + \tau) d\tau, & 0.6 < t \leq 1 \end{cases} \quad (3.90)$$

$$A_{210} = A_{210}(1), \quad B_{210} = B_{210}(1),$$

$$A_{211} = A_{211}(1), \quad B_{211} = B_{211}(1) \quad (3.91)$$

It should be noted that $A_{210}(t)$ and $B_{210}(t)$ are constant for $\sigma_1 < t \leq T$, namely,

$$A_{210}(t) = A_{210}(0.6), \quad B_{210}(t) = B_{210}(0.6), \quad 0.6 < t \leq 1 \quad (3.92)$$

On the other hand, $A_{211}(t)$ and $B_{211}(t)$ are constant for $0 < t \leq \sigma_1$, namely,

$$A_{211}(t) = 0, \quad B_{211} = 0, \quad 0 < t \leq 0.6 \quad (3.93)$$

For the above matrices $A_{ijh}(t)$ and $B_{ijh}(t)$ with the suffix $i = 2$, only the value at $t = 1$ is used in the calculation of the coefficient matrices of the discrete-time equation. This is due to the fact that there are only two stations in this example. If there were more than two stations, namely $p > 2$, the value for $0 < t < T$ will be used in the calculation of the coefficient matrices A_{ijh} and B_{ijh} with the suffix $i = 3$. From this point of view, the value of $A_{110}(t)$ and $B_{110}(t)$ are certainly used in the calculation of A_{ijh} and B_{ijh} with the suffix $i = 2$. Namely, A_{210} and B_{210} depend on $A_{110}(t)$ and $B_{110}(t)$ for $0.4 < t \leq 1$, respectively, while A_{211} and B_{211} depend on $A_{110}(t)$ and $B_{110}(t)$ for $0 < t \leq 0.4$, respectively.

As seen from the above equations, the coefficient matrices A_{221} and B_{221} do not vanish in this example. Thus the size of the discrete-time stations $\mathcal{S}_1$ and $\mathcal{S}_2$ turn out to be

$$\begin{aligned} n_1 &= 2, \quad m_1 = 1, \quad r_1 = 3 \\ n_2 &= 5, \quad m_2 = 1, \quad r_2 = 2 \end{aligned} \quad (3.94)$$

and the length of the discrete-time delay line $\mathcal{D}_1$ is

$$d_1 = 3 \quad (3.95)$$

Hence the overall size of the discrete-time equation is given by

$$n = n_1 + n_2 + d_1 r_1 = 16, \quad m = m_1 + m_2 = 2 \quad (3.96)$$

Chapter 4

Continuous-time optimal control of unilateral time-delay systems

This chapter is concerned with the infinite-time optimal control of unilateral time-delay systems. For a delay-free system, the optimal control law which minimizes the quadratic performance index is obtained by solving the Riccati equation. The same method cannot be directly applied to the unilateral time-delay system, but the optimal solution can be effectively obtained by considering the unilateral structure of the time delays. First, an imaginary delay-free system is introduced, which is obtained by eliminating the time-delays of the original system. The behavior of the imaginary system represents that of the original system except that the time is shifted according to the delay time of $\mathcal{D}_i$. Standard optimal control theory can be applied to obtain the solution for the imaginary system. Then the optimal solution is converted to the solution of the original unilateral time-delay system. Since the initial part of the original system is not expressed by the imaginary system, the optimal solution for the initial part cannot be determined by this method. Thus a finite-time optimal control problem is considered, and the solution for the initial part is derived separately. For simplicity, the case of $p = 2$ in Fig. 4.1 is explained first. Then the results for the case of general p is derived.

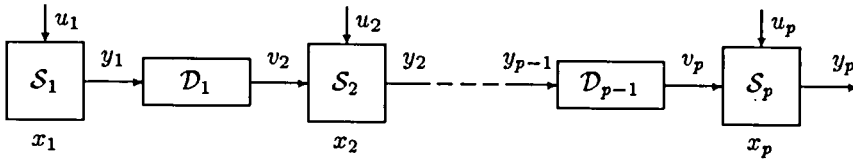


Fig. 4.1 Unilateral time-delay system $\mathcal{S}$

4.1 Two-station case

The case of two-station systems is studied in detail, as a preparation for the multi-station case. Since only one d_i (i.e., d_1) appears in this case, the subscript i of d_i will be omitted: i.e., $d = d_1$ throughout this section.

Consider the unilateral time-delay system $\mathcal{S}$ described by:

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + B_1 u_1(t) , \\ y_1(t) &= C_1 x_1(t)\end{aligned}\tag{4.1}$$

$$\begin{aligned}\dot{x}_2(t) &= A_2 x_2(t) + B_2 u_2(t) + E v(t) , \\ y_2(t) &= C_2 x_2(t)\end{aligned}\tag{4.2}$$

$$v(t) = y_1(t - d) .\tag{4.3}$$

Here, d is a scalar representing the delay time; x_i , y_i , u_i ($i = 1, 2$) are the state, output, external input vectors of dimensions n_i , r_i , m_i ($i = 1, 2$), respectively; v is the connecting input vector of dimension r_1 ; and A_i , B_i , C_i ($i = 1, 2$), E are the coefficient matrices of appropriate size. This is the case where $p = 2$, and the subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected by the time-delay $\mathcal{D}$ of length d . In this section, the initial condition of the time-delay $\mathcal{D}$ is neglected for simplicity. The suffixes $i = 1, 2$ correspond to subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Let the performance index be given as:

$$J = \int_0^\infty \sum_{i=1}^2 \{x'_i(t) Q_i x_i(t) + u'_i(t) R_i u_i(t)\} dt .\tag{4.4}$$

Our problem is to find the optimal input $u^*(t)$ which minimizes J subject to (4.1)–(4.3).

Observe that due to the time-delay between subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, the state and input of $\mathcal{S}_1$ has no influence on the behavior of $\mathcal{S}_2$ in the interval $0 < t \leq d$. Thus, according to the principle of optimality, the minimization problem of J can be carried out in two steps. To see this, put

$$\begin{aligned}J_0 &= \int_0^d \{x'_2(t) Q_2 x_2(t) + u'_2(t) R_2 u_2(t)\} dt , \\ \bar{J} &= \int_d^\infty \{x'_1(t) Q_1 x_1(t) + u'_1(t) R_1 u_1(t)\} dt \\ &\quad + \int_0^\infty \{x'_2(t) Q_2 x_2(t) + u'_2(t) R_2 u_2(t)\} dt .\end{aligned}\tag{4.5}$$

Then it follows that

$$J = J_0 + \bar{J} . \quad (4.6)$$

Let U , $\bar{U}$, and U_0 denote the input included in J , $\bar{J}$, and J_0 , respectively. Since the input $\bar{U}$, i.e., $u_1(t)$ for $t > 0$ and $u_2(t)$ for $0 < t \leq d$, has no effect upon the value of J_0 , the minimum value J^* of the performance index J can be expressed as

$$\begin{aligned} J^* &= \min_{\bar{U}} J \\ &= \min_{U_0} \left(J_0 + \min_{\bar{U}} \bar{J} \right) . \end{aligned} \quad (4.7)$$

In the following, the minimization of $\bar{J}$ over $\bar{U}$ is treated as the first step, followed by the minimization of J_0 over U_0 .

Now, consider a new system $\bar{\mathcal{S}}$ described by:

$$\begin{aligned} \dot{\bar{x}}_1(t) &= A_1 \bar{x}_1(t) + B_1 \bar{u}_1(t) , \\ \dot{\bar{x}}_2(t) &= EC_1 \bar{x}_1(t) + A_2 \bar{x}_2(t) + B_2 \bar{u}_2(t) . \end{aligned} \quad (4.8)$$

Here, $\bar{x}_i$ and $\bar{u}_i$ ($i = 1, 2$) are the state and input vectors of dimensions n_i and r_i ($i = 1, 2$), respectively. System $\bar{\mathcal{S}}$ has the structure that subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ of system $\mathcal{S}$ are directly connected without time-delay. System $\bar{\mathcal{S}}$ will be referred to as the Imaginary System in the following. The behavior of the imaginary system $\bar{\mathcal{S}}$ is closely related to that of the original time-delay system $\mathcal{S}$. In fact, let the initial state of $\bar{\mathcal{S}}$ be

$$\bar{x}_1(0) = x_1(0) , \quad \bar{x}_2(0) = x_2(d) \quad (4.9)$$

and the input be

$$\bar{u}_1(t) = u_1(t) , \quad \bar{u}_2(t) = u_2(t + d) , \quad t \geq 0 \quad (4.10)$$

then it is evident that the following relation holds:

$$\bar{x}_1(t) = x_1(t) , \quad \bar{x}_2(t) = x_2(t + d) , \quad t \geq 0 . \quad (4.11)$$

It should be noted that the state $x_2(t)$ for $0 \leq t < d$ is not included in the above relation (4.11), and that it is not affected by the input described in (4.10). Using (4.10) and (4.11), $\bar{J}$ defined in (4.5) can be expressed as

$$\bar{J} = \int_0^\infty \sum_{i=1}^2 \{ \bar{x}_i'(t) Q_i \bar{x}_i(t) + \bar{u}_i'(t) R_i \bar{u}_i(t) \} dt . \quad (4.12)$$

This is to be considered as the performance index for the imaginary system $\bar{\mathcal{S}}$. Since $\bar{\mathcal{S}}$ is a delay-free system, the optimal control which minimizes $\bar{J}$ can be obtained by applying the standard optimal regulator theory. The solution for the imaginary system $\bar{\mathcal{S}}$ takes the following form:

$$\begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \end{bmatrix} = \bar{F} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} \quad (4.13)$$

Here, $\bar{F}$ is the optimal feedback gain, which can be calculated by solving the Riccati equation associated with the optimal regulator problem for the imaginary system $\bar{\mathcal{S}}$.

In order to apply the control law to the original time-delay system $\mathcal{S}$, it must be expressed in terms of the variables in $\mathcal{S}$. Substituting (4.10) and (4.11) into (4.13), it follows that

$$\begin{bmatrix} u_1(t) \\ u_2(t+d) \end{bmatrix} = \bar{F} \begin{bmatrix} x_1(t) \\ x_2(t+d) \end{bmatrix} \quad (4.14)$$

Observe that the state $x_2(t+d)$ is used in the calculation of the input $u_1(t)$. This means that the future value $x_2(t+d)$ must be known at time t in order to implement the control law (4.14). As seen from the state equation (4.2), the value of $x_2(t+d)$ is determined by $u_2(\tau)$, $v(\tau)$ ($t < \tau \leq t+d$), and $x_2(t)$. The input $u_2(t+d)$ is calculated by (4.14) at time t , and the value of $v(t+d)$ is given by (4.3) as

$$v(t+d) = y_1(t) \quad (4.15)$$

Thus, $x_2(t+d)$ can be calculated at time t as follows:

$$\begin{aligned} x_2(t+d) &= e^{A_2 d} x_2(t) \\ &+ \int_{t-d}^t e^{A_2(t-\tau)} \{ B_2 u_2(\tau+d) + v(\tau+d) \} d\tau . \end{aligned} \quad (4.16)$$

The control law (4.14) gives a part of the solution for the original system $\mathcal{S}$. This is referred to as the steady state solution in the following. The optimal control $u_2(t)$ for $0 \leq t < d$ remains to be calculated, and

the solution for this part will be referred to as the solution for the initial part. This corresponds to the minimization of J_0 over U_0 . The value of J_0 depends on the initial state $x_2(0)$ of the subsystem S_1 and the value of $v(t)$ for $0 < t \leq d$ which corresponds to the initial state of the time-delay $\mathcal{D}$. A partial differential equation is involved in the minimization problem of J_0 , and it is not practical to implement the controller corresponding to the solution of the initial part.

In the following sections, the problem is formulated for the case of general p , and the optimal solution for the unilateral time-delay system is derived via the solution for the imaginary delay-free system.

4.2 Formulation of the problem

Consider the unilateral time-delay system $\mathcal{S}$ shown in Fig. 4.1. Each subsystem S_i ($i = 1, \dots, p$) is described by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + E_i v_i(t) \quad (4.17)$$

$$y_i(t) = C_i x_i(t) \quad (4.18)$$

and delay $\mathcal{D}_i$ ($i = 1, \dots, p-1$) by

$$v_{i+1}(t) = y_i(t - d_i) \quad (4.19)$$

where d_i is a scalar representing delay time; x_i , y_i , u_i , and v_i are state, output, external input and connecting input vectors and have the dimensions n_i , r_i , m_i , and r_{i-1} , respectively; A_i , B_i , E_i , and C_i are matrices of size $n_i \times n_i$, $n_i \times m_i$, $n_i \times r_{i-1}$, and $r_i \times n_i$, respectively; and v_1 does not exist.

Let the performance index J for system $\mathcal{S}$ be

$$J = \int_0^\infty \sum_{i=1}^p \{x'_i(t) Q_i x_i(t) + u'_i(t) R_i u_i(t)\} dt \quad (4.20)$$

where Q_i are positive semidefinite matrices of size $n_i \times n_i$, and R_i are $m_i \times m_i$ positive definite matrices. Our problem is to find the optimal control $u^*(t)$ which minimizes the performance index J subject to (4.17)–(4.19).

4.3 Introduction of imaginary delay-free system

Consider a new system $\bar{\mathcal{S}}$ shown in Fig. 4.2. Note that in the system $\bar{\mathcal{S}}$ subsystems $\mathcal{S}_i$ are directly connected in series, whereas in the system $\mathcal{S}$ they are connected by pure delay. We refer to $\bar{\mathcal{S}}$ as the Imaginary Delay-Free System for the original time-delay system $\mathcal{S}$. The state transition of the imaginary system $\bar{\mathcal{S}}$ is expressed as

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \quad (4.21)$$

$$\dot{\bar{x}}(t) = [\bar{x}'_p(t), \dots, \bar{x}'_1(t)]' \quad (4.22)$$

$$\dot{\bar{u}}(t) = [\bar{u}'_p(t), \dots, \bar{u}'_1(t)]' \quad (4.23)$$

$$\bar{A} = \begin{bmatrix} A_p & E_p C_{p-1} & & 0 \\ & \ddots & \ddots & \\ & & A_2 & E_2 C_1 \\ 0 & & & A_1 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} B_p & & & 0 \\ & \ddots & & \\ & & B_2 & \\ 0 & & & B_1 \end{bmatrix} \quad (4.24)$$

where $\bar{x}_i$ and $\bar{u}_i$ ($i = 1, \dots, p$) are state and input vectors and have the same dimensions with x_i and u_i , respectively. $\bar{x}$, $\bar{u}$ are vectors of dimensions $\bar{n}$ and $\bar{m}$; and $\bar{A}$ and $\bar{B}$ are matrices of size $\bar{n} \times \bar{n}$ and $\bar{m} \times \bar{m}$, respectively. Here, n and m are given by

$$\bar{n} = n_1 + \dots + n_p, \quad \bar{m} = m_1 + \dots + m_p \quad (4.25)$$

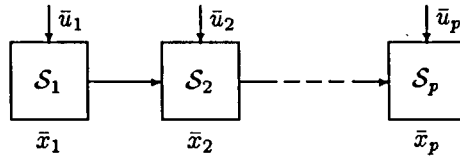


Fig. 4.2 Imaginary delay-free system $\bar{\mathcal{S}}$

Define the output $\bar{y}$ of the imaginary system as follows:

$$\bar{y}(t) = \bar{C}\bar{x}(t) \quad (4.26)$$

$$\bar{C} = \tilde{C}e^{-\bar{A}\bar{d}} \quad (4.27)$$

$$\bar{d} = d_1 + \dots + d_{p-1} \quad (4.28)$$

$$\tilde{C} = [C_p \ 0 \dots 0] \quad (4.29)$$

where $\bar{y}$ is a vector of dimension r_p , $\bar{C}$ and $\tilde{C}$ are matrices of size $r_p \times \bar{n}$. Let the performance index $\bar{J}$ for the imaginary system $\bar{\mathcal{S}}$ be

$$\bar{J} = \int_0^\infty \{ \bar{x}'(t)\bar{Q}\bar{x}(t) + \bar{u}'(t)\bar{R}\bar{u}(t) \} dt \quad (4.30)$$

$$\bar{Q} = \text{block diag} (Q_p, \dots, Q_1) \quad (4.31)$$

$$\bar{R} = \text{block diag} (R_p, \dots, R_1) \quad (4.32)$$

where $\bar{Q}$ and $\bar{R}$ are block diagonal matrices of size $\bar{n} \times \bar{n}$ and $\bar{m} \times \bar{m}$, respectively.

Now, consider the relationship between the original unilateral time-delay system $\mathcal{S}$ and the imaginary system $\bar{\mathcal{S}}$. Define $\bar{d}_i$ concerning the length of time-delay as

$$\bar{d}_i = \begin{cases} 0 & (i = 1) \\ d_1 + \dots + d_{i-1} & (i = 2, \dots, p) \end{cases} \quad (4.33)$$

and set the initial value of the state $\bar{x}(0)$ and input $\bar{u}(t)$ for $t \geq 0$ of the imaginary system $\bar{\mathcal{S}}$ as

$$\bar{x}_i(0) = x_i(\bar{d}_i) \quad (4.34)$$

$$\bar{u}_i(t) = u_i(t + \bar{d}_i), \quad t \geq 0. \quad (4.35)$$

$$(i = 1, \dots, p)$$

Then, following relations hold for $t \geq 0$:

$$\bar{x}_i(t) = x_i(t + \bar{d}_i) \quad (i = 1, \dots, p). \quad (4.36)$$

The performance index J can be expressed using $\bar{J}$ as

$$J = \bar{J} + J_0 \quad (4.37)$$

$$J_0 = \sum_{i=2}^p \int_0^{\bar{d}_i} \{x'_i(t)Q_i x_i(t) + u'_i(t)R_i u_i(t)\} dt . \quad (4.38)$$

Note that the input $\bar{u}_i(t)$, $t \geq 0$ of the imaginary system $\bar{S}$ has no effect upon J_0 , from which and the principle of optimality it follows that the optimal input $\bar{u}_i(t)$ which minimizes $\bar{J}$ is identical to the optimal input $u_i(t + \bar{d}_i)$ which minimizes J .

Next, consider the relations of the output. From equation (4.17), $\bar{x}(t)$ can be expressed as

$$\bar{x}(t) = e^{\bar{A}\bar{d}}\bar{x}(t - \bar{d}) + \int_{t-\bar{d}}^t e^{\bar{A}(t-\tau)}\bar{B}\bar{u}(\tau)d\tau \quad (4.39)$$

and the following relation holds:

$$\tilde{C}e^{-\bar{A}\bar{d}}\bar{x}(t) = \tilde{C}\bar{x}(t - \bar{d}) + \int_{t-\bar{d}}^t \tilde{C}e^{\bar{A}(t-\bar{d}-\tau)}\bar{B}\bar{u}(\tau)d\tau . \quad (4.40)$$

The first term of the right hand side of (4.40) can be transformed as

$$\tilde{C}\bar{x}(t - \bar{d}) = C_p \bar{x}_p(t - \bar{d}) = C_p x_p(t) = y_p(t) . \quad (4.41)$$

Thus, the outputs of unilateral time-delay system S and those of imaginary system $\bar{S}$ are related as follows:

$$\bar{y}(t) = y_p(t) + \int_{t-\bar{d}}^t \tilde{C}e^{\bar{A}(t-\bar{d}-\tau)}\bar{B}\bar{u}(\tau)d\tau . \quad (4.42)$$

The right hand side of (4.42) contains an integral element of finite time interval. Here a notation for such a finite interval integral is introduced. Define $W[A, d, z]$ for general A , d , and $z(t)$ by

$$W[A, d, z] = \int_{t-d}^t e^{A(t-\tau)}z(\tau)d\tau . \quad (4.43)$$

Then (4.42) is expressed as

$$\bar{y}(t) = y_p(t) + \tilde{C}e^{-\bar{A}\bar{d}}W[\bar{A}, \bar{d}, \bar{B}\bar{u}] . \quad (4.44)$$

Hence, the output $\bar{y}(t)$ of imaginary system $\bar{S}$ can be calculated from the output $y(t)$ of time-delay system S and the input $\bar{u}(\tau)$, $t - \bar{d} \leq \tau \leq t$ of the imaginary system $\bar{S}$.

Output of the imaginary system is not needed as far as the state feedback control is concerned, but it will be taken into consideration when the estimation of the state variables is required.

Controllability of the pair $(\bar{A}, \bar{B})$ and observability of the pair $(\bar{C}, \bar{A})$ are assumed in the following. In other words, the original time-delay system $\mathcal{S}$ is assumed to have the property such that the imaginary system $\bar{\mathcal{S}}$ will be completely controllable and observable.

Since the imaginary system $\bar{\mathcal{S}}$ is a delay-free system, the optimal control $\bar{u}(t)$ that minimizes the performance index $\bar{J}$ can be obtained by use of the well known optimal control theory. Namely, solve the discrete-time matrix algebraic Riccati equation

$$\bar{P}\bar{A} + \bar{A}'\bar{P} - \bar{P}\bar{B}\bar{R}^{-1}\bar{B}'\bar{P} + \bar{Q} = 0 \quad (4.45)$$

then using the positive definite symmetric solution $\bar{P}$, the optimal control law is given by

$$\bar{u}(t) = -\bar{F}\bar{x}(t) \quad (4.46)$$

$$\bar{F} = \bar{R}^{-1}\bar{B}'\bar{P} \quad (4.47)$$

4.4 Solution for time-delay system

Since (4.46) is a control law for the imaginary system $\bar{\mathcal{S}}$, following steps should be followed for implementing the control law to the time-delay system $\mathcal{S}$:

- a) estimate $\bar{x}_i(t)$ from $x_i(t)$
- b) calculate $\bar{u}(t)$ by (4.46)
- c) transform $\bar{u}_i(t)$ into $u_i(t)$

Step a) is achieved by use of a prediction mechanism in which a finite interval integral is employed. Step c) is realized by merely using a time-delay element. Now this procedure is considered in detail.

By the standard formula of linear systems, the state transition of the system $\mathcal{S}$ given by (4.17)–(4.19) is described as

$$x_i(t + \bar{d}_i) = e^{A_i \bar{d}_i} x_i(t) + \int_{t-\bar{d}_i}^t e^{A_i(t-\tau)} \{ B_i u_i(\tau + \bar{d}_i) + E_i C_{i-1} x_{i-1}(\tau + \bar{d}_{i-1}) \} d\tau. \quad (4.48)$$

According to the relationship between system $\mathcal{S}$ and system $\bar{\mathcal{S}}$, it can be written using the variables of the imaginary system as

$$\bar{x}_i(t) = e^{A_i \bar{d}_i} x_i(t) + \int_{t-\bar{d}_i}^t e^{A_i(t-\tau)} \{ B_i \bar{u}_i(\tau) + E_i C_{i-1} \bar{x}_{i-1}(\tau) \} d\tau. \quad (4.49)$$

Using the finite interval integral element defined before, it can be expressed as

$$\begin{aligned} \bar{x}_i(t) = & e^{A_i \bar{d}_i} x_i(t) + W [A_i, \bar{d}_i, \bar{B}_i \bar{u}_i] \\ & + W [A_i, \bar{d}_i, E_i C_{i-1} \bar{x}_{i-1}]. \end{aligned} \quad (4.50)$$

Note that the right hand side of (4.50) consists of variables $x_i(t)$ and $\bar{u}_i(\tau)$, $\bar{x}_{i-1}(\tau)$, $t - \bar{d}_i \leq \tau \leq t$. Since the latter two variables are available in the controller, $\bar{x}_i(t)$ is obtained by the prediction mechanism described by (4.50). By applying this prediction mechanism to all $\bar{x}_i(t)$ ($i = 2, \dots, p$) the state variable $\bar{x}(t)$ can be estimated. For the input, observe that (4.35) implies

$$u_i(t) = \bar{u}_i(t - d_i) \quad (i = 1, \dots, p) \quad (4.51)$$

Then it is evident that connecting a time-delay element of length $\bar{d}_i$ to $\bar{u}_i(t)$ will yield $u_i(t)$.

Thus the optimal control law (4.46) for the imaginary system $\bar{\mathcal{S}}$ can be applied to the time-delay system $\mathcal{S}$, and the controller using prediction mechanism and time-delay element can be implemented, provided the state variables $x_i(t)$ of the time-delay system $\mathcal{S}$ are directly measurable. The structure of the controller for the time-delay system $\mathcal{S}$ is shown in Fig. 4.3. When the state variables $x_i(t)$ are not measurable, a controller using an observer can be constructed, which will be treated later in detail.

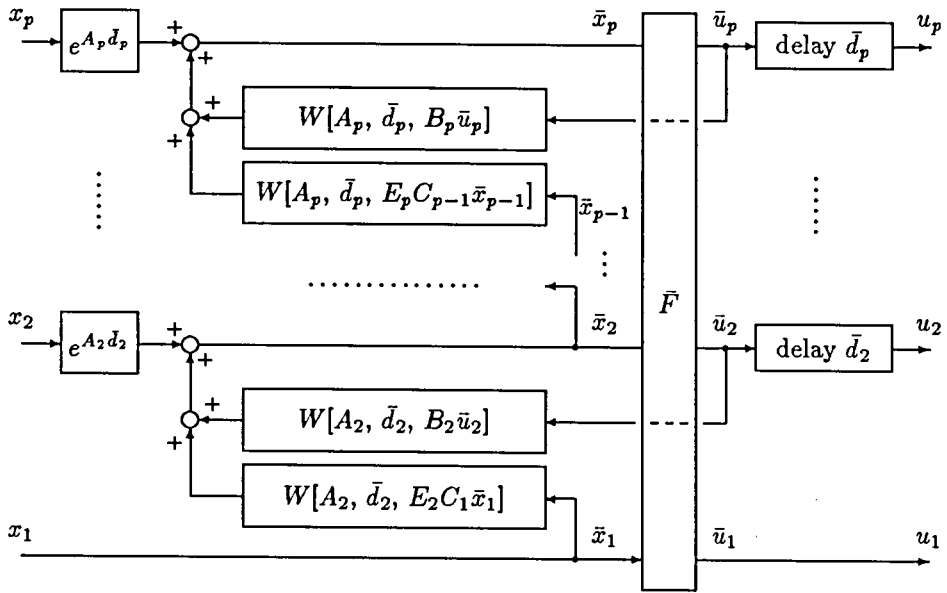


Fig. 4.3 Structure of the controller for time-delay system $\mathcal{S}$

4.5 Solution for the initial part

The controller shown in Fig. 4.3 is designed to minimize the performance index $\bar{J}$ of the imaginary system $\bar{\mathcal{S}}$. The control law (4.46) gives the optimal value for the input $u_i(t)$, $\bar{d}_i \leq t$. But the input for the initial part $u_i(t)$, $0 \leq t < \bar{d}_i$ is undefined. The input for the initial part cannot be determined by considering the imaginary system $\bar{\mathcal{S}}$, since it has effect only upon J_0 part of the performance index J and not upon $\bar{J}$, the performance index for the imaginary system $\bar{\mathcal{S}}$. The optimal input for the initial part is determined by the minimization of J_0 . In order to solve the minimization problem of J_0 , (4.38) is rewritten as

$$J_0 = \sum_{j=2}^p J_j \quad (4.52)$$

$$J_j = \sum_{i=j}^p \int_{\bar{d}_i - \bar{d}_j}^{\bar{d}_i - \bar{d}_{j-1}} \{x'_i(t)Q_i x_i(t) + u'_i(t)R_i u_i(t)\} dt. \quad (4.53)$$

In (4.38) J_0 was expressed as the summation of the performance index for each subsystem $\mathcal{S}_i$. Here, the performance index for each subsystem $\mathcal{S}_i$ is divided to integration intervals of length $\bar{d}_i - \bar{d}_j$ and J_0 is rearranged so that in each J_j parts of the performance index of the same length of integration interval is included. Note that in each J_j the integration is performed at different time for different subsystem $\mathcal{S}_i$, i.e., $\bar{d}_i - \bar{d}_j \leq t \leq \bar{d}_i - \bar{d}_{j-1}$ for subsystem $\mathcal{S}_i$. Only the length of integration interval is the same in each J_j . Refer to Fig. 4.4 for the time relationship among J_j and subsystems $\mathcal{S}_i$.

Now let the input included in J_j , i.e., $u_i(t)$, $\bar{d}_i - \bar{d}_j \leq t \leq \bar{d}_i - \bar{d}_{j-1}$ be denoted by U_j , and similarly, the input included in J , $\bar{J}$, J_0 by U , $\bar{U}$, U_0 , respectively. Consider a fixed value k for the subscript j of J_j . Then it is observed that the value of J_j for $j \geq k$ is not affected by the input U_j of J_j for $j < k$. This results from of the unilateral time-delay structure of the system $\mathcal{S}$. In the same way the initial part J_0 of the performance index is not affected by the input $\bar{U}$ of the imaginary system $\bar{\mathcal{S}}$, which has been already mentioned.

According to the principle of optimality, the minimization of the performance index J can be accomplished by minimizing $\bar{J}$ and J_j in a sequence. Thus the minimal value of the performance index J for the time-delay system $\mathcal{S}$ can be expressed as

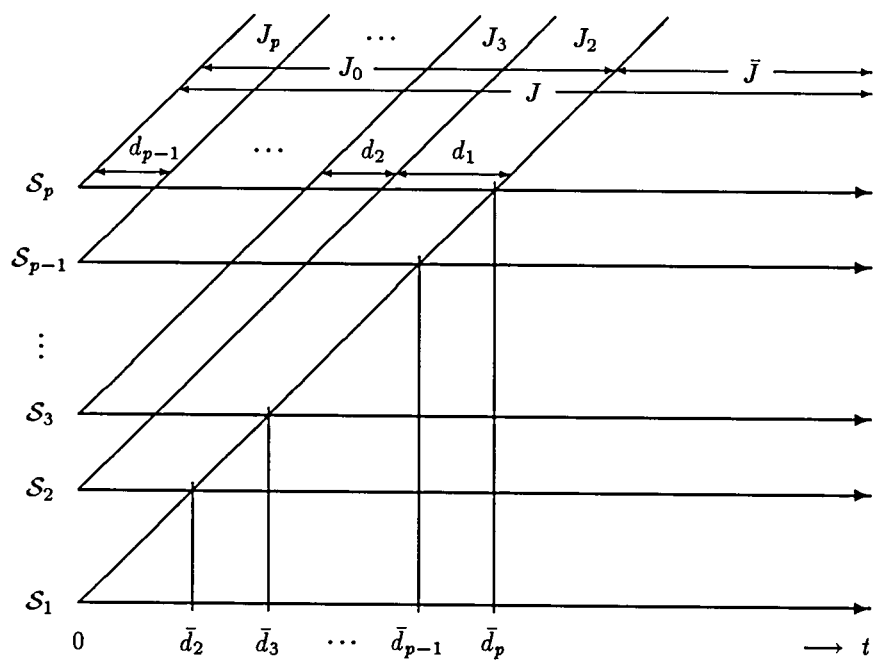


Fig. 4.4 Relationship among S_i and J_j

$$\begin{aligned}
\min_U J &= \min_{U_0} \left(J_0 + \min_{\bar{U}} \bar{J} \right) \\
&= \min_{U_p} \left(J_p + \cdots + \min_{U_2} \left(J_2 + \min_{\bar{U}} J \right) \cdots \right)
\end{aligned} \tag{4.54}$$

To find the input U which minimizes the performance index J , the input $\bar{U}$ which minimizes the performance index $\bar{J}$ is calculated first. This is accomplished by following the standard procedure of solving the infinite time optimal control problem. Next, J_2 is added to the minimal value of $\bar{J}$, which is to be minimized by the optimal input U_2 . This step can be formulated as the finite time optimal control problem. Then, $J_3, \dots, J_p$ is added to the performance index, which are to be minimized by the optimal input $U_3, \dots, U_p$, respectively.

Now, the finite time optimal control problem to find the optimal input U_j is considered in detail. Define matrices $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{Q}_j$, and $\tilde{R}_j$ as follows, which are related to the matrices $\bar{A}$, $\bar{B}$, $\bar{Q}$, and $\bar{R}$ of the imaginary system $\bar{S}$, respectively.

$$\begin{aligned}
\tilde{A}_j &= \begin{bmatrix} A_p & E_p C_{p-1} & & 0 & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & E_{j+1} C_j & \vdots \\ 0 & & & A_j & 0 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \\
\tilde{B}_j &= \begin{bmatrix} B_p & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & B_j \\ 0 & \dots & \dots & 0 \end{bmatrix}
\end{aligned} \tag{4.55}$$

$$\tilde{Q}_j = \text{block diag} (Q_p, \dots, Q_j, 0, \dots, 0) \tag{4.56}$$

$$\tilde{R}_j = \text{block diag} (R_p, \dots, R_j) \tag{4.57}$$

In $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{Q}_j$, and $\tilde{R}_j$, entries of the rows and columns corresponding to state $\bar{x}_1, \dots, \bar{x}_{j-1}$ are zero, while the columns corresponding to input $\bar{u}_1, \dots, \bar{u}_{j-1}$ are eliminated. Other entries of $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{Q}_j$, and $\tilde{R}_j$ are identical to those of $\bar{A}$, $\bar{B}$, $\bar{Q}$, and $\bar{R}$, respectively. By this definition the

size of the matrices $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{Q}_j$, and $\tilde{R}_j$ are $\bar{n} \times \bar{n}$, $\bar{n} \times \bar{m}_j$, $\bar{n} \times \bar{n}$, and $\bar{m}_j \times \bar{m}_j$, respectively, where $\bar{m}_j$ is defined by

$$\bar{m}_j = m_j + \dots + m_p \quad (j = 2, \dots, p). \quad (4.58)$$

Let the variables included in J_j be denoted by

$$\begin{aligned} \tilde{x}_j(t) = & \left[x'_p(\bar{d}_p - \bar{d}_j + t), \dots, x'_j(\bar{d}_j - \bar{d}_j + t), \right. \\ & \left. w_{j-1}(t), \dots, w_1(t) \right]' \end{aligned} \quad (4.59)$$

$$\tilde{u}_j(t) = \left[u'_p(\bar{d}_p - \bar{d}_j + t), \dots, u'_j(\bar{d}_j - \bar{d}_j + t) \right]' \quad (4.60)$$

where w_i is a vector of dimension n_i defined as

$$w_i(t) = x_i(0) = \text{const.}, \quad 0 \leq t \leq \bar{d}_i. \quad (4.61)$$

Define also matrices $\tilde{E}$ and $\tilde{E}_j$ ($j = 2, \dots, p$) as

$$\begin{aligned} \tilde{E} &= \text{block diag} (E_p, \dots, E_1) \\ &= \begin{bmatrix} \tilde{E}_p & \dots & \tilde{E}_1 \end{bmatrix}. \end{aligned} \quad (4.62)$$

From the above definitions the following relations hold.

$$\tilde{x}_j(t) = \tilde{A}_j \tilde{x}_j(t) + \tilde{B}_j \tilde{u}_j(t) + \tilde{E}_j v_j(t) \quad (4.63)$$

$$J_j = \int_0^{\bar{d}_{j-1}} \left\{ \tilde{x}'_j(t) \tilde{Q}_j \tilde{x}_j(t) + \tilde{u}'_j(t) \tilde{R}_j \tilde{u}_j(t) \right\} dt. \quad (4.64)$$

Note that w_i is defined as a dummy variable to adjust the dimension of $\tilde{x}_j$, and the corresponding rows and columns of $\tilde{A}_j$ are all zero.

In order to find the input U_j which minimizes J_j the variable v_j in (4.63) must be taken into consideration. v_j is the input from the preceding subsystem $\mathcal{S}_{j-1}$ through the time-delay $\mathcal{D}_{j-1}$ and $v_j(t)$, $0 < t < d_j$ affects J_j . The state of the time-delay elements $\mathcal{D}_{j-1}$ is described as $\xi_j(t, s)$, $0 \leq s \leq 1$. The signal enters the time-delay element $\mathcal{D}_{j-1}$ at location $s = 0$, and exits at $s = 1$. The length of the time-delay element is normalized to unity, and the propagation time of the signal from $s = 0$ to $s = 1$ is d_{j-1} . The state $\xi_j(t, s)$ of the time-delay element $\mathcal{D}_{j-1}$ satisfies the partial differential equation

$$\frac{\partial}{\partial t} \xi_j(t, s) + \frac{1}{d_{j-1}} \frac{\partial}{\partial s} \xi_j(t, s) = 0, \quad 0 < s < 1 \quad (4.65)$$

and (4.63) can be rewritten using ξ as

$$\frac{d}{dt} \tilde{x}_j(t) = \tilde{A}_j \tilde{x}_j(t) + \tilde{B}_j \tilde{u}_j(t) + \tilde{E}_j \xi_j(t, 1). \quad (4.66)$$

Since the value of $\xi_j(t, 0)$ for $t > 0$ has no influence upon J_j , it can be defined arbitrary as far as the minimization of J_j is concerned. For simplicity it is defined as

$$\xi_j(t, 0) = 0, \quad 0 < t < d_{j-1}. \quad (4.67)$$

The initial state of the time-delay element $\xi_j(0, s)$, $0 \leq s \leq 1$ is related to $v_j(t)$ in (4.63) as

$$v_j(t) = \xi_j(0, 1 - \frac{t}{d_{j-1}}), \quad 0 < t \leq d_{j-1} \quad (4.68)$$

So far a finite time optimal regulator problem has been set, i.e., to find U_j which minimizes

$$J_j + \min_{U_{j-1}, \dots, U_2, \bar{U}} (J_{j-1} + \dots + J_2 + \bar{J}) \quad (4.69)$$

subject to (4.65) and (4.66).

It is known that U_j , i.e.,

$$\tilde{u}_j(t), \quad 0 \leq t < d_{j-1} \quad (4.70)$$

is given by

$$\tilde{u}_j(t) = -\tilde{R}_j^{-1} \tilde{B}_j' \tilde{P}_j(t) \tilde{x}_j(t) + \int_0^1 \tilde{B}_j' \hat{P}_j(t, s) \xi_j(t, s) ds \quad (4.71)$$

where $\tilde{P}_j(t)$ and $\tilde{P}_j(t, s)$ satisfy the following equations:

$$\begin{aligned} \frac{d}{dt} \tilde{P}_j(t) + \tilde{P}_j(t) \tilde{A}_j + \tilde{A}_j' \tilde{P}_j(t) + \tilde{Q}_j \\ - \tilde{P}_j(t) \tilde{B}_j \tilde{R}_j^{-1} \tilde{B}_j' \tilde{P}_j(t) = 0 \end{aligned} \quad (4.72)$$

$$\begin{aligned} \frac{\partial}{\partial t} \hat{P}_j(t, s) + \frac{1}{d_{j-1}} \frac{\partial}{\partial s} \hat{P}_j(t, s) + \tilde{A}_j \hat{P}_j(t, s) \\ - \tilde{P}_j(t) \tilde{B}_j \tilde{R}_j^{-1} \tilde{B}_j' \hat{P}_j(t, s) = 0 \end{aligned} \quad (4.73)$$

$$\tilde{P}_j(t)\tilde{E}_j - \frac{1}{d_{j-1}}\hat{P}_j(t, 1) = 0 \quad (4.74)$$

$$\tilde{P}_j(d_{j-1}) = \begin{cases} \bar{P} & (j = 2) \\ \tilde{P}_{j-1}(0) & (j = 3, \dots, p) \end{cases} \quad (4.75)$$

$$2\tilde{x}'_j(d_{j-1}) \int_0^1 \hat{P}_j(d_{j-1}, s)\xi_j(d_{j-1}, s)ds = 0 \quad (4.76)$$

The optimal control law has the form that the optimal input $\tilde{u}_j(t)$ is given as a linear combination of the states $\tilde{x}_j(t)$ and $\xi_j(t, s)$. Note that the feedback gain concerning the state of the delay-free part $\tilde{x}_j$ is determined independent of the time-delay part of the system. In fact the equation (4.72) which determines $\tilde{P}_j(t)$ is a Riccati type ordinary differential matrix equation, and it is of the same form as in the finite time optimal control problem of a delay-free system.

4.6 Modification of the controller

Recall the structure of the controller shown in Fig. 4.3. It is of the form that a time-delay is placed at each manipulating variable $u_i(t)$ in order to adjust the time between the delay-free imaginary system and the time-delay system. Among the manipulating variables, no time-delay exist at u_1 , since

$$u_1(t) = \bar{u}_1(t) \quad (4.77)$$

On the other hand, time-delay of maximum length $\bar{d}$ is placed at u_p , corresponding to the relation

$$u_p(t) = \bar{u}_p(t - \bar{d}) \quad (4.78)$$

This is a natural consequence of implementing the control law given in (4.30), and the initial value of the time-delay is to be determined by the solution for the initial part described in the previous section. However, from practical point of view where the feedback controller is implemented based on the steady state solution of the optimal control problem, the existence of time-delay in the controller is not desirable, since it leads to delayed response against disturbances. For instance, if a perturbation

is detected at x_p , it will take at least time $\bar{d}$ before a correcting action appears at u_p . This delay can be avoided by modifying the structure of the controller as explained in the following.

According to the optimal control law for the imaginary system $\bar{\mathcal{S}}$, $\bar{u}_p$ is given by

$$\bar{u}_p(t) = F_p \begin{bmatrix} \bar{x}_p(t) \\ \vdots \\ \bar{x}_1(t) \end{bmatrix} \quad (4.79)$$

where F_p is the first row of F . Substituting the variables of the imaginary system $\bar{\mathcal{S}}$ by those of the time-delay system $\mathcal{S}$, it can be rewritten as

$$u_p(t + \bar{d}) = F_p \begin{bmatrix} x_p(t + \bar{d}_p) \\ \vdots \\ x_1(t + \bar{d}_1) \end{bmatrix} = F_p \begin{bmatrix} x_p(t + \bar{d}) \\ \vdots \\ x_1(t) \end{bmatrix} \quad (4.80)$$

Now, the time is shifted by $\bar{d}$, from which follows

$$u_p(t) = F_p \begin{bmatrix} x_p(t + \bar{d}_p - \bar{d}) \\ \vdots \\ x_1(t + \bar{d}_1 - \bar{d}) \end{bmatrix} = F_p \begin{bmatrix} x_p(t) \\ \vdots \\ x_1(t - \bar{d}) \end{bmatrix} \quad (4.81)$$

Thus the manipulating variable $u_p(t)$ of the time-delay system $\mathcal{S}$ can be calculated from the past state variables

$$\begin{aligned} x_{p-1}(t + \bar{d}_{p-1} - \bar{d}) &= x_{p-1}(t - d_{p-1}) , \\ &\vdots \\ x_2(t + \bar{d}_2 - \bar{d}) &= x_2(t - d_{p-1} - d_{p-2} - \cdots - d_2) , \\ x_1(t + \bar{d}_1 - \bar{d}) &= x_1(t - d_{p-1} - d_{p-2} - \cdots - d_1) \\ &= x_1(t - \bar{d}) \end{aligned} \quad (4.82)$$

together with the present state variable $x_p(t)$. The past value of the state variables is obtained by using a time-delay, which acts as a memory, to each state variable. The characteristics of the closed loop system is not altered by shifting the time. Hence a modified controller can be implemented according to the relation (4.64). It is easily understood

that now the perturbation at x_p is directly reflected to u_p . For the manipulating variables u_i , $i = 2, \dots, p-1$, similar modification is possible by considering the time shift by $\bar{d}_i$. In this case, the control law is expressed as

$$\bar{u}_i(t) = F_i \begin{bmatrix} x_p(t + \bar{d}_p - \bar{d}_i) \\ \vdots \\ x_i(t + \bar{d}_i - \bar{d}_i) \\ \vdots \\ x_1(t + \bar{d}_1 - \bar{d}_i) \end{bmatrix} \quad (4.83)$$

where F_i is the $(p+1-i)$ -th row of F . Thus $u_i(t)$ is determined by the future state

$$\begin{aligned} x_p(t + \bar{d}_p - \bar{d}_i) &= x_p(t - d_{p-1} + \dots + d_i), \\ x_{p-1}(t + \bar{d}_{p-1} - \bar{d}_i) &= x_p(t - d_{p-2} + \dots + d_i), \\ &\vdots \\ x_{i+1}(t + \bar{d}_{i+1} - \bar{d}_i) &= x_i(t + d_i) \end{aligned} \quad (4.84)$$

which must be calculated by the prediction mechanism, present state $x_i(t)$, and the past state

$$\begin{aligned} x_{i-1}(t + \bar{d}_{i-1} - \bar{d}_i) &= x_i(t - d_{i-1}), \\ &\vdots \\ x_2(t + \bar{d}_2 - \bar{d}_i) &= x_2(t - d_{p-1} - d_{p-2} - \dots - d_2), \\ x_1(t + \bar{d}_1 - \bar{d}_i) &= x_1(t - d_{i-1} - d_{i-2} - \dots - d_1) \\ &= x_1(t - \bar{d}_i) \end{aligned} \quad (4.85)$$

which must be memorized in the controller.

It should be noted that the modified controller becomes more complicated. In fact, the prediction mechanism of the original controller is still required in order to yield $u_1(t)$, and additional time-delays are needed to memorize the state variables. When modification of the manipulating variable u_i , $i = 2, \dots, p-1$ is concerned, another prediction mechanism and a set of time-delay must be added for each i . Since the modification

of the controller originates in the practical point of view, the trade off between the response of the controlled system and the complexity of the controller should be taken into account.

4.7 Controller with observer

In this section the case where the state variables $x_i(t)$ of the subsystems S_i are not directly accessible is considered. Two cases are assumed in the following. The initial part J_0 of the performance index is not considered in this section.

First, it is assumed that the output $y_i(t)$ of each subsystem S_i is measured, and each subsystem is observable. In this case an observer can be constructed for each subsystem S_i in the usual way, since each subsystem S_i is delay-free. As for the input v_i from the preceding subsystem, it is easily obtained by connecting the time-delay of length d_{i-1} to y_{i-1} . Then the controller can be constructed using the prediction mechanism and time-delay elements as described in Section 4.3. The present value of the state variables are substituted by the estimated values of the observers.

As the second case, it is assumed that only the output $y(t)$, i.e., the output $y_p(t)$ of subsystem S_p is measured. A natural method in this case is to estimate $x_i(t)$ in a similar way. If the state variables $x_i(t)$ could be estimated, the optimal control law can be implemented by using the prediction mechanism and time-delay elements as in the first case. However, a difficulty arises in the construction of the observer, due to the existence of time-delay between the subsystems. Here, a different method to implement the optimal control law is introduced, i.e., to estimate the state variables $\bar{x}(t)$, using an observer for the imaginary system $\bar{S}$.

Since the imaginary system $\bar{S}$ is a delay-free system, an observer can be constructed as follows:

$$\dot{z}(t) = \hat{A}z(t) + K\bar{y}(t) + M\bar{B}\bar{u}(t) \quad (4.86)$$

$$\hat{x}(t) = Gz(t) + H\bar{y}(t) \quad (4.87)$$

Here, $\hat{A}$, K , M , and H are matrices satisfying

$$\hat{A}M = M\bar{A} - K\bar{C} \quad (4.88)$$

$$I = GM + H\bar{C} \quad (4.89)$$

$z(t)$ is a vector representing the state variable of the observer, and $\hat{x}(t)$ is the estimation of state variable $\bar{x}(t)$. Output $\bar{y}(t)$ of the imaginary system $\bar{\mathcal{S}}$ can be calculated from the output $y(t)$ of the time-delay system $\mathcal{S}$ and the past value of input $\bar{u}_i(\tau)$, $t - \bar{d}_i < \tau < t$ of the imaginary system $\bar{\mathcal{S}}$ by (4.28). Then the state variable of the imaginary system is estimated as $\hat{x}(t)$ by the observer. The optimal input of the imaginary system $\bar{\mathcal{S}}$ is calculated by

$$\bar{u}(t) = -\bar{F}\hat{x}(t) \quad (4.90)$$

which is the state feedback law for the imaginary system $\bar{\mathcal{S}}$. Then time-delay element of length $\bar{d}_i$ is attached to each $\bar{u}_i(t)$ to yield the input $u_i(t)$. The structure of the controller using an observer for the imaginary system $\bar{\mathcal{S}}$ is shown in Fig. 4.5.

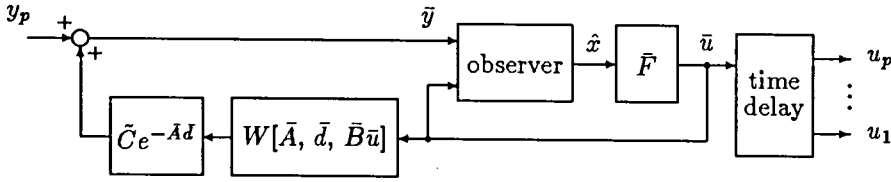


Fig. 4.5 Structure of the controller with an observer

4.8 Examples

Example 1

Consider the unilateral time-delay system shown in Fig. 4.6. This system has two stations, which are connected in series by a delay line of length $d_1 = 1$. Station $\mathcal{S}_1$ is described by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + B_1 u_1(t) \\ y_1(t) &= C_1 x_1(t) \end{aligned} \quad (4.91)$$

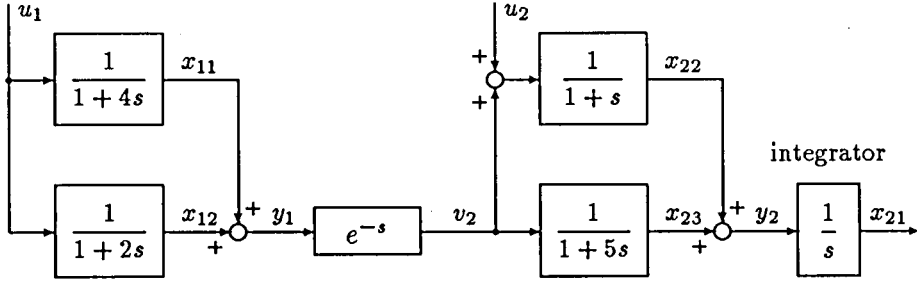


Fig. 4.6 Two-station example

and Station $\mathcal{S}_2$ by

$$\dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) + E_2 v_2(t). \quad (4.92)$$

The delay line $\mathcal{D}_1$ is described by

$$v_2(t) = y_1(t - d_1). \quad (4.93)$$

An integrator is added at the output y of the system in order to cancel the offset for a step disturbance at v_1 . This integrator will be included in the controller to be designed, but it will be regarded as a part of the plant in the design of the controller. In this example, the station $\mathcal{S}_2$ is defined as to include the integrator. Thus, the parameters of this system are given as follows:

$$A_1 = \begin{bmatrix} -0.25 & 0 \\ 0 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}, \quad (4.94)$$

$$C_1 = [1 \ 1],$$

$$x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} \quad (4.95)$$

$$A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 1 \\ 0.2 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (4.96)$$

$$x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix} \quad (4.97)$$

The performance index to be minimized is of the following form:

$$J = \int_0^\infty \{y'(t)Qy(t) + u'(t)Ru(t)\} dt \quad (4.98)$$

As an example, the weighting matrices are chosen as:

$$Q = I, \quad R = I. \quad (4.99)$$

Both the steady state solution and the solution for the initial part is required to implement the controller which minimizes the performance index (4.99). But the measurement of the infinite dimensional state in the delay line is required in order to implement the optimal control law for the initial part, and this is not desirable from practical point of view. Thus, only the steady state solution will be considered in the following.

First, the delay-free imaginary system is introduced. The imaginary system is described by:

$$\begin{aligned} \dot{\bar{x}}(t) &= A\bar{x}(t) + B\bar{u}(t), \\ \bar{y}(t) &= C\bar{x}(t) \end{aligned} \quad (4.100)$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & -0.25 & 0 \\ 0 & 0 & 0 & 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0.25 \\ 0 & 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad (4.101)$$

$$\bar{x}(t) = [\bar{x}_{21}(t), \bar{x}_{22}(t), \bar{x}_{23}(t), \bar{x}_{11}(t), \bar{x}_{12}(t)]',$$

$$\bar{u}(t) = [\bar{u}_2(t), \bar{u}_1(t)]' \quad (4.102)$$

The performance index for the imaginary system is defined according to that of the original system as:

$$\bar{J} = \int_0^\infty \{ \bar{y}'(t) Q \bar{y}(t) + \bar{u}'(t) R \bar{u}(t) \} dt \quad (4.103)$$

The optimal control problem for the imaginary system is solved in the usual way. For the weighting matrices (4.99), the solution of the Riccati equation turns out to be

$$\bar{P} = \begin{bmatrix} 1.887 & 0.887 & 1.310 & 0.652 & 0.597 \\ 0.887 & 0.887 & 1.310 & 0.652 & 0.597 \\ 1.310 & 1.310 & 2.249 & 1.497 & 1.255 \\ 0.652 & 0.652 & 1.497 & 1.400 & 1.063 \\ 0.597 & 0.597 & 1.255 & 1.063 & 0.857 \end{bmatrix} \quad (4.104)$$

and the optimal feedback gain is calculated as

$$\bar{F} = \begin{bmatrix} 0.887 & 0.887 & 1.310 & 0.652 & 0.597 \\ 0.461 & 0.461 & 1.002 & 0.882 & 0.694 \end{bmatrix} \quad (4.105)$$

Next, the steady state optimal solution for the imaginary system is converted to the solution for the original time-delay system. Since the states corresponding to station $\mathcal{S}_1$ in the imaginary system is identical to those of the original system, only the station $\mathcal{S}_2$ need to be considered. Station $\mathcal{S}_2$ can be expressed as

$$\begin{aligned} \bar{x}_2(t) &= e^{A_2 d_1} x_2(t) + W[A_2, d_1, z], \\ d_1 &= 1, \\ z(t) &= B_2 \bar{u}_2(t) + E_2 C_1 x_1(t) \end{aligned} \quad (4.106)$$

where $W[A_2, d_1, z]$ is the finite interval integral element defined in (4.27), i.e.,

$$W[A_2, d_1, z] = \int_{t-d_1}^t e^{A_2(t-\tau)} z(\tau) d\tau. \quad (4.107)$$

Calculation of the finite interval integral can be performed by using the model of station $\mathcal{S}_2$ as:

$$\dot{w}(t) = A_2 w(t) + z(t) \quad (4.108)$$

$$W[A_2, d_1, z] = w(t) - e^{A_2 d_1} w(t-d) \quad (4.109)$$

Thus, the state of the imaginary system is obtained. Then, it is multiplied by the optimal feedback gain, which yields the control signal for the imaginary system. Now, the relation between $\bar{u}$ and u is

$$\begin{bmatrix} u_2(t) \\ u_1(t) \end{bmatrix} = \begin{bmatrix} \bar{u}_2(t - d_1) \\ \bar{u}_1(t) \end{bmatrix} \quad (4.110)$$

Thus, $\bar{u}_1$ is directly applied to the plant as u_1 . As for $\bar{u}_2$, a delay line of length d is used to adjust the time gap between the imaginary system and the original time-delay system.

The structure of the controller is shown in Fig. 4.7. It should be noted that the gain block used in the model performing the finite interval integral is identical to the gain block between x_2 and $\bar{x}_2$. Thus the gain block is used in common.

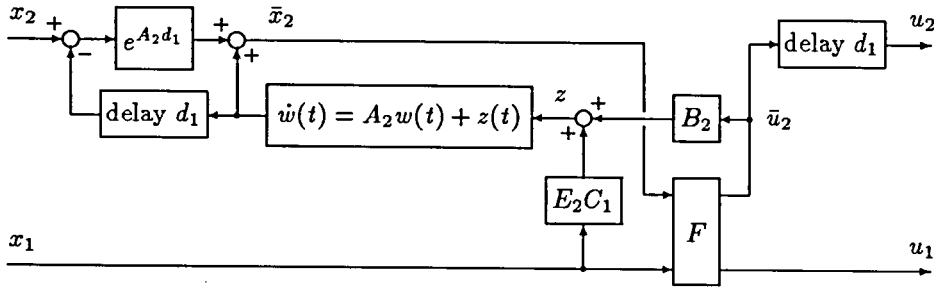


Fig. 4.7 Structure of the controller

Example 2

Modification of the controller is illustrated by an example in this section. The same controlled object as in the preceding example is used for this purpose. Since there are only two stations, namely $p = 2$, in the preceding example, the controller can be modified as described in Section 4.4 without adding so much complexity. In the modified controller, the time-delay element between $\bar{u}_2$ and u_2 is deleted, and new

time-delay element is added to obtain $x_1(t-d)$ from $x_1(t)$. The structure of the modified controller is shown in Fig. 4.8.

Now, the case of using an observer is considered. It is assumed that only y_2 is measurable. Thus the observer for the imaginary system is implemented. Kalman filter type full order observer is used in this example. This implies that the matrices M , G , and H of the observer are set as

$$M = G = I, \quad H = 0 \quad (4.111)$$

Thus, the observer is given by

$$\dot{z}(t) = (\bar{A} - K\bar{C})z(t) + K\bar{y}(t) + \bar{B}\bar{u}(t) \quad (4.112)$$

$$\hat{x}(t) = z(t) \quad (4.113)$$

The gain matrix K is determined by solving the optimal regulator problem for the dual system described by

$$\dot{x}(t) = \bar{A}'x(t) + \bar{C}'\bar{u}(t) \quad (4.114)$$

In this example, the weighting matrices of the performance index were set as

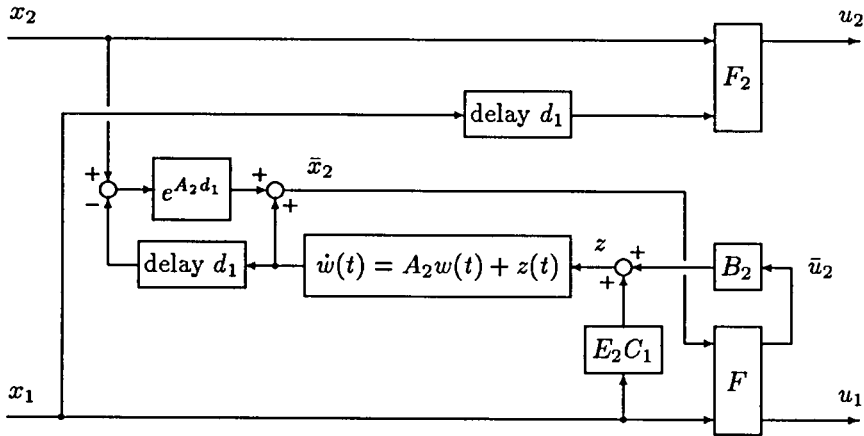


Fig. 4.8 Modified controller

$$Q = I, \quad R = 10^{-2} \times I \quad (4.115)$$

and the optimal gain K turned out to be

$$K = \begin{bmatrix} -5.97 & -4.87 \\ 1.54 & -10.2 \\ -1.09 & -0.841 \\ -1.80 & -0.273 \\ -2.77 & 0.093 \end{bmatrix} \quad (4.116)$$

The structure of the controller with the observer is shown in Fig. 4.9.

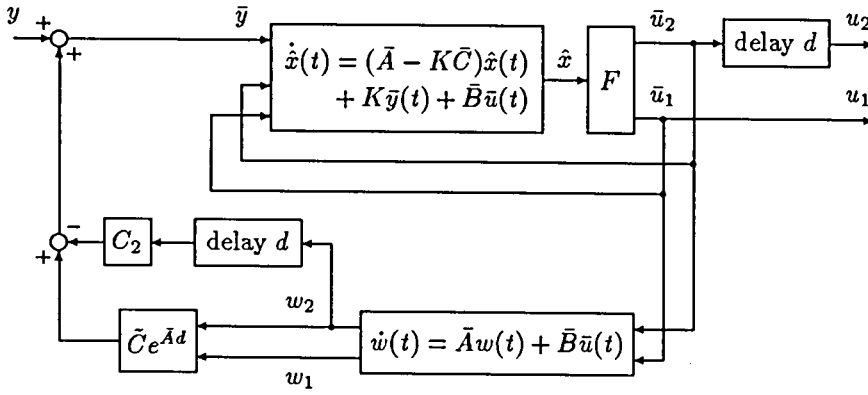


Fig. 4.9 Controller with an observer

Chapter 5

Discrete-time optimal control of unilateral time-delay systems

In this chapter, application of the discrete-time optimal control theory to the unilateral time-delay systems shown in Fig. 5.1 is considered. First, an imaginary delay-free system is introduced as in the continuous-time case. Then the solution for the imaginary system is converted to the solution for the time-delay system.

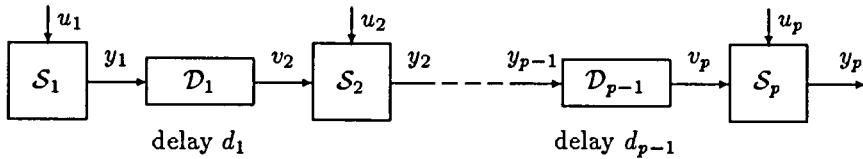


Fig. 5.1 Discrete-time unilateral time-delay system $\mathcal{S}$

When a digital controller is used as a sampled data controller, it often happens that the computation time in the controller cannot be neglected. The measurement of output variables may also require additional time. These extra time can be regarded as time-delay along the signal path at the input or the output of the plant. It can be considered that the plant has these time-delay elements, while there are no delays in the measurement and computation. This class of systems is described in Fig. 5.2. Time-delays between the stations are also considered, since it can be treated in the same framework. The class of systems mentioned here is certainly included in the discretized unilateral time-delay systems described in Chapter 3. It is a restricted case that the manipulating input exists only at station $\mathcal{S}_1$. Thus, it will be classified as the 'single input case' in this chapter. The solution for the general unilateral time-delay systems can be applied to the single input case as well. But it will be

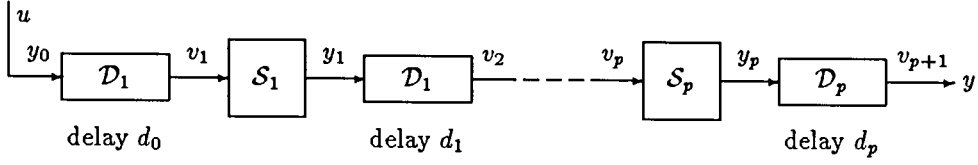


Fig. 5.2 Single input case

treated separately, since the solution for the single input case becomes considerably simpler.

It should be noted that the discrete-time unilateral time-delay system is expressed in the same form as a delay-free system. The results of this chapter contributes to the reduction of the computational difficulties in the design of the controllers for unilateral time-delay systems.

5.1 Problem formulation

5.1.1 General case

Consider the unilateral time-delay system shown in Fig. 5.1. Each S_i , $i = 1, \dots, p$ is described by

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u_i(k) + E_i v_i(k) \\ y_i(k) &= C_i x_i(k) + D_i u_i(k) \end{aligned} \quad (5.1)$$

where x_i , u_i , v_i , and y_i are vectors of dimension n_i , m_i , r_i , and r_{i-1} , respectively. Each D_i , $i = 1, \dots, p-1$ represents a time-delay of length d_i step, and is described by

$$\begin{aligned} z_i(k+1) &= F_i z_i(k) + G_i y_i(k) \\ v_{i+1}(k) &= H_i z_i(k) \end{aligned} \quad (5.2)$$

Here, z_i is a vector of dimension $d_i r_i$, and coefficient matrices F_i , G_i , and H_i are of the following form:

$$F_i = \begin{bmatrix} 0 & I & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I \\ 0 & & & 0 \end{bmatrix}, \quad G_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$$H_i = [I \ 0 \ \dots \ 0] \quad (5.3)$$

By putting

$$x(k) = [x'_p(k), z'_{p-1}(k), x'_{p-1}, \dots, z'_1(k), x'_1(k)]' \quad (5.4)$$

$$u(k) = [u'_p(k), u'_{p-1}(k), \dots, u'_1(k)]' \quad (5.5)$$

$$y(k) = y_p(k) \quad (5.6)$$

the overall system is described by

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k) \quad (5.7)$$

$$A = \begin{bmatrix} A_p & E_p H_{p-1} & 0 & \dots & 0 & 0 \\ 0 & F_{p-1} & G_{p-1} C_{p-1} & \dots & 0 & 0 \\ 0 & 0 & A_{p-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & F_1 & G_1 C_1 \\ 0 & 0 & 0 & \dots & 0 & A_1 \end{bmatrix} \quad (5.8)$$

$$B = \begin{bmatrix} B_p & 0 & \dots & \dots & 0 \\ 0 & G_{p-1} D_{p-1} & \dots & \dots & 0 \\ 0 & B_{p-1} & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & G_1 D_1 \\ 0 & 0 & \dots & \dots & B_1 \end{bmatrix} \quad (5.9)$$

$$C = [C_p \ 0 \ 0 \ \dots \ 0 \ 0] \quad (5.10)$$

$$D = [D_p \ 0 \ \dots \ \dots \ 0] \quad (5.11)$$

The variables x and u are vectors of dimension n and m , respectively, where

$$n = \sum_{i=1}^p n_i + \sum_{i=1}^{p-1} d_i r_i, \quad m = \sum_{i=1}^p m_i \quad (5.12)$$

The performance index to be minimized is set as

$$J = \sum_{k=0}^{\infty} \sum_{i=1}^p \{x'_i(k) Q_i x_i(k) + u'_i(k) R_i u_i(k)\} \quad (5.13)$$

where Q_i is a positive semidefinite matrix and R_i is a positive definite matrix. By putting

$$Q = \text{block diag}(Q_p, 0, Q_{p-1}, \dots, Q_2, 0, Q_1) \quad (5.14)$$

$$R = \text{block diag}(R_p, R_{p-1}, \dots, R_2, R_1) \quad (5.15)$$

the performance index J can be expressed as

$$J = \sum_{k=0}^{\infty} \{x'(k) Q x(k) + u'(k) R u(k)\} \quad (5.16)$$

where $x(k)$ is given by (5.4). The performance index is of the form that the subvectors $z_i(k)$ are not weighted. It is considered to be still general enough, since $z_i(k)$ are nothing but the delayed values of $x_j(k)$ and $u_j(k)$. In order to avoid complication, weighting the subvectors $z_i(k)$ will be considered only in the single input case.

It is well-known that the optimal input $u^*(k)$ which minimizes J subject to (5.1), (5.2) (i.e., to (5.7)) is given by

$$u^*(k) = -F x(k) \quad (5.17)$$

$$F = (B' P B + R)^{-1} B' P A \quad (5.18)$$

where P is the positive semidefinite solution of the discrete-time Riccati equation

$$P = A' P A - A' P B (B' P B + R)^{-1} B' P A + Q \quad (5.19)$$

The minimum value J^* of J is given by

$$J^* = x'(0)Px(0) \quad (5.20)$$

Now, the size of the matrix A is $n \times n$ where n is given by (5.12). In general, the sampling period T should be chosen short enough compared with the minimum time constant of $\mathcal{S}_{\alpha_i}$. So, if the delay times τ_{α_i} are large compared with the time constant of $\mathcal{S}_{\alpha_i}$, the integers d_i become very large. In such a case, to solve (5.19) directly becomes a formidable work. Thus the problem is to reduce computational difficulties in obtaining the solution P of the Riccati equation (5.19).

5.1.2 Single input case

For the single input case, the equations which describe the general case should be modified such that the terms concerning $u_i(k)$, $i = 1, \dots, p$ are deleted, and the delay lines $\mathcal{D}_0$ and $\mathcal{D}_p$ are added. Thus, the single input unilateral time-delay systems are described by the following equations. Each $\mathcal{S}_i$, $i = 1, \dots, p$ is described by

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + E_i v_i(k) \\ y_i(k) &= C_i x_i(k) \end{aligned} \quad (5.21)$$

and each $\mathcal{D}_i$, $i = 1, \dots, p$ by

$$\begin{aligned} z_i(k+1) &= F_i z_i(k) + G_i y_i(k) \\ v_{i+1}(k) &= H_i z_i(k) \end{aligned} \quad (5.22)$$

Put

$$\begin{aligned} x(k) &= [z'_p(k), x'_p(k), z'_{p-1}(k), x'_{p-1}, \\ &\quad \dots, z'_1(k), x'_1(k), z'_0(k)]' \end{aligned} \quad (5.23)$$

$$y_0(k) = u(k) \quad (5.24)$$

$$y(k) = v_{p+1}(k) \quad (5.25)$$

Then the overall system is described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \quad (5.26)$$

$$A = \begin{bmatrix} F_p & G_p C_p & 0 & \dots & \dots & 0 & 0 \\ 0 & A_p & E_p H_{p-1} & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & F_1 & G_1 C_1 & 0 \\ 0 & 0 & \dots & \dots & 0 & A_1 & E_1 H_0 \\ 0 & 0 & \dots & \dots & 0 & 0 & F_0 \end{bmatrix} \quad (5.27)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ G_0 \end{bmatrix} \quad (5.28)$$

$$C = [H_p \ 0 \ \dots \ \dots \ 0 \ 0 \ 0] \quad (5.29)$$

The vectors x and u are of dimensions n and m , respectively, where

$$n = \sum_{i=1}^p n_i + \sum_{i=0}^p d_i r_i, \quad m = m_1 \quad (5.30)$$

Note that the single input case is formulated in the form of having delay lines $\mathcal{D}_0$ and $\mathcal{D}_p$ at the beginning and the end of the signal path (cf. Fig. 5.2), while delay-free subsystems are located at both ends in the general unilateral time-delay case (cf. Fig. 5.1). This formulation is taken so that it will be easier to apply the result to the simplest single input case, in which there is only one station $\mathcal{S}_1$ and the time-delay is concentrated to either input or output of the plant.

It should also be noted that the result of the general case can be applied to the systems having the allocation of the time-delay as in Fig. 5.2. In such a case, stations $\mathcal{S}_0$ and $\mathcal{S}_{p+1}$ are introduced, which are actually delay lines of length 1. Namely, $\mathcal{S}_0$ is given by

$$\begin{aligned} x_0(k+1) &= u_0(k) \\ y_0(k) &= x_0(k) \end{aligned} \quad (5.31)$$

and $\mathcal{S}_{p+1}$ by

$$\begin{aligned}x_{p+1}(k+1) &= v_{p+1}(k) \\ y_{p+1}(k) &= x_{p+1}(k)\end{aligned}\tag{5.32}$$

Accordingly, the delay lines $\mathcal{D}_0$ and $\mathcal{D}_p$ should be redefined such that the length of the time-delay is shorter by 1. Then the system is indeed described in the form of Fig. 5.1. If the length of the time-delay in $\mathcal{D}_0$ or $\mathcal{D}_p$ is short enough, the delay line can be eliminated by redefinition of the station $\mathcal{S}_1$ or $\mathcal{S}_p$, respectively, such that the time-delay is included in the station. This is another way of modifying the description of the system into the form of Fig. 5.1. These methods of modification can be applied to the discrete-time systems only, since the continuous-time delay line cannot be expressed in the same form as a delay-free system.

For the single input case, the weighting matrix Q of the form

$$Q = \text{block diag}(W_p, Q_p, \dots, W_1, Q_1, W_0)\tag{5.33}$$

$$W_i = \text{block diag}(W_{i1}, W_{i2}, \dots, W_{id_i})\tag{5.34}$$

is considered, corresponding to the definition of the state vector $x(k)$ and manipulating vector $u(k)$. It may be noted that the form of the performance index for the single input case is slightly extended from that for the general input case. Namely, this formulation is valid when the subvectors $z_i(k)$ are also weighted using diagonal matrices W_i .

5.2 Introduction of imaginary delay-free system

Here, an imaginary delay-free system is introduced. As shown in Fig. 5.3, it is defined such that the time-delays are eliminated, and each subsystem is directly connected in series. The imaginary system is described by

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k)\tag{5.35}$$

$$\bar{A} = \begin{bmatrix} A_p & E_p C_{p-1} & & 0 \\ & \ddots & \ddots & \\ & & A_2 & E_2 C_1 \\ 0 & & & A_1 \end{bmatrix}\tag{5.36}$$

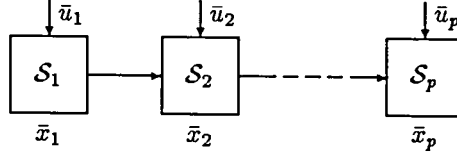


Fig. 5.3 Imaginary delay-free system $\bar{S}$

$$\bar{B} = \begin{bmatrix} B_p & E_p D_{p-1} & & 0 \\ & \ddots & \ddots & \\ & & B_2 & E_2 D_1 \\ 0 & & & B_1 \end{bmatrix} \quad (5.37)$$

Set the performance index $\bar{J}$ of the imaginary system according to that of the actual time-delay system as

$$\bar{J} = \sum_{k=0}^{\infty} \{ \bar{x}'(k) \bar{Q} \bar{x}(k) + \bar{u}'(k) \bar{R} \bar{u}(k) \} \quad (5.38)$$

$$\bar{Q} = \text{block diag} (Q_p, \dots, Q_2, Q_1), \quad \bar{R} = R \quad (5.39)$$

For the single input case, the coefficient matrices are defined as

$$\bar{A} = \begin{bmatrix} A_p & E_p C_{p-1} & & 0 \\ & \ddots & \ddots & \\ & & A_2 & E_2 C_1 \\ 0 & & & A_1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_1 \end{bmatrix} \quad (5.40)$$

and the weighting matrices are defined as

$$\bar{Q} = \text{block diag} (\bar{Q}_p, \dots, \bar{Q}_2, \bar{Q}_1) \quad (5.41)$$

$$\bar{Q}_i = Q_i + C_i' \bar{W}_i C_i \quad (5.42)$$

$$\bar{R} = R + \bar{W}_0 \quad (5.43)$$

$$\bar{W}_i = W_{i1} + W_{i2} + \dots + W_{id_i} \quad (5.44)$$

The differences of the above definitions correspond to the differences in the definitions of the plant given in the previous section.

Now, consider the relation between the original unilateral time-delay system $\mathcal{S}$ and the imaginary system $\bar{\mathcal{S}}$. Define integers $\bar{d}_i$ concerning the length of time-delay as

$$\bar{d}_i = \begin{cases} 0 & (\text{if } i = 1) \\ d_1 + \cdots + d_{i-1} & (\text{if } i = 2, \dots, p) \end{cases} \quad (5.45)$$

and set the initial value of the state $\bar{x}(0)$ and the input $\bar{u}(k)$ for $k \geq 0$ of the imaginary system $\bar{\mathcal{S}}$ as

$$\begin{bmatrix} \bar{x}_p(0) \\ \vdots \\ \bar{x}_2(0) \\ \bar{x}_1(0) \end{bmatrix} = \begin{bmatrix} x_p(\bar{d}_p) \\ \vdots \\ x_2(\bar{d}_2) \\ x_1(\bar{d}_1) \end{bmatrix}, \quad \begin{bmatrix} \bar{u}_p(k) \\ \vdots \\ \bar{u}_2(k) \\ \bar{u}_1(k) \end{bmatrix} = \begin{bmatrix} u_p(k + \bar{d}_p) \\ \vdots \\ u_2(k + \bar{d}_2) \\ u_1(k + \bar{d}_1) \end{bmatrix} \quad (5.46)$$

Then, following relations hold for $k \geq 0$.

$$\begin{bmatrix} \bar{x}_p(k) \\ \vdots \\ \bar{x}_2(k) \\ \bar{x}_1(k) \end{bmatrix} = \begin{bmatrix} x_p(k + \bar{d}_p) \\ \vdots \\ x_2(k + \bar{d}_2) \\ x_1(k + \bar{d}_1) \end{bmatrix} \quad (5.47)$$

The performance index J can be expressed using $\bar{J}$ as

$$\begin{aligned} J &= \bar{J} + J_0 \\ &= \bar{J} + \sum_{i=2}^{p-1} \sum_{k=0}^{\bar{d}_i-1} \{x'_i(k)Q_i x_i(k) + u'_i(k)R_i u_i(k)\} \\ &= \bar{J} + \sum_{\mu=1}^{p-1} \sum_{k=d_\mu}^{\bar{d}_{\mu+1}-1} \left[\sum_{i=\mu+1}^p \{x'_i(k)Q_i x_i(k) + u'_i(k)R_i u_i(k)\} \right] \end{aligned} \quad (5.48)$$

Note that the input $\bar{u}_i(k)$, $k \geq 0$ of the imaginary system $\bar{\mathcal{S}}$ has no effect upon J_0 , from which and the principle of optimality it follows that the

optimal input $\bar{u}_i(k)$ which minimizes $\bar{J}$ is identical to the optimal input $u_i(k + \bar{d}_i)$ which minimizes J .

For the single input case, definition of the integers $\bar{d}_i$ should be modified, due to the existence of $\mathcal{D}_0$ and $\mathcal{D}_p$. The relationship between the time-delay system and the imaginary system will also be different. Detailed description for the single input case is deferred to the next section.

The optimal control for the imaginary system is given by

$$\bar{u}^*(k) = -\bar{F}\bar{x}(k) \quad (5.49)$$

$$\bar{F} = (\bar{B}'\bar{P}\bar{B} + \bar{R})^{-1}\bar{B}'\bar{P}\bar{A} \quad (5.50)$$

where $\bar{P}$ is the positive semidefinite solution of

$$\bar{P} = \bar{A}'\bar{P}\bar{A} - \bar{A}'\bar{P}\bar{B}(\bar{B}'\bar{P}\bar{B} + \bar{R})^{-1}\bar{B}'\bar{P}\bar{A} + \bar{Q} \quad (5.51)$$

The minimum value of the performance index is given by

$$\bar{J}^* = \bar{x}'(0)\bar{P}\bar{x}(0) \quad (5.52)$$

The size of the Riccati equation for the imaginary system is $\bar{n} \times \bar{n}$ where

$$\bar{n} = \sum_{i=1}^p n_i \quad (5.53)$$

It is one of the main purposes of this thesis to present a new method to calculate the matrix P and the optimal control $u^*(k)$ from the solution of the Riccati equation for the imaginary system. The method will be described in the following section.

5.3 Solution for time-delay system

5.3.1 Single input case

As a relatively simple case, the single input case is considered first. Define integers $\bar{d}_i$ as

$$\bar{d}_i = d_0 + d_1 + \cdots + d_{i-1}, \quad i = 1, \dots, p \quad (5.54)$$

Note that this definition is for the single input case only, and the definition for the general case is already given in the last section. Define matrices T and T_i , $i = 1, \dots, p$ as follows:

$$T = \begin{bmatrix} T_p A^{d_p} \\ \vdots \\ T_2 A^{d_2} \\ T_1 A^{d_1} \end{bmatrix} : \bar{n} \times n$$

$$T_p = [0 \ I \ 0 \ \dots \ 0 \ 0 \ 0 \ 0 \ 0] : n_p \times n$$

$$\vdots$$

$$T_2 = [0 \ 0 \ 0 \ \dots \ 0 \ I \ 0 \ 0 \ 0] : n_2 \times n$$

$$T_1 = [0 \ 0 \ 0 \ \dots \ 0 \ 0 \ 0 \ I \ 0] : n_1 \times n \quad (5.55)$$

Here, the partitions of T_i correspond to that of $x(k)$. Each T_i extracts the rows corresponding to subvector $x_i(k)$. Define also matrices W , $\tilde{W}_i$, and $\bar{W}_{ij}$ as follows:

$$W = \text{block diag} (\tilde{W}_p, 0, \dots, \tilde{W}_1, 0, \tilde{W}_0) : n \times n \quad (5.56)$$

$$\tilde{W}_i = \text{block diag} (\bar{W}_{i1}, \dots, \bar{W}_{id_i}) : m_i d_i \times m_i d_i \quad (5.57)$$

$$\bar{W}_{ij} = W_{i1} + \dots + W_{ij} : n \times n \quad j = 1, \dots, d_i \quad (5.58)$$

Here, the partition of W corresponds to that of $x(k)$.

Now, set the initial value of the state $\bar{x}(0)$ and the input $\bar{u}(k)$ for $k \geq 0$ of the imaginary system $\bar{\mathcal{S}}$ as

$$\bar{x}(0) = T x(0) \quad (5.59)$$

$$\bar{u}(k) = u(k) \quad (5.60)$$

Note that from (5.26) and (5.55) it follows that

$$x_i(k+j) = T_i A^j x(k), \quad j = 0, 1, \dots, \bar{d}_i \quad (5.61)$$

Thus, (5.59) is equivalent to

$$\begin{bmatrix} \bar{x}_p(0) \\ \vdots \\ \bar{x}_2(0) \\ \bar{x}_1(0) \end{bmatrix} = \begin{bmatrix} x_p(\bar{d}_p) \\ \vdots \\ x_2(\bar{d}_2) \\ x_1(\bar{d}_1) \end{bmatrix} \quad (5.62)$$

Then, the following relations hold for $k \geq 0$.

$$x_i(k + \bar{d}_i) = \bar{x}_i(k), \quad i = 1, \dots, p \quad (5.63)$$

The subvectors concerning the time-delay satisfy the following relations:

$$z_i(k) = \begin{bmatrix} y_i(k - d_i) \\ \vdots \\ y_i(k - 1) \end{bmatrix} \quad i = 1, \dots, p \quad (5.64)$$

$$z_0(k) = \begin{bmatrix} u(k - d_0) \\ \vdots \\ u(k - 1) \end{bmatrix} \quad (5.65)$$

Thus, the terms concerning subvector $x_i(k)$ in the performance index J can be expressed as

$$\begin{aligned} & \sum_{k=0}^{\infty} x_i'(k) Q_i x_i(k) \\ &= \sum_{k=0}^{\infty} \bar{x}_i'(k) Q_i \bar{x}_i(k) + \sum_{j=0}^{d_i-1} x_i'(j) Q_i x_i(j) \\ &= \sum_{k=0}^{\infty} \bar{x}_i'(k) Q_i \bar{x}_i(k) + \sum_{j=0}^{d_i-1} x_i'(0) (T_i A^j)' Q_i (T_i A^j) x_i(0) \quad (5.66) \end{aligned}$$

The terms concerning subvector $z_i(k)$ can be expressed as

$$\begin{aligned}
& \sum_{k=0}^{\infty} z'_i(k) W_i z_i(k) \\
&= \sum_{k=0}^{\infty} \bar{y}'_i(k) \bar{W}_i \bar{y}_i(k) + \sum_{j=0}^{d_i-1} z'_i(j) W_i z_i(j) \\
&= \sum_{k=0}^{\infty} x'_i(k) C'_i \bar{W}_i C_i x_i(k) + z'_i(0) \tilde{W}_i z_i(0) \\
& i = 1, \dots, p
\end{aligned} \tag{5.67}$$

$$\begin{aligned}
& \sum_{k=0}^{\infty} z'_0(k) W_0 z_0(k) \\
&= \sum_{k=0}^{\infty} \bar{u}'(k) \bar{W}_0 \bar{u}(k) + \sum_{j=0}^{d_0-1} z'_0(j) W_0 z_0(j) \\
&= \sum_{k=0}^{\infty} u'(k) \bar{W}'_0 u(k) + z'_0(0) \tilde{W}_0 z_0(0)
\end{aligned} \tag{5.68}$$

Note that $x_i(k)$ appears in the right-hand side of (5.67). It can be expressed using $\bar{x}_i(k)$ as in (5.66). Thus (5.67) can be further modified as:

$$\begin{aligned}
\sum_{k=0}^{\infty} z'_i(k) W_i z_i(k) &= \sum_{k=0}^{\infty} \bar{x}'_i(k) C'_i \bar{W}_i C_i \bar{x}_i(k) \\
&+ \sum_{j=0}^{d_i-1} x'_i(0) (T_i A^j)' C'_i Q_i C_i (T_i A^j) x_i(0) \\
&+ z'_i(0) \tilde{W}_i z_i(0)
\end{aligned} \tag{5.69}$$

Using the above equations, the relation between the performance indices of the time-delay system and the imaginary system are expressed as follows:

$$J = \bar{J} + J_0 \tag{5.70}$$

$$\begin{aligned}
J_0 &= \sum_{i=1}^p \sum_{j=0}^{d_i-1} x'(0) (T_i A^j)' \bar{Q}_i (T_i A^j) x(0) + \sum_{i=0}^p z'_i(0) \tilde{W}_i z_i(0) \\
&= x'(0) \left\{ \sum_{i=1}^p \sum_{j=0}^{d_i-1} (T_i A^j)' \bar{Q}_i (T_i A^j) \right\} x(0) + x'(0) W x(0)
\end{aligned} \tag{5.71}$$

As seen from (5.71), J_0 is determined by the initial value $x(0)$ of the time-delay system $\mathcal{S}$, and it is not affected by the manipulating variable $u(k)$. Hence follows that the optimal input $\bar{u}^*(k)$ which minimizes $\bar{J}$ is at the same time the optimal input $u^*(k)$ which minimizes J . Namely,

$$J^* = \bar{J}^* + J_0 \quad (5.72)$$

$$u^*(k) = \bar{u}^*(k) \quad (5.73)$$

Now the optimal solution of the time-delay system is expressed in terms of the solution for the imaginary system. Substituting (5.71) into (5.72) and using (5.20) and (5.52), it follows that

$$\begin{aligned} x'(0)Px(0) &= x'(0)T'\bar{P}Tx(0) \\ &\quad + x'(0) \left\{ \sum_{i=1}^p \sum_{j=0}^{d_i-1} (T_i A^j)' \bar{Q}_i (T_i A^j) \right\} x(0) \\ &\quad + x'(0)Wx(0) \end{aligned} \quad (5.74)$$

Hence, the following relation is obtained for the solution of the Riccati equation.

$$P = T'\bar{P}T + \sum_{i=1}^p \sum_{j=0}^{d_i-1} (T_i A^j)' \bar{Q}_i (T_i A^j) + W \quad (5.75)$$

Next, note that (5.55), (5.59), and (5.63) leads to

$$\bar{x}(k) = Tx(k) \quad (5.76)$$

Then substitute (5.17) and (5.49) into (5.73), from which and (5.76) follows

$$-Fx(k) = -\bar{F}Tx(k) \quad (5.77)$$

Hence, the following relation is obtained for the optimal feedback gain.

$$F = \bar{F}T \quad (5.78)$$

It should be noted that in the single input case the unilateral time-delay system having the form of Fig. 5.2 can be transformed into a simpler form in which there is only one station and the time-delay is concentrated

at the input or the output of the station. This is accomplished by applying a similarity transformation to the system. However, the coefficient matrix Q of the performance index is also subject to change by the similarity transformation, and it results in a rather complicated form. Hence the method proposed for the system with time-delays concentrated at the input or output cannot be directly applied to the systems having the form of Fig. 5.2. The influence of the similarity transformation upon the matrix Q will be shown by an example in Section 5.4.1.

5.3.2 Two-station case

The case of two-station systems is studied in detail, as a preparation for the multi-station case. Since only one d_i (i.e., d_1) appears in this case, the subscript i of d_i will be omitted: i.e., $d = d_1$ throughout this section.

To reduce the optimal control problem to a smaller size problem, the method of dynamic programming is employed. Let h be an integer such that $0 \leq h \leq n$, and define $\tilde{J}_h$ by

$$\begin{aligned} \tilde{J}_h = \sum_{k=0}^{\infty} \{ & x_1'(k)Q_1x_1(k) + u_1'(k)R_1u_1(k) \\ & + x_2'(h+k)Q_2x_2(h+k) \\ & + u_2'(h+k)R_2u_2(h+k) \} \end{aligned} \quad (5.79)$$

Then,

$$J = \tilde{J}_0 \quad (5.80)$$

$$\tilde{J}_{h-1} = x_2'(h-1)Q_2x_2(h-1) + u_2'(h-1)R_2u_2(h-1) + \tilde{J}_h \quad (5.81)$$

By examining (5.1) and (5.2), it is easily verified that $\tilde{J}_h$ does not include $u_2(0), \dots, u_2(h-1)$. Therefore, the next formula is obtained by the principle of optimality:

$$\begin{aligned} \tilde{J}_{h-1}^* &= \min_{U[h-1]} \tilde{J}_{h-1} \\ &= \min_{u_2(h-1)} \left\{ x_2'(h-1)Q_2x_2(h-1) \right. \\ &\quad \left. + u_2'(h-1)R_2u_2(h-1) + \min_{U[h]} \tilde{J}_h \right\} \end{aligned} \quad (5.82)$$

where $U[h]$ is the set of inputs

$$U[h] = \left\{ u_1(0), \dots, u_1(h-1), \right. \\ \left. \begin{bmatrix} u_1(h) \\ u_2(h) \end{bmatrix}, \begin{bmatrix} u_1(h+1) \\ u_2(h+1) \end{bmatrix}, \dots \right\} \quad (5.83)$$

Now, the problem of minimizing $\tilde{J}_d$ over $U[d]$ subject to (5.7) is solved. Put

$$\bar{x}(k) = [x'_2(d+k), x'_1(k)]' \quad (5.84)$$

$$\bar{u}(k) = [u'_2(d+k), u'_1(k)]' \quad (5.85)$$

From (5.1)–(5.2)

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) \quad (5.86)$$

is obtained, where

$$\bar{A} = \begin{bmatrix} A_2 & E_2 C_1 \\ 0 & A_1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_2 & E_2 D_1 \\ 0 & B_1 \end{bmatrix} \quad (5.87)$$

$\tilde{J}_h$ can be given as

$$\tilde{J}_d = \sum_{k=0}^{\infty} \left\{ \bar{x}'(k) \bar{Q} \bar{x}(k) + \bar{u}'(k) \bar{R} \bar{u}(k) \right\} \quad (5.88)$$

where

$$\bar{Q} = \begin{bmatrix} Q_2 & 0 \\ 0 & Q_1 \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R_2 & 0 \\ 0 & R_1 \end{bmatrix} \quad (5.89)$$

Thus, the problem of minimizing $\tilde{J}_d$ subject to (5.7) is expressed as the problem of minimizing (5.38) subject to (5.35), which is the optimal control problem for the imaginary system introduced in the last section. The minimum of $\tilde{J}_d$ is given as

$$\tilde{J}_d^* = \bar{x}(0) \bar{P} \bar{x}(0) \quad (5.90)$$

where $\bar{P}$ is the positive definite solution of the discrete-time Riccati equation

$$\bar{P} = \bar{A}'\bar{P}\bar{A} - \bar{A}'\bar{P}\bar{B}(\bar{B}'\bar{P}\bar{B} + \bar{R})^{-1}\bar{B}'\bar{P}\bar{A} + \bar{Q} \quad (5.91)$$

The optimal $\bar{u}^*(k)$ is given by

$$\bar{u}^*(k) = -(\bar{B}'\bar{P}\bar{B} + \bar{R})^{-1}\bar{B}'\bar{P}\bar{A}\bar{x}(k) \quad (5.92)$$

Next, define

$$\tilde{x}_h(k) = \begin{bmatrix} x_2(h+k) \\ z_d(h+k) \\ \vdots \\ z_{h+1}(h+k) \\ x_1(k) \end{bmatrix} \quad (5.93)$$

Now, by induction, it is shown that $\tilde{J}_h^*$ is given as the quadratic form in $\tilde{x}_h(0)$

$$\tilde{J}_h^* = \tilde{x}_h'(0)\tilde{P}_h\tilde{x}_h(0) \quad (5.94)$$

and also derive the formula to calculate $\tilde{P}_h$. For $h = d$, set $\tilde{P}_h = \bar{P}$. Then equation (5.94) is true as shown above. If (5.94) is true for a certain h , (5.82) becomes

$$\begin{aligned} \tilde{J}_{h-1}^* = \min_{u_2(h-1)} \{ & x_2'(h-1)Q_2x_2(h-1) \\ & + u_2'(h-1)R_2u_2(h-1) + \tilde{x}_h'(0)\tilde{P}_h\tilde{x}_h(0) \} \end{aligned} \quad (5.95)$$

From (5.7)

$$\tilde{x}_h(0) = \tilde{A}_{h-1}\tilde{x}_{h-1}(0) + \tilde{B}_{h-1}u_2(h-1) \quad (5.96)$$

is obtained, where

$$\tilde{A}_{h-1} = \begin{bmatrix} A_2 & E_2 & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (5.97)$$

$$\tilde{B}_{h-1} = \begin{bmatrix} B_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (5.98)$$

By substituting (5.96) into (5.95)

$$\begin{aligned} \tilde{J}_{h-1}^* = \min_{u_2(h-1)} & \left\{ \tilde{x}'_{h-1}(0) \left(\tilde{Q}_{h-1} + \tilde{A}'_{h-1} \tilde{P}_h \tilde{A}_{h-1} \right) \tilde{x}_{h-1}(0) \right. \\ & + 2\tilde{x}'_{h-1}(0) \tilde{A}'_{h-1} \tilde{P}_h \tilde{B}_{h-1} u_2(h-1) \\ & \left. + u'_2(h-1) \left(\tilde{B}'_{h-1} \tilde{P}_h \tilde{B}_{h-1} + R_2 \right) u_2(h-1) \right\} \end{aligned} \quad (5.99)$$

is obtained, where

$$\tilde{Q}_{h-1} = \begin{bmatrix} Q_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (5.100)$$

It is easy to show that the above minimum is attained for the value of $u_2(h-1)$ given by

$$u_2^*(h-1) = \left(\tilde{B}'_{h-1} \tilde{P}_h \tilde{B}_{h-1} R_2 \right)^{-1} \tilde{B}'_{h-1} \tilde{P}_h \tilde{A}_{h-1} \tilde{x}_{h-1}(0) \quad (5.101)$$

and that the minimum is given by

$$\tilde{J}_{h-1}^* = \tilde{x}'_{h-1}(0) \tilde{P}_{h-1} \tilde{x}_{h-1}(0) \quad (5.102)$$

where

$$\begin{aligned} \tilde{P}_{h-1} = & \tilde{Q}_{h-1} + \tilde{A}'_{h-1} \tilde{P}_h \tilde{A}_{h-1} \\ & - \tilde{A}'_{h-1} \tilde{P}_h \tilde{B}_{h-1} \left(\tilde{B}'_{h-1} \tilde{P}_h \tilde{B}_{h-1} + R_2 \right)^{-1} \tilde{B}'_{h-1} \tilde{P}_h \tilde{A}_{h-1} \end{aligned} \quad (5.103)$$

Thus, it has been shown that (5.94) is true for $h = d, d-1, \dots, 0$, and that $\tilde{P}_h$ ($h = d-1, \dots, 0$) can be obtained by using (5.103) recurrently. Here note that (5.80) implies

$$\tilde{P}_0 = P \quad (5.104)$$

where P is the matrix which is needed to give the optimal control (i.e., the positive semidefinite solution of (5.19)). The optimal gain F is obtained by substituting (5.104) into (5.18). Thus, the problem posed in the last section has been solved. The procedure is summarized as follows:

- Step 1** Solve the discrete-time Riccati equation (5.51), and obtain the positive semidefinite solution $\bar{P}$ where the coefficient matrices are given by (5.87) and (5.89).
- Step 2** Put $\tilde{P}_d = \bar{P}$. Compute $\tilde{P}_h$ ($h = d - 1, d - 2, \dots, 1, 0$) by (5.103) where the coefficient matrices are given by (5.97), (5.98), and (5.100). (Note that R_2 is the coefficient of the original performance index (5.79).)
- Step 3** Put $P = \bar{P}_0$. Then, the optimal gain and the minimum value of the performance index J is given by (5.18) and (5.20), respectively.

5.3.3 Multi-station case

Now, the general case where $p \geq 3$ is considered. In the last section the size of the Riccati equation was first reduced to $\bar{n} \times \bar{n}$ by excluding the delay d_1 , and then calculated the solution of the original problem by recurrence formulae. The latter half of the above procedure is nothing but the operation to 'restore' the excluded delay. The same method can be applied to the multi-station case as well. The only difference is that multiple delays must be treated now, and that a 'jump' should be made whenever each delay is restored.

First consider the imaginary system described by (5.35)–(5.37), and solve the optimal control problem of this system with respect to the performance index (5.38). Namely, compute $\bar{P}$ which is the positive semidefinite solution of the Riccati equation (5.51). Then, let h be an integer taking the descending values $\bar{d}_p - 1, \bar{d}_p - 2, \dots, 1, 0$, where $\bar{d}_p$ is defined by (5.45). Compute integers $\tilde{n}_h$, $\bar{n}_h$, $\hat{n}_h$, and $\tilde{m}_h$ by

$$\tilde{n}_h = \bar{n} - (d_i r_i + d_{i+1} r_{i+1} + \dots + d_{p-1} r_{p-1}) - (h - d_i) r_i \quad (5.105)$$

$$\bar{n}_h = n_1 + \dots + n_i \quad (5.106)$$

$$\hat{n}_h = \tilde{n}_h - \bar{n}_h - r_i \quad (5.107)$$

$$\tilde{m}_h = m_{i+1} + \dots + m_p \quad (5.108)$$

where the integer i satisfy for each h

$$\bar{d}_i \leq h < \bar{d}_{i+1} \quad (5.109)$$

Lastly, put

$$P_{\bar{d}_p} = \bar{P} \quad (5.110)$$

and compute $\tilde{P}_h$ ($h = \bar{d}_p - 1, \dots, 0$) in descending order by

$$\begin{aligned} \tilde{P}_h = & \tilde{A}'_h \tilde{P}_{h+1} \tilde{A}_h + \tilde{Q}_h \\ & - \tilde{A}'_h \tilde{P}_{h+1} \tilde{B}_h \left(\tilde{B}'_h \tilde{P}_{h+1} \tilde{B}_h + R_h \right)^{-1} \tilde{B}'_h \tilde{P}_{h+1} \tilde{A}_h \end{aligned} \quad (5.111)$$

where matrices $\tilde{A}_h$, $\tilde{B}_h$, $\tilde{Q}_h$, $\tilde{R}_h$, $\tilde{T}_h$, $\hat{T}_h$, and $\tilde{U}_h$ are

$$\tilde{A}_h = \begin{bmatrix} \hat{T}_h A \tilde{T}'_h & 0 \\ 0 & I \end{bmatrix} : \tilde{n}_{h+1} \times \tilde{n}_h \quad (5.112)$$

$$\tilde{B}_h = \begin{bmatrix} \hat{T}_h B \tilde{U}'_h \\ 0 \end{bmatrix} : \tilde{n}_{h+1} \times \tilde{m}_h \quad (5.113)$$

$$\tilde{Q}_h = \begin{bmatrix} \tilde{T}_h Q \tilde{T}'_h & 0 \\ 0 & 0 \end{bmatrix} : \tilde{n}_h \times \tilde{n}_h \quad (5.114)$$

$$\tilde{R}_h = \tilde{U}_h R \tilde{U}'_h : \tilde{m}_h \times \tilde{m}_h \quad (5.115)$$

$$\tilde{T}_h = [I \ 0] : (\tilde{n}_h - \bar{n}_h) \times n \quad (5.116)$$

$$\hat{T}_h = [I \ 0] : \hat{n}_h \times n \quad (5.117)$$

$$\tilde{U}_h = [I \ 0] : \tilde{m}_h \times m \quad (5.118)$$

Then the solution P of the Riccati equation (5.19) is given by

$$P = \tilde{P}_0 \quad (5.119)$$

Now, the validity of this method is verified. Using the definitions (5.116)–(5.118), put

$$\tilde{x}_h = \begin{bmatrix} \tilde{T}_h x(k) \\ \bar{T}_h \bar{x}(0) \end{bmatrix}, \quad \tilde{u}_h = \tilde{U}_h u(h) \quad (5.120)$$

$$\bar{T}_h = [0 \ I] : \bar{n}_h \times \bar{n} \quad (5.121)$$

Then it follows that

$$\tilde{x}_{h+1} = \tilde{A}_h \tilde{x}_h + \tilde{B}_h \tilde{u}_h, \quad h = \bar{d}_p - 1, \bar{d}_p - 2, \dots, 1, 0 \quad (5.122)$$

and the performance index J can be expressed as

$$J = \bar{J} + \sum_{h=0}^{\bar{d}_p-1} (\tilde{x}'_h \tilde{Q}_h \tilde{x}_h + \tilde{u}'_h \tilde{R}_h \tilde{u}_h) \quad (5.123)$$

Induction is used in the following step of verification. Assume that

$$J^* = \min_{U[j]} \left[\tilde{x}'_{j+1} \tilde{P}_{j+1} \tilde{x}_{j+1} + \sum_{h=0}^j (\tilde{x}'_h \tilde{Q}_h \tilde{x}_h + \tilde{u}'_h \tilde{R}_h \tilde{u}_h) \right] \quad (5.124)$$

$$U[h] = \{\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_h\} \quad (5.125)$$

holds. Then it follows that

$$\begin{aligned} J^* = \min_{U[j-1]} & \left[\sum_{h=0}^{j-1} (\tilde{x}'_h \tilde{Q}_h \tilde{x}_h + \tilde{u}'_h \tilde{R}_h \tilde{u}_h) \right. \\ & + \tilde{x}'_j (\tilde{Q}_j + \tilde{A}'_j \tilde{P}_{j+1} \tilde{A}_j) \tilde{x}_j \\ & + \min_{\tilde{u}_j} \left\{ 2\tilde{x}'_j \tilde{A}'_j \tilde{P}_{j+1} \tilde{B}_j \tilde{u}_j \right. \\ & \left. \left. + \tilde{u}'_j (\tilde{B}'_j \tilde{P}_{j+1} \tilde{B}_j + \tilde{R}_j) \tilde{u}_j \right\} \right] \quad (5.126) \end{aligned}$$

By putting

$$\frac{\partial J}{\partial \tilde{u}_j} = 0 \quad (5.127)$$

$\tilde{u}_j^*$ is given by

$$\tilde{u}_j^* = -(\tilde{B}'_j \tilde{P}_{j+1} \tilde{B}_j + \tilde{R}_j)^{-1} \tilde{B}'_j \tilde{P}_{j+1} \tilde{A}_j \tilde{x}_j \quad (5.128)$$

and J^* can be expressed as

$$J^* = \min_{U[j-1]} \left[\tilde{x}'_j \tilde{P}_j \tilde{x}_j + \sum_{h=0}^{j-1} (\tilde{x}'_h \tilde{Q}_h \tilde{x}_h + \tilde{u}'_h \tilde{R}_h \tilde{u}_h) \right] \quad (5.129)$$

Thus (5.124) implies (5.129). Since the assumption (5.124) clearly holds for $j = \bar{d}_p - 1$ it follows that

$$J^* = \tilde{x}'_0 \tilde{P}_0 \tilde{x}_0 \quad (5.130)$$

Here,

$$\tilde{x}_0 = x(0) \quad (5.131)$$

by definition, hence the result

$$P = \tilde{P}_0 \quad (5.132)$$

is obtained.

Using the solution P , the optimal gain F and the minimum value of the performance index J can be calculated by (5.18) and (5.20) as in the last section.

In executing the above method, the computation of the integers $\tilde{n}_h$, $\bar{n}_h$, $\hat{n}_h$, and $\tilde{m}_h$ should be performed in parallel with the computation of $\tilde{P}_h$. The 'jump' which was mentioned at the beginning of the section is brought about by the sudden increases of $\bar{n}_h$ and $\tilde{m}_h$ which occur at $h = \bar{d}_i$ ($i = p - 1, \dots, 1$).

5.4 Examples

Example 1

A discrete-time system shown in Fig. 5.4 is considered as an example. This can be classified as the single input case. The parameters of the stations $\mathcal{S}_1$, $\mathcal{S}_2$, and the delay lines $\mathcal{D}_0$, $\mathcal{D}_1$ are given as follows:

$$A_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\frac{4}{3} \\ \frac{4}{3} \end{bmatrix}, \quad C_1 = [1 \quad 1] \quad (5.133)$$

$$A_2 = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad B_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{4}{5} \end{bmatrix}, \quad C_2 = [1 \quad 1] \quad (5.134)$$

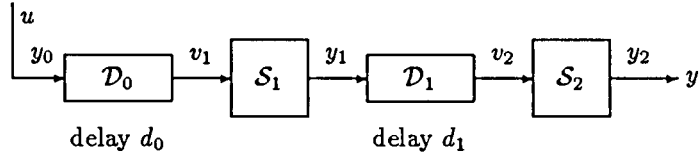


Fig. 5.4 Example of discrete-time unilateral time-delay system

$$d_0 = 3 \quad (5.135)$$

$$d_1 = 2 \quad (5.136)$$

Since there are two stations, this is the case of $p = 2$. But, as can be seen from Fig. 5.4, the delay line $\mathcal{D}_2$ at the output of station $\mathcal{S}_2$ is assumed not to exist in this example. Thus the whole system is described by

$$x(k+1) = Ax(k) + Bu(k) \quad (5.137)$$

$$y(k) = Cx(k) \quad (5.138)$$

$$A = \begin{bmatrix} \frac{3}{4} & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{4}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \quad (5.139)$$

The performance index to be minimized is set as

$$Q = \text{block diag}(Q_2, W_1, Q_1, W_0) = I$$

$$R = I = [1] \quad (5.140)$$

Then the submatrices of Q are

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (5.141)$$

$$W_1 = \text{block diag}(W_{11}, W_{12}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W_0 = \text{block diag}(W_{01}, W_{02}, W_{03}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.142)$$

The coefficient matrices of the imaginary system are given by

$$\bar{A} = \begin{bmatrix} \frac{3}{4} & 0 & \frac{4}{5} & \frac{4}{5} \\ 0 & -\frac{1}{2} & -\frac{4}{5} & -\frac{4}{5} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ \frac{3}{4} \end{bmatrix} \quad (5.143)$$

The matrices concerning the performance index are given by

$$\bar{W}_1 = [2], \quad \bar{W}_0 = [3] \quad (5.144)$$

$$\bar{Q}_2 = Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{Q}_1 = Q_1 + C_1' \bar{W}_1 C_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad (5.145)$$

Thus the coefficient matrices of the imaginary system are defined as

$$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 3 \end{bmatrix}, \quad \bar{R} = R + \bar{W}_0 = [4] \quad (5.146)$$

By solving the optimal control problem for the imaginary system, the solution $\bar{P}$ and the optimal feedback gain $\bar{F}$ are obtained as follows:

$$\bar{P} = \begin{bmatrix} 2.248311 & -0.001107 & 1.642283 & 2.019821 \\ -0.001107 & 1.333145 & 0.473136 & 0.420468 \\ 1.642283 & 0.473136 & 6.130037 & 5.812060 \\ 2.019821 & 0.420468 & 5.812060 & 8.143598 \end{bmatrix} \quad (5.147)$$

$$\bar{F} = \begin{bmatrix} 0.043344 & 0.004031 & 0.040515 & 0.231135 \end{bmatrix} \quad (5.148)$$

Now the solution for the imaginary system is transformed to the time-delay system. The matrices T , $\tilde{W}_1$, $\tilde{W}_2$, and W are given by

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (5.149)$$

$$T = \begin{bmatrix} T_2 A^5 \\ T_1 A^3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{243}{1024} & 0 & \frac{81}{320} & \frac{27}{80} & \frac{13}{20} & \frac{19}{20} & \frac{4}{15} & 0 & 0 \\ 0 & -\frac{1}{32} & -\frac{1}{20} & \frac{1}{10} & -\frac{3}{20} & -\frac{1}{5} & -\frac{4}{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{64} & 0 & -\frac{1}{12} & -\frac{1}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix} \quad (5.150)$$

$$\tilde{W}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \tilde{W}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$W = \text{block diag} (0, \tilde{W}_1, 0, \tilde{W}_0)$$

$$= \text{diag}(0, 0, 1, 2, 0, 0, 1, 2, 3) \quad (5.151)$$

Using the relation (5.75) and (5.78) given in Section 5.3.1, the solution P and the optimal feedback gain F for the time-delay system are calculated as follows:

$$P = \begin{bmatrix} 2.283608 & 0.000008 & 1.369195 & 1.025549 & 0.943551 \\ 0.000008 & 1.333333 & 0.533342 & -0.266654 & 0.118541 \\ 1.369195 & 0.533342 & 3.313821 & 0.667272 & 1.196120 \\ 1.025549 & -0.266654 & 0.667272 & 4.311853 & 0.962610 \\ 0.943551 & 0.118541 & 1.196120 & 0.962610 & 6.171979 \\ 1.220731 & 0.106723 & 1.472869 & 1.394638 & 5.980355 \\ 0.269640 & 0.007971 & 0.300370 & 0.357981 & 0.491492 \\ 0.189635 & -0.003831 & 0.196147 & 0.281963 & 0.446320 \\ 0.119455 & 0.002195 & 0.130930 & 0.162870 & 0.331109 \\ 1.220731 & 0.269640 & 0.189635 & 0.119455 & \\ 0.106723 & 0.007971 & -0.003831 & 0.002195 & \\ 1.472869 & 0.300370 & 0.196147 & 0.130930 & \\ 1.394638 & 0.357981 & 0.281963 & 0.162870 & \\ 5.980355 & 0.491492 & 0.446320 & 0.331109 & \\ 8.886275 & 2.462549 & 1.504441 & 0.880850 & \\ 2.462549 & 6.445061 & 2.525029 & 1.224532 & \\ 1.504441 & 2.525029 & 7.272910 & 2.213801 & \\ 0.880850 & 1.224532 & 2.213801 & 7.710249 & \end{bmatrix} \quad (5.152)$$

$$F = \begin{bmatrix} 0.010286 & -0.000126 & 0.010770 & 0.015032 & 0.028202 \\ 0.069263 & 0.084152 & 0.140585 & 0.254161 & \end{bmatrix} \quad (5.153)$$

As previously mentioned, the solution P can also be obtained by directly solving the Riccati equation for the time-delay system. This method was also tested, and no difference was seen in the solution, within the digits shown above.

It took 16 ms to solve the Riccati equation for the imaginary system, and 7 ms to calculate P and F from $\bar{P}$ and $\bar{F}$. On the other hand, it took 137 ms when the Riccati equation for the whole system was directly solved. Thus, the computation time of solving the optimal control problem is considerably reduced by the method proposed in this chapter. FACOM M382/M380 of Data Processing Center, Kyoto University, was used for the computation, and the iteration method was employed in solving the Riccati equation.

Now, the method of applying a similarity transformation to the system is considered. In this example the system can be transformed

into the form where the time-delay is concentrated at the input. For this purpose the similarity transformation matrix is defined as

$$\tilde{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{3} & -\frac{9}{4} & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{8} & -\frac{3}{2} & -\frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.154)$$

Applying the transformation

$$x(k) = \tilde{T}\tilde{x}(k) \quad (5.155)$$

the coefficient matrices of the system turn out to be

$$\begin{aligned} \tilde{A} &= \tilde{T}^{-1}A\tilde{T} \\ &= \begin{bmatrix} \frac{3}{4} & 0 & \frac{4}{5} & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{4}{5} & -\frac{4}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & -\frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (5.156)$$

$$\tilde{B} = \tilde{T}^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (5.157)$$

which is indeed of the form that the time-delay is concentrated at the input. On the other hand, the coefficient matrix Q of the performance index is transformed as

$$\begin{aligned}\tilde{Q} &= \tilde{T}' Q \tilde{T} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ \tilde{Q}_2' & \tilde{Q}_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{273}{256} & \frac{9}{8} & -\frac{1}{48} & -\frac{1}{12} & 0 & 0 & 0 \\ 0 & 0 & \frac{9}{8} & \frac{21}{16} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{48} & \frac{1}{6} & \frac{5}{9} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{12} & \frac{1}{3} & \frac{4}{3} & \frac{32}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.158)\end{aligned}$$

In the above matrix $\tilde{Q}$, submatrix $\tilde{Q}_1$ corresponds to the states of the delay-free station, while submatrix $\tilde{Q}_3$ corresponds to the states of the delay line concentrated at the input. The method proposed in Mita (1983) can be applied only when

$$\tilde{Q}_2 = 0, \quad \tilde{Q}_3 = 0 \quad (5.159)$$

Hence it cannot be applied to this example.

In this example the states of the delay line was weighted by the diagonal matrices W_1 and W_0 . It should be noted that nonzero elements appear in $\tilde{Q}_2$ and $\tilde{Q}_3$ even if the states of the delay line are not weighted. In fact, redefinition of the weighting matrix Q as

$$Q_2 = I, \quad W_1 = 0, \quad Q_1 = I, \quad W_0 = 0 \quad (5.160)$$

and the application of the similarity transformation yields

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{256} & 0 & -\frac{1}{48} & -\frac{1}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{16} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{48} & \frac{1}{6} & \frac{5}{9} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.161)$$

Example 2

A two-station system in which both stations have manipulating input is considered as another example. The parameters of the stations $\mathcal{S}_{c1}$, $\mathcal{S}_{c2}$, and the delay line $\mathcal{D}_{c1}$ are given as follows:

$$\begin{aligned} \mathcal{S}_{c1}: \quad A_{c1} &= \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & -\frac{2}{5} \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{5} \end{bmatrix}, \\ C_{c1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{c1} = 0 \end{aligned} \quad (5.162)$$

$$D_{c1}: \quad \tau_{c1} = 4 \quad (5.163)$$

$$\begin{aligned} \mathcal{S}_{c2}: \quad A_{c2} &= \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{2}{9} \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{9} \end{bmatrix}, \\ E_{c2} &= \begin{bmatrix} \frac{3}{20} & \frac{1}{10} \\ \frac{3}{27} & \frac{2}{27} \end{bmatrix}, \quad C_{c1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ D_{c2} &= 0 \end{aligned} \quad (5.164)$$

This is in fact the same system as that considered in Section 3.3. The controlled variables are y_1 and y_2 , the manipulating variables are u_1 and u_2 , and the major disturbance is w . It is assumed that all the state variables are directly measured. The sampling period is chosen to be $T = 1$ as in Section 3.3. Then the coefficient matrices of the discrete-time equation has been calculated as:

$$\begin{aligned} \mathcal{S}_1: \quad A_1 &= \begin{bmatrix} e^{-\frac{1}{3}} & 0 \\ 0 & e^{-\frac{2}{3}} \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2(e^{-\frac{1}{3}} - 1) \\ -(e^{-\frac{2}{3}} - 1) \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned} \quad (5.165)$$

$$\begin{aligned} \mathcal{D}_1: \quad F_1 &= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \\ H_1 &= [I \ 0 \ 0] \end{aligned} \quad (5.166)$$

$$\begin{aligned} \mathcal{S}_2: \quad A_2 &= \begin{bmatrix} e^{-\frac{1}{3}} & 0 \\ 0 & e^{-\frac{2}{3}} \end{bmatrix}, \quad B_2 = \begin{bmatrix} -(e^{-\frac{1}{3}} - 1) \\ -2(e^{-\frac{2}{3}} - 1) \end{bmatrix} \\ E_2 &= \begin{bmatrix} \frac{9}{8}(e^{-\frac{1}{3}} - e^{-\frac{2}{3}}) & \frac{1}{2}(e^{-\frac{1}{3}} - e^{-\frac{2}{3}}) \\ \frac{2}{3}(e^{-\frac{2}{3}} - e^{-\frac{1}{3}}) & \frac{5}{12}(e^{-\frac{2}{3}} - e^{-\frac{1}{3}}) \\ \frac{1}{2}e^{-\frac{2}{3}} - \frac{19}{4}e^{-\frac{1}{3}} + \frac{9}{4}e^{-\frac{1}{3}} + 2 & \\ -\frac{4}{11}e^{-\frac{2}{3}} + \frac{3}{4}e^{-\frac{1}{3}} + \frac{5}{12}e^{-\frac{2}{3}} + 1 & \end{bmatrix} \\ C_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (5.167)$$

By adding discrete-time integrator, the expanded system

$$\begin{aligned}\tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{u}(k) \\ \tilde{y}(k) &= \tilde{C}\tilde{x}(k)\end{aligned}\tag{5.168}$$

is obtained, where

$$\begin{aligned}\tilde{x}(k) &= \begin{bmatrix} \hat{x}(k) \\ x(k) \end{bmatrix}, \quad \tilde{u}(k) = u(k), \quad \tilde{y}(k) = \begin{bmatrix} \hat{y}(k) \\ y(k) \end{bmatrix} \\ \tilde{A} &= \begin{bmatrix} I & CA \\ 0 & A \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} CB \\ B \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\end{aligned}\tag{5.169}$$

Since the purpose is to control y_1 and y_2 , the weights of the performance index are naturally of the form

$$\begin{aligned}Q_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \text{diag}(q_1, q_2, q_3, q_4) \\ R_1 &= q_5, \quad R_2 = q_6\end{aligned}\tag{5.170}$$

The weighting matrices was set as

$$q_1 = q_2 = q_3 = q_4 = 1, \quad q_5 = q_6 = 10\tag{5.171}$$

by trial and error. The optimal feedback gain matrix F is

$$\begin{aligned}F &= \begin{bmatrix} 0.106 & 0.236 & 0.254 & 0.549 & 0.074 & 0.060 \\ 0.251 & -0.120 & 1.022 & -0.420 & 0.114 & 0.064 \\ 0.041 & 0.057 & 0.046 & 0.031 & 0.041 & 0.033 \\ 0.058 & 0.112 & 0.063 & 0.055 & 0.109 & 0.062 \\ 0.022 & 0.027 & 0.022 & 0.015 & 0.023 & 0.020 \\ 0.053 & 0.105 & 0.060 & 0.052 & 0.272 & 0.140 \end{bmatrix}\end{aligned}\tag{5.172}$$

In order to obtain F , it took 803 ms if the Riccati equation of the whole system was directly solved. On the other hand it took only 68 ms (52 ms to obtain $\bar{P}$ and 16 ms for the calculation of P from $\bar{P}$) when the proposed method was used.

The computation time depends on the parameters of the system and the algorithm of solving the Riccati equation. Thus it is difficult to evaluate the efficiency of a computation method with only a small number of examples, especially from the quantitative point of view. But the result of these examples show qualitatively that the computational difficulties are considerably reduced by using the method proposed in this chapter.

Chapter 6

Discrete-time optimal controller which assigns closed-loop poles in a prescribed region

This chapter is concerned with the problem for designing discrete-time optimal control systems with its closed-loop poles in a prescribed region of stability. First, by utilizing the property of Riccati equation with Q being zero matrix, we develop a method for allocating poles in a disc with its center at the origin of the complex plane and with radius less than one. Secondly, we deal with the pole placement in a disc which is in the unit disc and also contacts the point $1 + j0$ of the complex plane. To this end, a bilinear transformation and continuous-time regulator results are employed. In each case, the radius of the disc can be specified as a design parameter, and the weighting matrices of the performance index are obtained to fulfill the desired pole allocations. The design procedures are also illustrated by numerical examples.

6.1 Problem Formulation

Consider a linear discrete-time system described by

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k)\end{aligned}\tag{6.1}$$

where $x(k)$ is the state vector of dimension n , $u(k)$ is the input vector of dimension r , and $y(k)$ is the output vector of dimension m . The pair (A, B) is assumed to be stabilizable. Let the performance index be

$$J = \sum_{k=0}^{\infty} \{x'(k)Qx(k) + 2x'(k)Su(k) + u'(k)Ru(k)\}\tag{6.2}$$

and Q, R, S are weighting matrices of dimension $n \times n$, $n \times r$, $r \times r$, respectively, and

$$Q - SR^{-1}S' \geq 0, \quad R > 0. \quad (6.3)$$

Then the optimal control which minimizes J subject to (6.1) is

$$u(k) = -Fx(k) \quad (6.4)$$

with the optimal feedback gain

$$F = (B'PB + R)^{-1} (B'PA + S') \quad (6.5)$$

where P is the positive semidefinite solution of the algebraic Riccati equation

$$P = A'PA + Q - (A'PB + S)(B'PB + R)^{-1} (B'PA + S') \quad (6.6)$$

The closed-loop system is described by

$$x(k+1) = A_c x(k) \quad (6.7)$$

where

$$A_c = (BR^{-1}B'P + I)^{-1} (A - BR^{-1}S') \quad (6.8)$$

is guaranteed to be stable, i.e., all the eigenvalues of the matrix A_c are located in the unit disc of the complex plane.

Our objective is to present the methods of selecting the weighting matrices in (6.2) so that the optimal closed-loop poles are allocated in a more restricted region of stability.

In this chapter we consider two particular regions of stability, into which the closed-loop poles are to be allocated (see Fig. 6.1). One is the disc D_1 with its center at the origin of the complex plane and with radius $\alpha < 1$, and the other is the disc D_2 which has its center on the real axis and contacts the point $1 + j0$ of the complex plane. We observe that the former region of stability corresponds to placing emphasis on the fast settling of the system, while the latter region corresponds to avoiding undue oscillatory responses. In both cases the radius of the disc is to be specified as a design parameter.

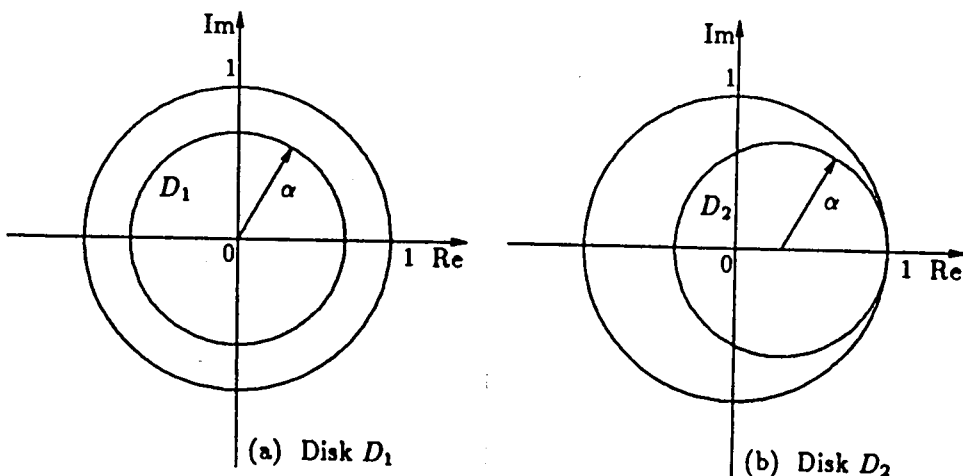


Fig. 6.1 Prescribed region of stability

6.2 Pole Placement in the Region D_1

Throughout this section we consider the performance index without the cross term, namely $S = 0$ in (6.2). Thus the Riccati equation (6.6) and the closed-loop matrix (6.8) are conveniently expressed as

$$P = A'PA - A'PB(B'PB + R)^{-1}B'PA + Q \quad (6.9a)$$

or

$$P = A'P(BR^{-1}B'P + I)^{-1}A + Q \quad (6.9b)$$

Also the closed-loop matrix (6.8) is

$$A_c = (BR^{-1}B'P + I)^{-1}A \quad (6.10)$$

The weighting matrix R may be specified arbitrarily. The objective of this section is to develop a method of determining the weighting matrix Q such that the poles of the optimal closed-loop system is placed inside the disc D_1 which has its center at the origin of the complex plane and is included in the unit disc.

We need several properties of the algebraic Riccati equation in order to derive the main result of this section. Among the eigenvalues of matrix

A , those located inside the disc of radius α will be denoted by λ_i ($i = 1, \dots, p$), and those outside the disc will be denoted by μ_j ($j = 1, \dots, n - p$). The eigenvectors of A corresponding to λ_i is denoted by v_i ($i = 1, \dots, p$). It is assumed that matrix A has no eigenvalues at the origin or on the circle of radius α .

Lemma 6.1

Let P be the maximal solution of the Riccati equation of (6.9), and let A_c be the coefficient matrix of the closed-loop system given by (6.10). If v_i belongs to the kernel of Q , namely

$$Qv_i = 0 \quad (6.11)$$

then λ_i is an eigenvalue of the closed-loop matrix A_c , and v_i is the corresponding eigenvector.

Proof

Let H be a symplectic matrix of size $2n \times 2n$ defined by

$$H = \begin{bmatrix} A + BR^{-1}B'(A')^{-1}Q & -BR^{-1}B'(A')^{-1} \\ -(A')^{-1}Q & (A')^{-1} \end{bmatrix} \quad (6.12)$$

From (6.11), (6.12) and the definition of λ_i and v_i , it follows that

$$H \begin{bmatrix} v_i \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} v_i \\ 0 \end{bmatrix} \quad (6.13)$$

It is well known (Pappas et al. 1980) that the optimal closed-loop poles are given by the eigenvalues of H inside the unit disc. Since $|\lambda_i| < \alpha < 1$, equation (6.13) shows that λ_i is an eigenvalue of the closed-loop matrix A_c and v_i is the corresponding eigenvector. $\square$

Lemma 6.2

Let P be the maximal solution of the algebraic Riccati equation (6.9) with $Q = 0$, namely

$$P = A'P(BR^{-1}B'P + I)^{-1}A \quad (6.14)$$

Then the following relation holds:

$$Pv_i = 0 \quad (i = 1, \dots, p) \quad (6.15)$$

Proof

Since $Q \geq 0$, it follows from (6.9a) that

$$P \leq A'PA \quad (6.16)$$

Observing that v_i is the eigenvector of A corresponding to the eigenvalue λ_i , we find that

$$v_i^H P v_i \leq v_i^H A' P A v_i = |\lambda_i|^2 v_i^H P v_i \quad (6.17)$$

where the superscript H denotes the conjugate transpose. Since $|\lambda_i| < \alpha < 1$, (6.17) implies

$$v_i^H P v_i = 0 \quad (6.18)$$

Thus (6.15) holds as claimed. $\square$

Lemma 6.3

Define the matrices $\tilde{A}$, $\tilde{B}$ by

$$\tilde{A} = \frac{1}{\alpha} A, \quad \tilde{B} = \frac{1}{\alpha} B \quad (6.19)$$

Let $\tilde{P}$ be the maximal solution of the Riccati equation

$$\tilde{P} = \tilde{A}' \tilde{P} (\tilde{B} \tilde{R}^{-1} \tilde{B}' \tilde{P} + I)^{-1} \tilde{A} \quad (6.20)$$

and let $\tilde{A}_c$ be given by

$$\tilde{A}_c = (\tilde{B} \tilde{R}^{-1} \tilde{B}' \tilde{P} + I)^{-1} A \quad (6.21)$$

Then the eigenvalues of $\tilde{A}_c$ are λ_i ($i = 1, \dots, p$) and α^2/μ_j ($j = 1, \dots, n-p$).

Proof

By transformation (6.19) the eigenvalues of $\tilde{A}$ are λ_i/α ($i = 1, \dots, p$) and μ_j/α ($j = 1, \dots, n-p$). Observe that the symplectic matrix H corresponding to the Riccati equation (6.20) takes the form

$$H = \begin{bmatrix} \tilde{A} & -\tilde{B} \tilde{R}^{-1} \tilde{B}' (\tilde{A}')^{-1} \\ 0 & (\tilde{A}')^{-1} \end{bmatrix} \quad (6.22)$$

Thus the eigenvalues of H are those of $\tilde{A}$ and its inverse $\tilde{A}^{-1}$. Here, observe that from (6.21)

$$\left(\tilde{B}R^{-1}\tilde{B}'\tilde{P} + I\right)^{-1}\tilde{A} = \frac{\tilde{A}_c}{\alpha} \quad (6.23)$$

and that the left hand side of (6.23) is the closed-loop matrix corresponding to (6.22). Thus the eigenvalues of $\tilde{A}_c/\alpha$ are given by those of H inside the unit disc. Hence it follows that the eigenvalues of $\tilde{A}_c$ are λ_i ($i = 1, \dots, p$) and α^2/μ_j ($j = 1, \dots, n-p$), and this completes the proof. $\square$

Above lemmas lead to the next theorem.

Theorem 6.1

Let $\tilde{P}$ be the maximal solution of the Riccati equation (6.20). Let P be the maximal solution of the Riccati equation (6.9) with the weighting matrix Q given by

$$Q = \frac{\tilde{P}}{\alpha^2} \quad (6.24)$$

Then A_c of (6.10) has eigenvalues λ_i ($i = 1, \dots, p$) and at least one eigenvalue (or a pair of complex eigenvalues) other than λ_i is located inside the disc D_1 of radius α .

Proof

First, applying Lemma 6.2 to the Riccati equation (6.20), we obtain

$$\tilde{P}v_i = 0 \quad (i = 1, \dots, p) \quad (6.25)$$

We observe from (6.19) that v_i , the eigenvector of A , is the eigenvector of $\tilde{A}$ corresponding to the eigenvalue λ_i/α . From (6.24) and (6.25) the weighting matrix $Q = \tilde{P}/\alpha^2$ satisfies the condition (6.11). Thus it follows from Lemma 6.1 that λ_i ($i = 1, \dots, p$) are the eigenvalues of A_c . Here we see from Lemma 6.3 that A_c and $\tilde{A}_c$ have eigenvalues λ_i ($i = 1, \dots, p$) in common.

Next, we consider the eigenvalues other than λ_i ($i = 1, \dots, p$). It follows from (6.9a) and (6.24) that

$$P \geq \frac{\tilde{P}}{\alpha^2} \quad (6.26)$$

Thus from (6.19) and (6.26)

$$\begin{aligned}
\det(BR^{-1}B'P + I) &= \det(R^{-1/2}B'PBR^{-1/2} + I) \\
&\geq \det\left(R^{-1/2}B'\frac{\tilde{P}}{\alpha^2}BR^{-1/2} + I\right) \\
&= \det(R^{-1/2}\tilde{B}'\tilde{P}\tilde{B}R^{-1/2} + I) \\
&= \det(\tilde{B}R^{-1}\tilde{B}'\tilde{P} + I)
\end{aligned} \tag{6.27}$$

We see from (6.10), (6.21) and (6.27) that

$$|\det(A_c)| \leq |\det(\tilde{A}_c)| \tag{6.28}$$

Let the eigenvalues of A_c other than λ_i be denoted by ν_j ($j = 1, \dots, n-p$). From Lemma 6.3 the eigenvalues of $\tilde{A}_c$ are λ_i ($i = 1, \dots, p$) and α^2/μ_j ($j = 1, \dots, n-p$). Thus, from (6.28)

$$\prod_{j=1}^{n-p} |\nu_j| \leq \prod_{j=1}^{n-p} \left| \frac{\alpha^2}{\mu_j} \right| \tag{6.29}$$

But, since

$$\left| \frac{\alpha^2}{\mu_j} \right| < \alpha \quad (j = 1, \dots, n-p) \tag{6.30}$$

it follows from (6.29) that at least one ν_j , or a complex conjugate pair, lies inside the disc D_1 of radius α . This completes the proof. $\square$

Next lemma is helpful for developing the design procedure for placing all the closed-loop poles inside the prescribed disc.

Lemma 6.4 (Amin 1984)

The coefficient matrices A , B , and the weighting matrices R , Q_i ($i = 1, \dots, p$) are assumed to be given. Let P_i be the maximal solution of the Riccati equation

$$P_i = A_i' P_i (BR_i^{-1}B'P_i + I)^{-1} A_i + Q_i \tag{6.31}$$

and define A_i and R_i ($i = 1, \dots, p$) by

$$\begin{aligned}
A_{i+1} &= (BR_i^{-1}B'P_i + I)^{-1} A_i \\
R_{i+1} &= B'P_iB + R_i
\end{aligned} \tag{6.32}$$

with

$$A_1 = A, \quad R_1 = R \quad (6.33)$$

Furthermore, define the weighting matrix Q by

$$Q = Q_1 + Q_2 + \cdots + Q_p \quad (6.34)$$

Then

$$P = P_1 + P_2 + \cdots + P_p \quad (6.35)$$

is the solution of the Riccati equation (6.9).

Above results lead to the following design procedure:

Procedure 1

Step 1 For an arbitrary weighting matrix Q_0 , solve the Riccati equation

$$P_0 = A'P_0(BR^{-1}B'P_0 + I)^{-1}A + Q_0 \quad (6.36)$$

Using the maximal solution P_0 , define the matrices A_1 and R_1 as

$$\begin{aligned} A_1 &= (BR^{-1}B'P_0 + I)^{-1}A \\ R_1 &= B'P_0B + R \end{aligned} \quad (6.37)$$

Let $i = 1$.

Step 2 Put

$$\tilde{A}_i = \frac{1}{\alpha}A_i, \quad \tilde{B} = \frac{1}{\alpha}B \quad (6.38)$$

and solve the Riccati equation

$$\tilde{P}_i = \tilde{A}_i'\tilde{P}_i(\tilde{B}R_i^{-1}\tilde{B}'\tilde{P}_i + I)^{-1}\tilde{A}_i \quad (6.39)$$

If $\tilde{P}_i = 0$ then go to Step 4.

Step 3 Put

$$Q_i = \frac{\tilde{P}_i}{\alpha^2} \quad (6.40)$$

and solve the Riccati equation (6.31). Define the matrices A_{i+1} and R_{i+1} by (6.32). Let $i = i + 1$ and go to Step 2.

Step 4 Put

$$\begin{aligned} Q &= Q_0 + Q_1 + \cdots + Q_{i-1} \\ P &= P_0 + P_1 + \cdots + P_{i-1} \end{aligned} \quad (6.41)$$

Then Q is the weighting matrix that attains the pole allocation in disc D_1 , and P is the solution of the corresponding Riccati equation (6.9). It should be noted that the weighting matrix $R > 0$ is arbitrary.

6.3 Pole Placement in the Region D_2

The objective of this section is to derive a method of determining the weighting matrices Q , R , and S of the performance index (6.2) such that the optimal closed-loop poles are allocated inside the disc D_2 (see Fig. 6.1(b)). We utilize a continuous Riccati equation for the purpose of designing the discrete-time optimal control system.

Consider the transformation from discrete-time system to continuous-time system defined by:

$$\begin{aligned} x(k+1) &:= -\dot{\bar{x}}(t) - \bar{x}(t) \\ x(k) &:= -\dot{\bar{x}}(t) + \bar{x}(t) \end{aligned} \quad (6.42)$$

Observe that the definition of the transformation (6.42) is similar to that in Kondo and Furuta (1986) except for the sign of the right hand side of (6.42). It should be noted, however, that the difference in the definition is relevant in deriving the result of the present chapter.

Let the transformed continuous time system be described by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \quad (6.43)$$

It can be easily verified that the coefficient matrices $\bar{A}$ and $\bar{B}$ are given by

$$\bar{A} = (A - I)^{-1}(A + I), \quad \bar{B} = (A - I)^{-1}B \quad (6.44)$$

In the following, the matrix $A - I$ is assumed to be nonsingular.

Lemma 6.5

If λ is an open-loop pole of the discrete-time system (6.1), then $\bar{\lambda}$ defined by

$$\bar{\lambda} = \frac{\lambda + 1}{\lambda - 1} \quad (6.45)$$

is an open-loop pole of the continuous-time system (6.43).

Proof

Consider Jordan canonical forms of A and $\bar{A}$. Then (6.45) readily follows from the first relation of (6.44). $\square$

Let $\bar{P}$ be the maximal solution of the Riccati equation

$$\bar{P}\bar{A} + \bar{A}'\bar{P} - \bar{P}\bar{B}\bar{R}^{-1}\bar{B}'\bar{P} + \bar{Q} = 0 \quad (6.46)$$

and put

$$\bar{p}(t) = \bar{P}\bar{x}(t) \quad (6.47)$$

Then the well-known continuous-time Hamilton equation is given by (Kwakernaak and Sivan 1972)

$$\begin{bmatrix} \dot{\bar{x}}(t) \\ \dot{\bar{p}}(t) \end{bmatrix} = \begin{bmatrix} \bar{A} & -\bar{B}\bar{R}^{-1}\bar{B}' \\ -\bar{Q} & -\bar{A}' \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{p}(t) \end{bmatrix} \quad (6.48)$$

Applying the transformation (6.42) to (6.48) yields

$$\begin{aligned} & \begin{bmatrix} I & 2U^{-1}\bar{B}\bar{R}^{-1}\bar{B}'(\bar{A}' - I)^{-1} \\ 0 & (I + 2U^{-1})' \end{bmatrix} \begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} \\ &= \begin{bmatrix} I + 2U^{-1} & 0 \\ -2(\bar{A}' - I)\bar{Q}U^{-1} & I \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} \end{aligned} \quad (6.49)$$

where the matrix U is defined by

$$U = (\bar{A} - I) + \bar{B}\bar{R}^{-1}\bar{B}'(\bar{A} - I)^{-1}\bar{Q} \quad (6.50)$$

and $p(k)$, $p(k+1)$ are defined similarly to (6.42).

Let the maximal solution of the discrete Riccati equation (6.6) be P , and put

$$p(k) = Px(k) \quad (6.51)$$

Then it is well known (Pappas et al. 1980) that the discrete-time Hamilton equation is given by

$$\begin{aligned} & \begin{bmatrix} I & BR^{-1}B' \\ 0 & A' - SR^{-1}B' \end{bmatrix} \begin{bmatrix} x(k+1) \\ p(k+1) \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}S' & 0 \\ -Q + SR^{-1}S' & I \end{bmatrix} \begin{bmatrix} x(k) \\ p(k) \end{bmatrix} \end{aligned} \quad (6.52)$$

Define the matrices Q , R , and S as

$$\begin{aligned} Q &= \frac{1}{2}(A - I)' \bar{Q} (A - I) \\ R &= 2\bar{R} + \frac{1}{2}B' \bar{Q} B \\ S &= \frac{1}{2}(A - I)' \bar{Q} B \end{aligned} \quad (6.53)$$

Then we see that the submatrices in (6.49) and (6.52) are related by:

$$\begin{aligned} A - BR^{-1}S' &= I + 2U^{-1} \\ BR^{-1}B' &= 2U^{-1} \bar{B} \bar{R}^{-1} \bar{B}' (\bar{A}' - I)^{-1} \\ Q - SR^{-1}S' &= 2(\bar{A}' - I) \bar{Q} U^{-1} \end{aligned} \quad (6.54)$$

Moreover it follows that

$$P = \bar{P} \quad (6.55)$$

In this case (6.45) in Lemma 6.5 holds between the continuous-time optimal closed-loop poles and the discrete-time optimal closed-loop poles.

The following is a well-known property concerning the continuous-time optimal control problem:

Lemma 6.6 (Anderson and Moore 1971)

Define the matrix $\hat{A}$ by

$$\hat{A} = \bar{A} + \bar{\alpha}I \quad (\bar{\alpha} > 0) \quad (6.56)$$

and let $\hat{P}$ be the maximal solution of the Riccati equation

$$\hat{P}\hat{A} + \hat{A}'\hat{P} - \hat{P}\bar{B}\bar{R}^{-1}\bar{B}'\hat{P} + \hat{Q} = 0 \quad (6.57)$$

where

$$\bar{Q} = 2\bar{\alpha}\hat{P} + \hat{Q} \quad (6.58)$$

Moreover, let $\bar{P}$ be the maximal solution of the Riccati equation (6.46).

Let $\bar{A}_c$ be defined by

$$\bar{A}_c = \bar{A} - \bar{B}\bar{R}^{-1}\bar{B}'\bar{P} \quad (6.59)$$

and let $\bar{\lambda}$ be an eigenvalue of $\bar{A}_c$. Then

$$\bar{P} = \hat{P} \quad (6.60)$$

and

$$\text{Re}(\bar{\lambda}) < -\bar{\alpha} \quad (6.61)$$

By putting

$$\bar{\alpha} = \frac{1 - \alpha}{\alpha} \quad (6.62)$$

and applying Lemmas 6.5 and 6.6, we have the following.

Theorem 6.2

Suppose that the eigenvalues of the continuous time optimal closed-loop system for a given set of weighting matrices satisfy the relation (6.61), where $\bar{\alpha}$ is determined from the design parameter α by (6.62). Then the eigenvalues of the discrete-time optimal closed-loop system for the weighting matrices defined by (6.53) are allocated inside the disc D_2 with its center at the point $(1 - \alpha) + j0$ and radius α .

Proof

Let λ be a pole of the discrete-time system (6.1) and $\bar{\lambda}$ be the corresponding pole of the continuous-time system transformed by (6.42). From the relations (6.45) and (6.62) it is evident that λ is inside the disc D_2 of radius α if and only if $\bar{\lambda}$ satisfies (6.61). We observe that corresponding to the transformation (6.42), the weighting matrices are given by (6.53). This completes the proof. $\square$

The design method based on this result is described in the following.

Procedure 2

Step 1 Compute matrices $\bar{A}$, $\bar{B}$ from the given coefficient matrices A , B using (6.44). Also compute $\bar{\alpha}$ from the design parameter α by (6.62).

Step 2 Put

$$\hat{A} = \bar{A} + \bar{\alpha}I \quad (6.63)$$

and solve the continuous Riccati equation (6.57).

Step 3 Calculate $\bar{Q}$, Q , R , and S by (6.58) and (6.53). Then Q , R , and S are the required weighting matrices, and the solution $\hat{P}$ of the continuous Riccati equation (6.57) is equal to the solution of the discrete Riccati equation (6.6).

It should be noted that since S in (6.53) is generally a nonzero matrix, the coupling term appears in the performance index for the present situation.

6.4 Examples

In this section the design procedures given in the previous sections are illustrated by numerical examples. Examples 1 and 2 show the application of Procedures 1 and 2, respectively. Example 3 shows the application of the same procedures to a more realistic system which has a larger dimension. The maximal solution of Riccati equations are obtained via the real Schur method due to Laub (1979).

Example 1

Consider a discrete-time system given by

$$x(k+1) = Ax(k) + Bu(k) \quad (6.64)$$

where

$$A = \begin{bmatrix} 0.9 & 0.2 & 0 & 0 \\ -0.2 & 0.9 & 1 & 0 \\ 0 & 0 & -0.2 & 0.4 \\ 0 & 0 & -0.4 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (6.65)$$

The open-loop poles of the system are located as follows: $0.9 \pm j0.2$, $-0.2 \pm j0.4$.

The design parameter α , namely the radius of the disc D_1 is chosen as $\alpha = 0.8$ in this example. The weighting matrix R , also a design parameter, is chosen as $R = I$.

In Step 1 the initial weighting matrix Q_0 is set as $Q_0 = 0$ for simplicity. Then, it follows from (6.36) and (6.37) that

$$P_0 = 0, \quad A_1 = A, \quad R_1 = I \quad (6.66)$$

In Step 2 matrices $\tilde{A}_1$ and $\tilde{B}$ are calculated as

$$\tilde{A}_1 = \frac{5}{4}A, \quad \tilde{B} = \frac{5}{4}B \quad (6.67)$$

and the Riccati equation (6.39) to be solved is

$$\tilde{P}_1 = \frac{25}{16}A'\tilde{P}_1 \left(\frac{25}{16}BB'\tilde{P}_1 + I \right)^{-1} A \quad (6.68)$$

The maximal solution $\tilde{P}_1$ turns out to be

$$\tilde{P}_1 = \begin{bmatrix} 0.2358 & -0.0268 & -0.0474 & -0.0284 \\ -0.0268 & 0.2880 & 0.2306 & 0.0805 \\ -0.0474 & 0.2306 & 0.1875 & 0.0667 \\ -0.0284 & 0.0805 & 0.0667 & 0.0244 \end{bmatrix} \quad (6.69)$$

In Step 3 matrix Q_1 is defined as

$$Q_1 = \frac{25}{16}\tilde{P}_1 \quad (6.70)$$

Then we have the Riccati equation

$$P_1 = A'P_1 (BB'P_1 + I)^{-1} A + Q_1 \quad (6.71)$$

The solution P_1 of (6.71) is

$$P_1 = \begin{bmatrix} 0.7337 & -0.0774 & -0.1427 & -0.0868 \\ -0.0774 & 0.9424 & 0.7534 & 0.2624 \\ -0.1427 & 0.7534 & 0.6113 & 0.2170 \\ -0.0868 & 0.2624 & 0.2170 & 0.0789 \end{bmatrix} \quad (6.72)$$

Matrices A_2 and R_2 are defined as

$$A_2 = (BR^{-1}B'P_1 + I)^{-1} A, \quad R_2 = B'P_1B + I \quad (6.73)$$

and Step 2 is again executed for $i = 2$. In this example the maximal solution of the Riccati equation

$$\tilde{P}_2 = \frac{25}{16} A_2' \tilde{P}_2 \left(\frac{25}{16} BR_2^{-1} B' \tilde{P}_2 + I \right)^{-1} A_2 \quad (6.74)$$

turns out to be $\tilde{P}_2 = 0$, and the iteration is terminated.

Hence the required result is obtained in Step 4 as

$$Q = Q_0 + Q_1 = \frac{25}{16} \tilde{P}_1$$

$$P = P_0 + P_1 = P_1 \quad (6.75)$$

The location of the optimal closed-loop poles corresponding to $Q = Q_0 + Q_1$ and $R = I$ are as follows: $0.5411 \pm j0.1174$, $-0.2 \pm j0.4$ (see Fig. 6.2). $\square$

Example 2

Procedure 2 is applied to the same system of (6.64) and (6.65). The design parameter α is chosen as $\alpha = 0.5$, so that no closed-loop poles should have negative real part.

In Step 1 the coefficient matrices $\bar{A}$ and $\bar{B}$ of the transformed continuous-time system are calculated by (6.44); the entries of $\bar{A}$, $\bar{B}$ are

$$\bar{A} = \begin{bmatrix} -3.0 & -8.0 & -6.0 & -2.0 \\ 8.0 & -3.0 & -3.0 & -1.0 \\ 0 & 0 & -0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} -2.0 & -3.0 \\ 4.0 & -1.5 \\ 0 & -0.75 \\ 0 & 0.25 \end{bmatrix} \quad (6.76)$$

The parameter $\bar{\alpha}$ is calculated by (6.62) to be $\bar{\alpha} = 1$.

In Step 2, the weighting matrices for the continuous-time system are chosen as

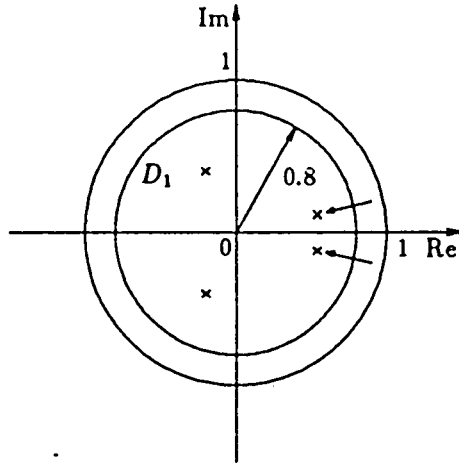


Fig. 6.2 Closed-loop poles of Example 1

$$\hat{Q} = 0, \quad \bar{R} = I \quad (6.77)$$

The matrix $\hat{A}$ is given by

$$\hat{A} = \bar{A} + I \quad (6.78)$$

Then the continuous Riccati equation (6.57) is solved, where the solution is

$$\hat{P} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5.76 & 4.48 \\ 0 & 0 & 4.48 & 7.04 \end{bmatrix} \quad (6.79)$$

In Step 3 the matrix $\bar{Q}$ is given by

$$\bar{Q} = 2\hat{P} \quad (6.80)$$

It follows from (6.53) that the desired weighting matrices Q , R , and S are calculated as

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 13.72 & 6.35 \\ 0 & 0 & 6.35 & 6.76 \end{bmatrix}, \quad R = \begin{bmatrix} 2.0 & 0 \\ 0 & 7.76 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -8.704 \\ 0 & -3.072 \end{bmatrix} \quad (6.81)$$

The location of the optimal closed-loop poles are as follows: $0.9 \pm j0.2$, $0.2308 \pm j0.1538$ (see Fig. 6.3). $\square$

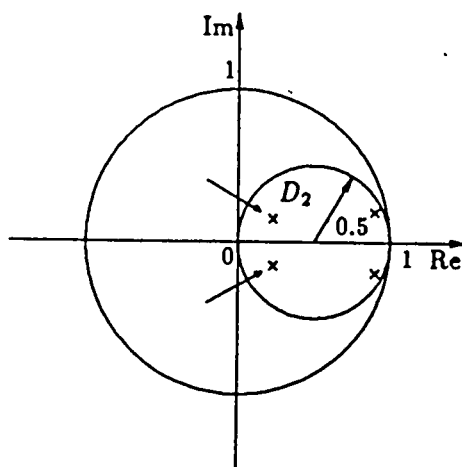


Fig. 6.3 Closed-loop poles of Example 2

Example 3

In this example, the design procedures are applied to the control of a power plant model (Katayama et al. 1985). The open-loop poles are allocated as shown in Fig. 6.4. Procedures 1 and 2 are applied to the model, where the design parameters R and α are chosen as in Examples 1

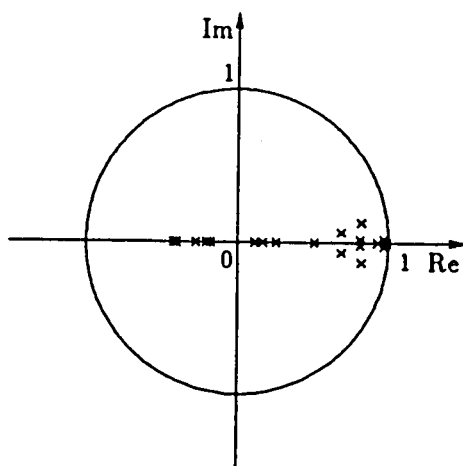


Fig. 6.4 Open-loop poles of Example 3

and 2. The closed-loop poles are successfully placed in D_1 and D_2 as shown in Figs. 6.5 and 6.6, respectively.

6.5 Remarks

Methods for allocating the poles of the optimal closed-loop system in two different desired circular regions have been presented in this chapter. One is concerned with a disc D_1 which has its center at the origin of the complex plane, and the other deals with a disc D_2 which contacts the point $1 + j0$. In the design method for the disc D_1 , the weighting matrix R and the parameter α can be specified arbitrary. Several discrete Riccati equations have to be solved iteratively. However, it is assured that the number of iterations do not exceed the number of the open-loop poles outside the desired region. For the prescribed region D_2 , the pole allocation procedure is developed with the aid of the continuous Riccati equation. Specifying the radius of the disc D_2 as the design parameter, the performance index is readily obtained. Note, however, that a cross term appears in the resultant performance index.

Due to the limitation of using the LQ technique where the performance index is nonnegative, the closed-loop poles cannot be allocated in

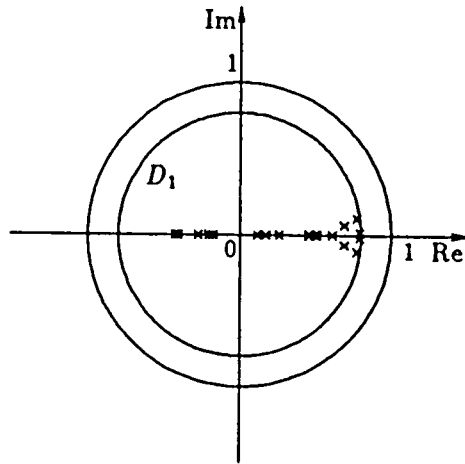


Fig. 6.5 Closed-loop poles of Example 3 in Disc D_1

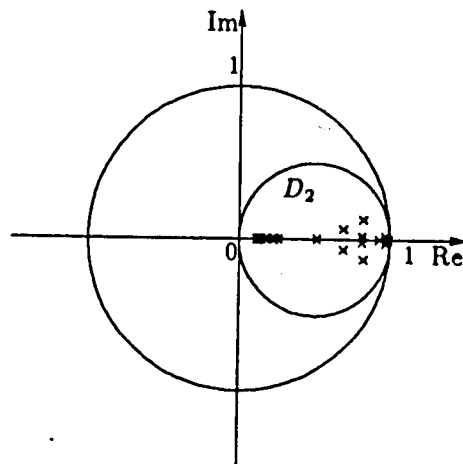


Fig. 6.6 Closed-loop poles of Example 3 in Disc D_2

an arbitrary prescribed region. However, the two procedures presented in this chapter could be combined, by which a more general region of stability can be considered. This can be done by first applying Procedure 2, followed by Procedure 1. In this case, the starting matrices Q_0 and R of Procedure 1 is determined by Procedure 2. It should be noted that the order of applying Procedures 1 and 2 cannot be reversed, since the weighting matrix R cannot be arbitrary specified in Procedure 2. Also, Procedure 1 should be modified to cope with the performance index with the cross term in Step 1.

Chapter 7

Assignable region for the closed-loop poles of discrete-time optimal controller systems

The objective of this chapter is to clarify the region of assignable closed-loop poles for shifting a single real pole or a pair of complex conjugate poles by the discrete-time optimal regulators. The modal decomposition is employed to the controlled object, and the weighting matrices are chosen so that only the specified mode is altered. The assignable region of closed-loop poles is determined by evaluating the characteristic equation of the symplectic matrix associated with the discrete-time optimal regulator problem. Numerical examples are given in which the assignable region is displayed on the complex plane.

7.1 Preliminaries

Consider the linear time-invariant discrete-time system described by

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k)\end{aligned}\tag{7.1}$$

where $x(k)$ is the state vector of dimension n , $u(k)$ is the input vector of dimension r , $y(k)$ is the output vector of dimension m , and A , B , C are coefficient matrices of size $n \times n$, $n \times r$, $m \times n$, respectively. The pair (A, B) is assumed to be controllable, and A is assumed to be invertible. Let the performance index be

$$J = \sum_{k=0}^{\infty} \{x(k)'Qx(k) + u(k)'Ru(k)\}\tag{7.2}$$

where the superscript T denotes the transpose, and Q , R are weighting matrices of dimension $n \times n$, $r \times r$, respectively, satisfying

$$Q \geq 0, \quad R > 0.\tag{7.3}$$

It is also assumed that the pair $(Q^{1/2}, A)$ is observable. Then it is well known (Kwakernaak and Sivan 1972) that the optimal control which minimizes J subject to system (7.1) is given by the linear feedback control law

$$u(k) = -Fx(k) \quad (7.4)$$

with the optimal feedback gain

$$F := (R + B'PB)^{-1}B'A \quad (7.5)$$

where P is the unique positive semidefinite solution of the matrix algebraic Riccati equation

$$P = A'PA - A'PB(R + B'PB)^{-1}B'PA + Q. \quad (7.6)$$

The closed-loop system described by

$$x(k+1) = A_c x(k) \quad (7.7)$$

where

$$A_c := (BR^{-1}B'P + I)^{-1}A \quad (7.8)$$

is guaranteed to be stable, i.e., all the eigenvalues of the matrix A_c are located in the unit disc of the complex plane.

Consider the symplectic matrix H associated with the optimal regulator problem:

$$H = \begin{bmatrix} A + VA^{-T}Q & -VA^{-T} \\ -A^{-T}Q & A^{-T} \end{bmatrix}. \quad (7.9)$$

Here, A^{-T} is the shorthand for $(A^{-1})'$, and the matrix V is defined as

$$V := BR^{-1}B' \quad (7.10)$$

The relationship between the weighting matrices Q, R and the optimal closed-loop poles can be investigated using the following lemma (Kučera 1972).

Lemma 7.1

Let the optimal closed-loop poles corresponding to the performance index (7.2) be $\{z_1, \dots, z_n\}$. Then the symplectic matrix H of (7.9) has the set of eigenvalues $\{z_1, \dots, z_n, 1/z_1, \dots, 1/z_n\}$.

By applying a similarity transformation, the system (7.1) can be expressed in the form where a specific mode is isolated from the others. Namely, choosing an appropriate nonsingular matrix M of dimension $n \times n$ yields

$$M^{-1}AM = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad (7.11)$$

where A_{11} is the submatrix corresponding to the specific mode. The dimension of A_{11} is either 1×1 or 2×2 , according as the specific mode is a single real pole or a pair of complex conjugate poles. By this transformation the matrices V and Q are transformed to $M^{-1}VM$ and $M'QM$, respectively, which are expressed in the divided form as:

$$M^{-1}VM = \begin{bmatrix} V_1 & V_2 \\ V_2 & V_3 \end{bmatrix} \quad (7.12)$$

$$M'QM = \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{bmatrix}. \quad (7.13)$$

Note that if the weighting matrix Q is chosen as to satisfy $Q_2 = Q_3 = 0$, only the mode corresponding to A_{11} is altered. This can be verified by substituting (7.11)–(7.13) into (7.9) and evaluating the characteristic equation of H . Thus, only the upper-left blocks A_{11} , V_1 , and Q_1 in (7.11)–(7.13) will be considered in the following development. The weighting matrix R , which affects V_1 , is assumed to be given *a priori*. And the matrix Q_1 is regarded as the free design parameter for achieving desired assignment of the closed-loop poles. The matrix Q_1 will be chosen such that $Q_1 \geq 0$ holds and that the pair $(Q_1^{1/2}, A_{11})$ is observable.

7.2 Assignable region for a real pole

First, the assignable region of optimal closed-loop pole for a single real pole is clarified. In this case the derivation of the result is quite straightforward, since it suffices to consider the system with only one state variable corresponding to the real pole.

Let $z = a$ be the open-loop pole to be shifted. Then A_{11} , V_1 , and Q_1 , which are all scalars, can be expressed as

$$A_{11} = [a], \quad V_1 = [1], \quad Q_1 = [q]. \quad (7.14)$$

It should be noted that multiplying the weighting matrices Q and R by a scalar ν does not affect the location of the closed-loop pole, whereas the solution of the Riccati equation and the minimum of the performance index are multiplied by ν . Thus, V_1 is assumed to be unity without loss of generality. Also, we assume that $q > 0$, because the weighting matrix Q_1 must be positive semidefinite and the choice of $q = 0$ is ruled out by the observability requirement of the pair $(Q_1^{1/2}, A_{11})$.

The symplectic matrix H_1 with A_{11} , V_1 , and Q_1 of (7.14) in place of A , V , and Q in (7.9) turns out to be

$$H_1 = \begin{bmatrix} a + \frac{q}{a} & -\frac{1}{a} \\ -\frac{q}{a} & \frac{1}{a} \end{bmatrix} \quad (7.15)$$

and the characteristic equation of H_1 is

$$\det(zI - H_1) = z^2 - \left(a + \frac{1}{a} + \frac{q}{a}\right)z + 1 = 0. \quad (7.16)$$

Let $z = a_c$ be the optimal closed-loop pole corresponding to a specific weight q . Then it follows from Lemma 7.1 that a_c is given as the stable eigenvalue of H_1 , which can be calculated by solving the quadratic equation (7.16).

Lemma 7.2

The optimal closed-loop pole $z = a_c$ and the corresponding weight q is related by

$$a_c + \frac{1}{a_c} = a + \frac{1}{a} + \frac{q}{a}. \quad (7.17)$$

Proof

Let the solution of equation (7.16) be a_{c1} , a_{c2} with $|a_{c1}| \leq |a_{c2}|$. Then by the relationship between the solution and the coefficients of the quadratic equation, we have

$$a_{c1} + a_{c2} = a + \frac{1}{a} + \frac{q}{a}, \quad a_{c1}a_{c2} = 1. \quad (7.18)$$

Putting $a_c := a_{c1}$ and eliminating a_{c2} in (7.18) yield (7.17). $\square$

The region of optimal poles for the case of shifting a single real pole is given by the following theorem.

Theorem 7.1

The closed-loop pole $z = a_c$ can be optimal if and only if the following two conditions are simultaneously satisfied:

$$a_c a > 0 \quad (7.19)$$

$$|a_c| < \min \left(|a|, \left| \frac{1}{a} \right| \right). \quad (7.20)$$

Proof

Suppose that given closed-loop pole $z = a_c$ is optimal. Then it follows from Lemma 7.2 that a_c is stable, and that there exists a weight $q > 0$ which satisfies (7.17). Since $q > 0$, it is evident that $a_c a > 0$ and

$$\left| a_c + \frac{1}{a_c} \right| > \left| a + \frac{1}{a} \right|. \quad (7.21)$$

Note that for a real number a the value of $|a + 1/a|$ takes its minimum value 2 when $|a| = 1$. This leads to (7.20), where 'min' on the right-hand side is due to the fact that a_c is stable while a can be either stable or unstable.

Conversely, suppose that a_c satisfies (7.19) and (7.20). Then (7.20) implies $|a_c| < 1$, so that (7.21) holds. Solving (7.17) for q yields

$$q = a \left(a_c + \frac{1}{a_c} - a - \frac{1}{a} \right) \quad (7.22)$$

where $q > 0$ is guaranteed by (7.19) and (7.21). Then by Lemma 7.2 $z = a_c$ is the optimal closed-loop pole corresponding to weight q given by (7.22). $\square$

By Theorem 7.1 the region of optimal closed-loop pole for shifting a single real pole is clarified, as shown in Fig. 7.1.

7.3 Assignable region for complex poles

The region of the optimal closed-loop poles is clarified for the case where a pair of complex conjugate poles is shifted. To this end, a system with two state variables is considered, corresponding to the mode to be altered. The result can be derived in a similar manner as in the previous section, but the manipulation is more complicated. First, the

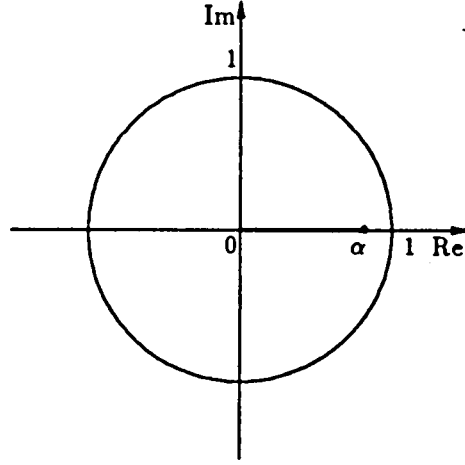


Fig. 7.1 The region of closed-loop poles for shifting a real pole

relationship between the weighting matrix and the closed-loop poles is established via the characteristic equation of the symplectic matrix. Next, temporarily assuming that Q_1 is singular, the range of the coefficients of the characteristic equation is determined. It is also shown that in some cases all the possible optimal poles can be attained by singular weighting matrices. Then, a more general result is derived, which clarifies the range of the coefficients for the case where Q_1 may be nonsingular. Finally, the range of the coefficients of the characteristic equation is interpreted to the region of optimal closed-loop poles.

7.3.1 Characteristic equation of the symplectic matrix

Let $z = \alpha \pm j\beta$ ($\beta \neq 0$) be the open-loop poles to be shifted. Then A_{11} , V_1 , and Q_1 , which are all matrices of size 2×2 , can be expressed as

$$A_{11} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 0 \\ 0 & v \end{bmatrix}, \quad Q_1 = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} \quad (7.23)$$

where v is a real number satisfying $0 \leq v \leq 1$. Note that the matrix V_1 in (7.12) is symmetric, so that it can be diagonalized by a suitable orthogonal transformation. Moreover, applying the orthogonal transformation to matrix A_{11} of the form in (7.23) yields either A_{11} itself

or A'_{11} . Furthermore, the weighting matrices Q and R can be multiplied by a scalar value ν without altering the location of the closed-loop poles, as in the case of shifting a real pole. Thus, V_1 is assumed to be of the form in (7.23) without loss of generality. The weighting matrix Q_1 must satisfy $Q_1 \geq 0$ and $Q_1 \neq 0$, where the latter constraint is due to the observability requirement of the pair $(Q_1^{1/2}, A_{11})$.

Substituting (7.23) into the symplectic matrix

$$H_1 = \begin{bmatrix} A_{11} + V_1 A_{11}^{-T} Q_1 & -V_1 A_{11}^{-T} \\ -A_{11}^{-T} Q_1 & A_{11}^{-T} \end{bmatrix} \quad (7.24)$$

and calculating the characteristic equation of H_1 yield

$$\det(zI - H_1) = z^4 + c_1 z^3 + c_2 z^2 + c_1 z + 1 = 0 \quad (7.25)$$

with the coefficients

$$c_1 = -2\alpha \left(1 + \frac{1}{\alpha^2 + \beta^2} \right) - \frac{1}{\alpha^2 + \beta^2} \bar{c}_1 \quad (7.26)$$

$$c_2 = \alpha^2 + \beta^2 + \frac{4\alpha^2 + 1}{\alpha^2 + \beta^2} + \frac{1}{\alpha^2 + \beta^2} \bar{c}_2 \quad (7.27)$$

where

$$\bar{c}_1 := \alpha q_1 + (1 - v)\beta q_2 + v\alpha q_3 \quad (7.28)$$

$$\begin{aligned} \bar{c}_2 := & (\alpha^2 + v\beta^2 + 1)q_1 + 2(1 - v)\alpha\beta q_2 + (v\alpha^2 + \beta^2 + v)q_3 \\ & + v(q_1 q_3 - q_2^2). \end{aligned} \quad (7.29)$$

Equation (7.25) enables us to calculate the optimal closed-loop poles when a specific weighting matrix Q_1 is given. This is conveniently accomplished by defining complex variable w by

$$w := z + \frac{1}{z}, \quad z \neq 0. \quad (7.30)$$

Note that $z = 0$ does not satisfy (7.25), regardless of the coefficients c_1 and c_2 . Substituting (7.30) into (7.25) yields the quadratic equation

$$w^2 + (c_2 - 2)w + c_1 = 0. \quad (7.31)$$

First, we solve the equation (7.31) for w . Then substituting the solution $w = w_i$ ($i = 1, 2$) into (7.30), we solve (7.30) for z ; namely,

$$z^2 - w_i z + 1 = 0, \quad i = 1, 2. \quad (7.32)$$

Finally, choose the stable solution of (7.32). Thus, we see that the optimal closed-loop poles for a specific weight Q_1 can be calculated by solving two quadratic equations (7.31) and (7.32), successively. It should be noted that the coefficient w_i in (7.32) may be complex in general, whereas (7.31) always has real coefficients.

The method of calculating the optimal closed-loop poles for a specific weighting matrix Q_1 can be utilized in clarifying the assignable region of optimal poles for arbitrary weights.

Lemma 7.3

Let $z = \alpha_c \pm j\beta_c$ be a pair of optimal closed-loop poles. Then the coefficients c_1, c_2 of the corresponding characteristic equation (7.25) of H_1 is given by

$$c_1 = -2\alpha_c \left(1 + \frac{1}{\alpha_c^2 + \beta_c^2} \right) \quad (7.33)$$

$$c_2 = \alpha_c^2 + \beta_c^2 + \frac{4\alpha_c^2 + 1}{\alpha_c^2 + \beta_c^2} \quad (7.34)$$

Proof

If $z = \alpha_c \pm j\beta_c$ are the optimal closed-loop poles satisfying the characteristic equation (7.25), it follows from Lemma (7.1) that $z = 1/(\alpha_c \pm j\beta_c)$ also satisfy (7.25). Thus (7.25) can be factored as

$$(z - z_{c1})(z - z_{c2}) \left(z - \frac{1}{z_{c1}} \right) \left(z - \frac{1}{z_{c2}} \right) = 0 \quad (7.35)$$

where

$$z_{c1} := \alpha_c + j\beta_c, \quad z_{c2} := \alpha_c - j\beta_c. \quad (7.36)$$

Expanding (7.35) and comparing the coefficients with (7.25) yield (7.33) and (7.34). $\square$

Equations (7.26)–(7.29) and (7.33), (7.34) show that the pair of coefficients (c_1, c_2) in the characteristic equation (7.25) can be expressed either in terms of the weighting matrix or in terms of the closed-loop poles. It should be noted that putting $\bar{c}_1 = 0$ and $\bar{c}_2 = 0$ in (7.26) and (7.27) results in the form of (7.33) and (7.34), respectively. Thus, the region of optimal closed-loop poles can effectively be investigated via the range of (c_1, c_2) . However, since only $\bar{c}_1$ and $\bar{c}_2$ depend on the weights q_1, q_2 , and q_3 , it suffices to consider the range of $(\bar{c}_1, \bar{c}_2)$ instead of the range of (c_1, c_2) .

7.3.2 Region of optimal poles for singular weight

It is assumed that $(Q_1^{1/2}, A_{11})$ is observable. Here, as a further constraint, we will temporarily restrict Q_1 to be singular matrices.

Lemma 7.4

Every nonzero, positive semidefinite, singular matrix of size 2×2 can be expressed in the form

$$Q_1 = \begin{bmatrix} \rho \cos^2 \theta & \rho \cos \theta \sin \theta \\ \rho \cos \theta \sin \theta & \rho \sin^2 \theta \end{bmatrix} \quad (7.37)$$

where $\rho > 0$ and $0 \leq \theta < \pi$.

Proof

Suppose Q_1 given in the form of (7.23) is nonzero, positive semidefinite, and singular. Then it follows that

$$0 \leq \frac{q_1}{q_1 + q_3} \leq 1 \quad (7.38)$$

and there exist ρ and θ that satisfy

$$\rho = q_1 + q_3, \quad \rho > 0 \quad (7.39)$$

$$\cos \theta = \text{sgn}(q_2) \sqrt{\frac{q_1}{q_1 + q_3}}, \quad \sin \theta = \sqrt{\frac{q_3}{q_1 + q_3}}, \quad 0 \leq \theta < \pi. \quad (7.40)$$

Substituting (7.39) and (7.40) into (7.37) yields

$$\rho \cos^2 \theta = q_1, \quad \rho \cos \theta \sin \theta = q_2, \quad \rho \sin^2 \theta = q_3 \quad (7.41)$$

which completes the proof. $\square$

Parameters $\bar{c}_1$ and $\bar{c}_2$ can be expressed in terms of ρ and θ as follows. Substituting (7.41) into (7.28) and (7.29) yields

$$\bar{c}_1 = \frac{\rho}{2} \left\{ (1+v)\alpha + (1-v)\sqrt{\alpha^2 + \beta^2} \cos(2\theta + \psi) \right\} \quad (7.42)$$

and

$$\begin{aligned} \bar{c}_2 = \frac{\rho}{2} \left\{ (1+v)(\alpha^2 + \beta^2 + 1) \right. \\ \left. + (1-v)\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2} \cos(2\theta + \phi) \right\} \end{aligned} \quad (7.43)$$

where ψ and ϕ satisfy

$$\cos \psi = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \quad \sin \psi = \frac{-\beta}{\sqrt{\alpha^2 + \beta^2}} \quad (7.44)$$

and

$$\begin{aligned} \cos \phi &= \frac{\alpha^2 - \beta^2 + 1}{\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2}} \\ \sin \phi &= \frac{-2\alpha\beta}{\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2}}. \end{aligned} \quad (7.45)$$

It is clear from (7.42) and (7.43) that when θ varies over $0 \leq \theta < \pi$ with ρ fixed, the trajectory of $(\bar{c}_1, \bar{c}_2)$ on the $\bar{c}_1\bar{c}_2$ -plane will be an ellipse. Moreover, the size of the ellipse is proportional to ρ , as shown in Fig. 7.2. Hence the envelope of these ellipses will give the boundary for the region of $(\bar{c}_1, \bar{c}_2)$.

Lemma 7.5

Parameter $\bar{c}_1$ does not change the sign for all θ if and only if the inequality

$$4v\alpha^2 - (1-v)^2\beta^2 > 0 \quad (7.46)$$

holds.

Proof

Since $\bar{c}_1$ in (7.42) is a continuous function of θ and $0 \leq v \leq 1$, it is clear that $\bar{c}_1$ does not vanish for any θ if and only if

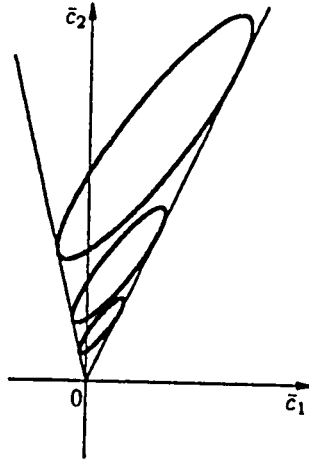


Fig. 7.2 The region of $\bar{c}_1, \bar{c}_2$

$$(1 + v)|\alpha| > (1 - v)\sqrt{\alpha^2 + \beta^2}. \quad (7.47)$$

Then (7.46) follows by squaring both sides of (7.47) and rearranging it. $\square$

Note that when (7.46) holds, the sign of $\bar{c}_1$ is fixed to that of α which is the real part of the open-loop pole.

Lemma 7.6

Inequality $\bar{c}_2 > 0$ holds for arbitrary θ .

Proof

Since $0 \leq v \leq 1$ in (7.43), it is clear that

$$(1 + v)(\alpha^2 + \beta^2 + 1) > 0$$

$$(1 - v)\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2} \geq 0. \quad (7.48)$$

Thus, it suffices to verify that

$$(1 + v)(\alpha^2 + \beta^2 + 1) > (1 - v)\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2}. \quad (7.49)$$

Direct calculation shows that

$$\begin{aligned}
& \left\{ (1+v)(\alpha^2 + \beta^2 + 1) \right\}^2 \\
& - \left\{ (1-v)\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2} \right\}^2 \\
& = 4(\alpha^4 v + 2\alpha^2 \beta^2 v + 2\alpha^2 v + \beta^4 v + \beta^2 v^2 + \beta^2 + v) \\
& > 0.
\end{aligned} \tag{7.50}$$

Consequently, $\bar{c}_2 > 0$ holds for arbitrary θ . $\square$

Now a result concerning the range of the parameters $\bar{c}_1, \bar{c}_2$ can be stated as follows.

Theorem 7.2

The region of $(\bar{c}_1, \bar{c}_2)$ corresponding to singular weight Q_1 is a sector described by

$$k_1 \leq \frac{\bar{c}_1}{\bar{c}_2} \leq k_2. \tag{7.51}$$

The value of $\bar{c}_1/\bar{c}_2$ takes its boundary values k_1 and k_2 when θ satisfies

$$\sin(2\theta + \sigma) = -\frac{1-v}{1+v} \cdot \frac{\alpha^2 + \beta^2 - 1}{\sqrt{(\alpha^2 - \beta^2 - 1)^2 + (2\alpha\beta)^2}} \tag{7.52}$$

where σ is determined by

$$\begin{aligned}
\sin \sigma &= \frac{\alpha^2 - \beta^2 - 1}{\sqrt{(\alpha^2 - \beta^2 - 1)^2 + (2\alpha\beta)^2}} \\
\cos \sigma &= \frac{2\alpha\beta}{\sqrt{(\alpha^2 - \beta^2 - 1)^2 + (2\alpha\beta)^2}}.
\end{aligned} \tag{7.53}$$

Proof

From (7.42) and (7.43) it is clear that $\bar{c}_1/\bar{c}_2$ depends only on θ and not on ρ . Moreover, $\bar{c}_2 > 0$ is guaranteed by Lemma 7.6. Thus, the boundary of $\bar{c}_1/\bar{c}_2$ is attained by the θ which satisfies

$$\frac{d}{d\theta} \left(\frac{\bar{c}_1}{\bar{c}_2} \right) = 0. \tag{7.54}$$

Substituting (7.42) and (7.43) into (7.54) yields

$$\begin{aligned}
& (1 - v^2)\alpha\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2} \sin(2\theta + \phi) \\
& - (1 - v^2)(\alpha^2 + \beta^2 + 1)\sqrt{\alpha^2 + \beta^2} \sin(2\theta + \psi) \\
& + (1 - v^2)^2\sqrt{(\alpha^2 - \beta^2 + 1)^2 + (2\alpha\beta)^2}\sqrt{\alpha^2 + \beta^2} \sin(\phi - \psi) \\
& = 0.
\end{aligned} \tag{7.55}$$

Equation (7.52) is established by carrying out a routine manipulation of trigonometric functions. $\square$

The result of Theorem 7.2 can be used for calculating the range of $\bar{c}_1$ and $\bar{c}_2$ as follows. Consider the case where the inequality (7.46) of Lemma 7.5 holds. Then the sign of $\bar{c}_1$ is fixed to that of α , and it follows that k_1 and k_2 in Theorem 7.2 are of the same sign. Thus, for a fixed value of $\bar{c}_1$, the bounds of $\bar{c}_2$ are given by $k_1/\bar{c}_1$ and $k_2/\bar{c}_1$. On the other hand, consider the case where the left hand side of (7.46) is negative. In this case, it follows that $k_1 < 0$ and $k_2 > 0$. Thus, for a fixed value of $\bar{c}_1$, only the lower bound exists for $\bar{c}_2$, which is given by either $k_1/\bar{c}_1$ or $k_2/\bar{c}_1$, according to the sign of $\bar{c}_1$. Another special case is when the left hand side of (7.46) vanishes. In this case, $\bar{c}_1$ remains either nonnegative or nonpositive, and either k_1 or k_2 vanishes. Only the lower bound of $\bar{c}_2$ exists for a fixed value of $\bar{c}_1$.

7.3.3 Region of optimal poles for general weight

Here, the restriction that the weighting matrix Q_1 is singular is removed, and the region of optimal poles for general weighting matrices $Q_1 \geq 0$ is considered. First, the results of Lemmas 7.5 and 7.6 are extended to the case of general Q_1 .

Lemma 7.7

Parameter $\bar{c}_1$ does not change the sign for all positive semidefinite, nonzero matrices Q_1 , if and only if the inequality (7.46) holds.

Proof

Suppose inequality (7.46) holds. Then by Lemma 7.5 $\bar{c}_1$ is either positive or negative for all singular Q_1 , where $q_2 = \pm\sqrt{q_1q_3}$. Since $\bar{c}_1$ is linear in q_2 , it follows that $\bar{c}_1$ does not vanish for all q_2 satisfying

$$-\sqrt{q_1 q_3} \leq q_2 \leq \sqrt{q_1 q_3} . \quad (7.56)$$

The converse is obvious from Lemma 7.5. $\square$

Lemma 7.8

Inequality $\bar{c}_2 > 0$ holds for arbitrary positive semidefinite, nonzero matrices Q_1 .

Proof

Applying Lemma 7.6 to (7.29) shows that

$$(\alpha^2 + v\beta^2 + 1)q_1 + 2(1 - v)\alpha\beta q_2 + (v\alpha^2 + \beta^2 + v)q_3 > 0 \quad (7.57)$$

holds for $q_2 = \pm\sqrt{q_1 q_3}$. Hence (7.57) holds for all q_2 satisfying (7.56). Moreover, $v(q_1 q_3 - q_2^2)$ is always nonnegative since $0 \leq v \leq 1$. Therefore $\bar{c}_2 > 0$ for arbitrary positive semidefinite, nonzero matrices Q_1 , as claimed. $\square$

Consider the bounds of $\bar{c}_2$ for a fixed value of $\bar{c}_1$. It can be shown that the minimum of $\bar{c}_2$ is still attained by a singular weight Q_1 .

Lemma 7.9

For a fixed value of $\bar{c}_1$, the minimum value of $\bar{c}_2$ is attained by a singular weighting matrix Q_1 .

Proof

In the proof of Lemma 7.8 the left hand side of (7.57) is linear in q_2 , so that its minimum is attained when Q_1 is singular. Moreover, the minimum of $v(q_1 q_3 - q_2^2)$ is also attained when Q_1 is singular. Hence the minimum of $\bar{c}_2$ defined by (7.29) is attained by a singular weight. $\square$

Lemma 7.7 shows that the existence of the upper bound for $\bar{c}_2$ depends on the inequality (7.46) as in the restricted case of singular weighting matrices Q_1 . If inequality (7.46) does not hold, the upper bound of $\bar{c}_2$ does not exist. In this case, all the range of $\bar{c}_1$, $\bar{c}_2$ can be covered by singular weighting matrices Q_1 . Single input systems always fall into this case.

Theorem 7.3

Consider the region of optimal poles for the case of shifting a pair of complex poles. If the given system is of single input, all the possible closed-loop poles can be attained by singular weighting matrices Q_1 .

Proof

If (7.1) is a single input system, V defined by (7.10) must be of rank 1, since B is a column vector. Then, it follows that $v = 0$ in (7.23), and inequality (7.46) does not hold. This means that the upper bound of $\bar{c}_2$ for a fixed $\bar{c}_1$ does not exist, and all the range of $\bar{c}_1, \bar{c}_2$ is covered by singular weighting matrices Q_1 . $\square$

For the case where the inequality (7.46) does hold, there may be some region where the upper bound of $\bar{c}_2$ for nonsingular Q_1 is greater than that for singular Q_1 . Next lemma is concerned with this possibility.

Lemma 7.10

Assume that the inequality (7.46) holds, and the value of $\bar{c}_1$ is fixed to be $\bar{c}_1 = \tilde{c}_1$. Then there exists a value of $\bar{c}_2$ that can be attained by nonsingular Q_1 and not by any singular Q_1 , if and only if the matrix Q_1 with its entries

$$\begin{aligned} q_1 &= \frac{1}{v} (\lambda v \alpha - v \alpha^2 - \beta^2 - v) \\ q_2 &= \frac{1-v}{2v} (-\lambda \beta + 2\alpha \beta) \\ q_3 &= \frac{1}{v} (\lambda \alpha - \alpha^2 - v \beta^2 - 1) \end{aligned} \quad (7.58)$$

is positive semidefinite, where

$$\lambda := \frac{2v \{ \tilde{c}_1 + 2\alpha(\alpha^2 + \beta^2 + 1) \}}{4v\alpha^2 - (1-v)^2\beta^2}. \quad (7.59)$$

Proof

Let $\bar{c}_1$ be fixed to $\bar{c}_1 = \tilde{c}_1$. Since Lemma 7.9 shows that the minimum of $\bar{c}_2$ is attained by a singular weight Q_1 regardless of the inequality (7.46), the problem is to determine the maximum of

$$\begin{aligned} \bar{c}_2 &= (\alpha^2 + v\beta^2 + 1)q_1 + 2(1-v)\alpha\beta q_2 + (v\alpha^2 + \beta^2 + v)q_3 \\ &\quad + v(q_1 q_3 - q_2^2) \end{aligned} \quad (7.60)$$

with respect to q_1, q_2, q_3 , subject to the constraint

$$\alpha q_1 + (1-v)\beta q_2 + v\alpha q_3 = \tilde{c}_1. \quad (7.61)$$

This can be solved by employing Lagrange's method of indeterminate coefficients. It turns out that the maximum is attained by putting q_1 , q_2 , q_3 as in (7.58), where λ is the Lagrange multiplier. Inequality (7.46) guarantees that the denominator of (7.59) does not vanish, and that $v \neq 0$. It should be noted that the weighting matrix Q_1 with its entries given by (7.58) may not be positive semidefinite. Therefore, the allowable maximum of $\bar{c}_2$ is attained by (7.58) if and only if $Q_1 \geq 0$. $\square$

Now the region of optimal poles for general positive semidefinite, nonzero weighting matrices Q_1 can be described in terms of the range of parameters $\bar{c}_1$ and $\bar{c}_2$.

Theorem 7.4

If inequality (7.46) does not hold, the region of $(\bar{c}_1, \bar{c}_2)$ corresponding to general positive semidefinite, nonzero weight Q_1 is identical to the sector given in Theorem 7.2. If inequality (7.46) holds, the minimum of $\bar{c}_2$ for a fixed $\bar{c}_1$ is still attained by a singular weight Q_1 . The maximum of $\bar{c}_2$ is attained by the weighting matrix with its entries given by (7.58) and (7.59), provided $Q_1 \geq 0$. Otherwise, the maximum is also attained by a singular weight Q_1 .

Proof

It has been shown by Lemmas 7.7 and 7.9 that if the inequality (7.46) does not hold, the range of $\bar{c}_1, \bar{c}_2$ can be covered by singular Q_1 . If the inequality (7.46) holds, the results of Lemmas 7.9 and 7.10 respectively determine the minimum and maximum of $\bar{c}_2$. This is nothing but what is claimed in the rest of the theorem. $\square$

The region of optimal closed-loop poles is not explicitly shown in Theorem 7.4. However, once the parameters α , β , and v of the open-loop system is given, it is straightforward to visualize the region by numerical computation. First, the region of $\bar{c}_1$ and $\bar{c}_2$ is calculated by substituting the values of α , β , and v into equations (7.52), (7.53), (7.58), and (7.59). It may be noted that the inequality (7.46) can be easily checked when specific values of α , β , and v are given. Next, the region of $(\bar{c}_1, \bar{c}_2)$ is translated into that of (c_1, c_2) via (7.26) and (7.27). Then, the characteristic equation (7.25) is solved for the boundary values of (c_1, c_2) . The characteristic equation is of 4th order, but the solution is obtained

by solving two quadratic equations consecutively, as mentioned before. Finally, plotting the stable solution of the characteristic equation on the complex plane gives a picture of the region of optimal closed-loop poles.

7.4 Examples

In this section, the region of assignable optimal closed-loop poles is shown by numerical examples. Two examples for shifting a pair of complex conjugate poles are considered. Example 1 considers a single input system. In this case, all the assignable closed-loop poles are attained by singular a weighting matrix. Example 2 corresponds to the case of multi-input system where singular weighting matrices do not cover all the assignable region. In both examples, it is assumed that the modal decomposition has already been carried out.

Example 1

Consider the case of

$$\alpha = 0.6, \quad \beta = 0.4, \quad v = 0 \quad (7.62)$$

in (7.23). Substituting these values into (7.42)–(7.45) yields

$$\bar{c}_1 = \frac{\rho}{2} \{0.6 + 0.7211 \cos(2\theta + \psi)\} \quad (7.63)$$

$$\bar{c}_2 = \frac{\rho}{2} \{1.52 + 1.292 \cos(2\theta + \phi)\} \quad (7.64)$$

where

$$\cos \psi = 0.8321, \quad \sin \psi = -0.5547 \quad (7.65)$$

$$\cos \phi = 0.9285, \quad \sin \phi = -0.3714 . \quad (7.66)$$

The left hand side of (7.46) turns out to be -0.16 ; thus inequality (7.46) does not hold. Then, as stated in Theorem 7.3, it follows that all the assignable closed-loop poles are attained by singular weighting matrices. Moreover, the region of $\bar{c}_1, \bar{c}_2$ is determined by the sector described in Theorem 7.2. Solving (7.52) for θ yields

$$\theta = 0.5404, 1.5708 \quad (7.67)$$

and the range of $\bar{c}_1/\bar{c}_2$ turns out to be

$$-0.5 \leq \frac{\bar{c}_1}{\bar{c}_2} \leq 0.5 . \quad (7.68)$$

The region of $\bar{c}_1, \bar{c}_2$ can be mapped into the region of closed-loop poles on the complex plane, as shown in Fig. 7.3.

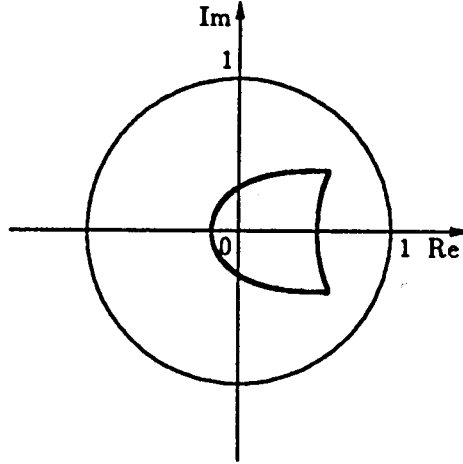


Fig. 7.3 The region of closed-loop poles (Example 1)

Example 2

Consider the case of

$$\alpha = 0.6, \quad \beta = 0.4, \quad v = 0.2 \quad (7.69)$$

in (7.23). Note that the open-loop poles are the same as in Example 1. Substituting these values into (7.42)–(7.45) yields

$$\bar{c}_1 = \frac{\rho}{2} \{0.72 + 0.5769 \cos(2\theta + \psi)\} \quad (7.70)$$

$$\bar{c}_2 = \frac{\rho}{2} \{1.824 + 1.034 \cos(2\theta + \phi)\} \quad (7.71)$$

where

$$\cos \psi = 0.8321, \quad \sin \psi = -0.5547 \quad (7.72)$$

$$\cos \phi = 0.9285, \quad \sin \phi = -0.3714 . \quad (7.73)$$

In this example, the left hand side of (7.46) turns out to be 0.1856; thus inequality (7.46) does hold, and $\bar{c}_1$ is always positive. Hence, the region of closed-loop poles must be determined according to Theorem 7.4.

First, the range of $\bar{c}_1/\bar{c}_2$ for singular weighting matrices is clarified. Solving (7.52) for θ yields

$$\theta = 0.4453, 1.6660 \quad (7.74)$$

and it follows that

$$0.1741 \leq \frac{\bar{c}_1}{\bar{c}_2} \leq 0.4722 . \quad (7.75)$$

Thus, for a fixed value of $\bar{c}_1 = \tilde{c}_1$, the range of $\bar{c}_2$ corresponding to a singular weighting matrix is

$$2.118\tilde{c}_1 \leq \bar{c}_2 \leq 5.744\tilde{c}_1 . \quad (7.76)$$

Next, the maximum of $\bar{c}_2$ for a fixed value of $\bar{c}_1 = \tilde{c}_1$ is calculated. Substituting (7.69) into (7.58) and (7.59) yields

$$\begin{aligned} q_1 &= 1.293\tilde{c}_1 + 0.199 \\ q_2 &= -1.724\tilde{c}_1 - 2.185 \\ q_3 &= 6.466\tilde{c}_1 + 4.833 \end{aligned} \quad (7.77)$$

A simple calculation shows that the matrix Q_1 with these entries becomes positive definite if and only if $\tilde{c}_1 > 0.8413$. Substituting (7.77) into (7.29) yields

$$\bar{c}_2 \leq 1.078\tilde{c}_1^2 + 3.931\tilde{c}_1 + 0.763 . \quad (7.78)$$

Consequently, we have

$$\begin{aligned} 2.118\bar{c}_1 &\leq \bar{c}_2 \leq 5.744\bar{c}_1 && (\text{ if } \bar{c}_1 \leq 0.8413) \\ 2.118\bar{c}_1 &\leq \bar{c}_2 \leq 1.078\bar{c}_1^2 + 3.931\bar{c}_1 + 0.763 && (\text{ if } \bar{c}_1 > 0.8413) . \end{aligned} \quad (7.79)$$

Mapping this region into the region of closed-loop poles on the complex plane results in Fig. 7.4.

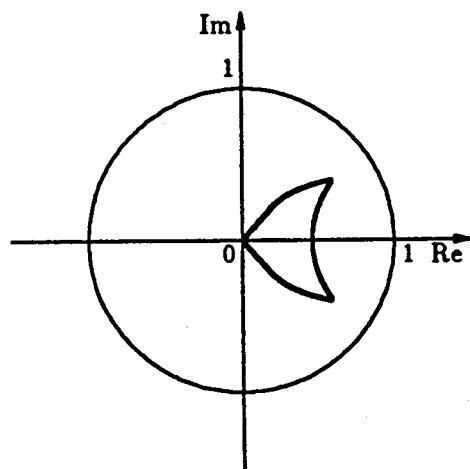


Fig. 7.4 The region of closed-loop poles (Example 2)

Chapter 8

Conclusions

The results presented in this thesis are summarized in this chapter. In Chapter 3, continuous-time unilateral time-delay systems were formulated using difference-differential equations. Unilateral time-delay systems of a relatively simple structure were also represented in the form using a partial differential equation. It was shown that the unilateral time-delay systems belong to the class of retarded-type time-delay systems. Then, the continuous-time difference-differential equation was discretized, and the formula to calculate the coefficients of the discrete-time difference equations were presented. It was shown that the discretization is possible without any approximating assumptions about the behavior of the state variables between sampling instants, and that the dimension of the resultant discrete-time equation is finite. This result is due to the special structure of the unilateral time-delay systems that the time-delays are allocated in series to the signal flow.

The optimal control problem of continuous-time unilateral time-delay systems was solved in Chapter 4. The difficulty of directly solving the optimal control problem of a system having time-delays was avoided by reducing the original system into an imaginary delay-free system. The imaginary system is obtained by eliminating the time-delays of the original system. According to the optimal control problem for the original time-delay system, a related optimal control problem for the imaginary system was set up, which can be solved by usual methods. Then the optimal steady-state solution for the original time-delay system was derived from the solution for the imaginary system. Since the optimal solution for the initial part cannot be obtained by this method, it was separately solved as a finite-time optimal control problem. For this purpose, the unilateral time-delay system was described using a partial differential equation. It was shown that only a very simple form need be considered for the unilateral time-delay systems, as far as the optimal solution for the initial part is concerned. Thus the solution of

the optimal control problem can be obtained with less difficulties by the proposed method than by solving it directly for general unilateral time-delay systems. As for the steady state solution, the structure of the controller was examined from practical point of view. A modification of the controller in relation to shifting the time in the control law was proposed. Also, the structure of a controller with an observer for the imaginary system was presented, assuming the case where only the final output of the unilateral time-delay system are available for measurement.

In Chapter 5, the optimal control problem of discrete-time unilateral time-delay systems is considered, and a method to reduce computational difficulties in solving the Riccati equation was presented. First, a delay-free imaginary system was introduced and a related optimal control problem was set up for the imaginary system, as in the continuous-time case. The size of coefficient matrices is smaller than that of the original system, since the state variables corresponding to delay lines are eliminated in the imaginary system. Thus the solution of the optimal control problem for the imaginary system can be solved with less computational difficulties. Then the optimal solution for the original time-delay system is calculated from the optimal solution for the imaginary system by using the formula given in Section 5.3.2. The formula was derived from the relation of the performance indices between the original time-delay system and the imaginary delay-free system, and it is given in the form of a recurrence formula. The formula becomes much simpler for the single input case in which no manipulating inputs exist at intermediate stations. Thus, the formula for the single input case is presented separately in Section 5.3.1.

In Chapter 6, the optimal control theory was applied to the problem of allocating the closed-loop poles in a desired region. This method is effective when the exact location of each closed-loop pole is of little concern. Two different circular regions was considered as the desired region. One is a disc which has its center at the origin of the complex plane. This region may be selected when the fast settling of the system is desired. The other region is a disc which contacts the point $1 + j0$. This region may be selected when undue oscillatory responses should be avoided.

In Chapter 7, the region of assignable closed-loop poles for shifting a single real pole or a pair of complex conjugate poles by the discrete-

time optimal regulators has been clarified. For a real pole, the assignable closed-loop poles lie on the real axis closer to the origin but not across the imaginary axis of the complex plane. For a pair of complex conjugate poles, the assignable closed loop poles lie on a region closer to the origin and sometimes across the imaginary axis, as shown in the example. The result of this chapter leads to a design method which enables the exact placement of the closed-loop poles. The designer can check whether or not a specific location of the closed-loop poles can be attained by the discrete-time optimal regulator, i.e., if there exists a positive semidefinite weighting matrix corresponding to the specified closed-loop poles.

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